MATHEMATICS

PROPERTIES OF THE SET OF TOPOLOGICALLY INVARIANT MEANS ON P. EYMARD'S, W^*-ALGEBRA $VN(G)$

BY

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(Communicated by Prof. T. A. SPRINGER at the meeting of September 29, 1973)

Let G be any locally compact group and $VN(G)$ its associated Von Neumann algebra as in EYMARD [3]. We show, for nondiscrete G, that the set of topologically invariant means (states) on $VN(G)$ (denoted by $TIM(\hat{G})$) is not norm separable. In case G is second countable and nondiscrete, then $TIM(\hat{G})$ does not have w^* exposed points. These results improve a result of P. F. RENAUD [7] who showed that $VN(G)$ admits a unique topologically invariant mean if and only if G is discrete. We make at the end the conjecture which roughly states that for *any* nondiscrete G, $TIM(\hat{G})$ does not possess w^* exposed points. (For exact statement see end of paper.)

INTRODUCTION

Let G be *any* locally compact group. The Fourier algebra $A(G)$ and its dual Von Neumann algebra $VN(G)$ (isomorphic to $L^1(\hat{G})$, $L^\infty(\hat{G})$ resp in the abelian case) have been defined and extensively studied by P. EYMARD in [3], whose notations we follow in general (see especially [3] p. 185).

For $f \in L^1(G)$ let $\|f\|_\varrho$ denote the operator norm of ϱf: $L^2(G) \to L^2(G)$ defined by $(\varrho f)(g) = f \star g$ (the convolution formulae are as in [4], consistent with [3]). $\| \ \|_\varrho$ is a C^*-norm on the convolution algebra $L^1(G)$. The dual Banach space $B(G)$, of the normed space $(L^1(G), \| \ \|_\varrho)$, is identified in [3] with the linear span of $P(G)$, the set of continuous positive definite functions on G. $B(G)$ with the dual norm and pointwise multiplication is a commutative Banach algebra. The closed subalgebra $A(G)$ (in fact ideal) generated by the elements of $B(G)$ with compact support is called the Fourier algebra of G. We denote $P_1(G) = \{u \in P(G) \cap A(G); u(e) = 1\}$ where e is the unit of G.

The Banach space dual of $A(G)$ is identified by EYMARD in [3] to be the Von Neumann algebra on $L^2(G)$ generated by $\varrho[L^1(G)]$. Denote the pairing of this duality by $T(u)$ (or Tu) for $T \in VN(G)$ and $u \in A(G)$. We digress here from the notations in [3]. No ambiguity will arise since we do not use anywhere the notations on p. 213 of [3].

$VN(G)$ becomes a $A(G)$-module by defining for $u, v \in A(G)$ $T \in VN(G)$, $(u \cdot T)(v) = T(uv)$. One has $\|u \cdot T\| \leqslant \|u\| \ \|T\|$.

Let $\psi \in VN(G)^*$ (the Banach space dual of $VN(G)$). ψ is said to be topologically invariant if $\psi(u \cdot T) = \psi(T)$ for all $u \in P_1(G)$ and $T \in VN(G)$.

$\psi \in VN(G)$ is a mean (or a state) if $\psi \geqslant 0$ and $\psi(I)=1$ where $I \in VN(G)$ is the identity. If ψ possesses both the above properties then ψ is said to be a topologically invariant mean (TIM) and the set of all such $\psi \in VN(G)^*$ is denoted by $TIM(\hat{G})$. In case G is abelian, it has been shown by Dunkl and Ramirez that $\psi \in TIM(\hat{G}) \subset L^\infty(\hat{G})^*$ iff $\psi \geqslant 0$, $\psi(1)=1$ and $\psi(f \star g)=\psi(g)$ for all $0 \leqslant f \in L^1(\hat{G})$ of norm 1 and all $g \in L^\infty(\hat{G})$, [9].

If (X, τ) is a locally convex space (with topology τ) with dual X' and $K \subset X$ is convex then $x_0 \in K$ is a τ-exposed point of K if for some $x_0' \in X'$ one has $x_0'(x) < x_0'(x_0)$ for all $x_0 \neq x \in K$. $x_0 \in K$ is a $\tau - G_\delta$ point of K if $\{x_0\} = \bigcap_1^\infty V_n \cap K$ for some open $V_n \subset X$, $n=1, 2, \ldots$.

The main result of our paper implies the

THEOREM: (a) If G is not discrete, $TIM(\hat{G})$ is not norm separable. (If G is descrete then $\psi(T)=T(\delta_e)$ is the unique TIM on $VN(G)$ (see [7] p. 286) where $\delta_e \in P_1(G)$ is one at e and zero otherwise).

(b) If G is second countable nondiscrete then $TIM(\hat{G})$ does not posses w^*-exposed points.

We conjecture in the end that (b) holds for any nondiscrete locally compact G and show that it is enough to prove this for compactly generated G. We do not know to prove this conjecture even for the case that G is compact abelian nonsecond countable.

The results in the body of the paper are somewhat stronger than the ones in this introduction and the main conjecture is slightly different from the one above.

THEOREM 1: Let G be a separable metric locally compact group. Let $F \subset P_1(G)$ be convex and $\phi_n \in VN(G)$, $n=1, 2, \ldots$. If $A = [w^* \text{ cl } F] \cap \cap \{\psi \in TIM(\hat{G}); \psi(\phi_n)=0, n=1, 2, \ldots\} \neq \emptyset$ is norm separable or has w^*-exposed points then G is discrete. (If G is discrete $TIM(\hat{G})$ contains one element).

NOTATION: Define for $u \in A(G)$, $t_u' : A(G) \to A(G)$ by $t_u' v = uv$. Let $t_u = (t_u')^* : VN(G) \to VN(G)$ be the adjoint and define $T_u = t_u^* : VN(G)^* \to \to VN(G)^*$.

PROOF: Let D be a countable norm dense subset of $P_1(G)$ (use [3] p. 207). If $\psi \in VN(G)^*$ satisfies $T_u \psi = \psi$ for all $u \in D$ then $T_u \psi = \psi$ for all $u \in P_1(G)$. Since if $u \in P_1(G)$ and $u_n \in D$, $\|u_n - u\| \to 0$, then $\|u_n \cdot \phi - u \cdot \phi\| \to 0$ for all $\phi \in VN(G)$, thus

$$\psi(\phi) = (T_{u_n} \psi)\phi \to (T_u \psi)(\phi) \text{ as } n \to \infty.$$

Hence, if $D = \{v_n\}_1^\infty$ then

$$A = [w^* \text{ cl } K] \cap \{\psi \in VN(G)^*, (t'_{v_n} - I')^{**}\psi = 0, \psi(\phi_n) = 0, n \geqslant 1\}$$

where $I' : A(G) \to A(G)$ is the identity. Apply now our Corolary 1.3 on p. 21 in [5]. It follows that there exists some $\psi_0 \in A \cap w^*$ seq cl K. Let

$w_n \in F$ be such that for all $\phi \in VN(G)$, $\phi(w_n) \to \psi_0(\phi)$. Then $w_n \in P_1(G)$ is a weak Cauchy sequence. By a theorem of SAKAI [8] the predual of a W^* algebra is weakly sequentially complete. There is hence some $v_0 \in A(G)$ such that $\phi(v_0) = \psi_0(\phi)$ for all $\phi \in VN(G)$ (i.e. ψ_0 is represented by an element of $A(G)$). Proposition 5 of RENAUD [7] p. 288 implies that G is discrete.

REMARK: Assuming the continuum hypothesis we get that for any non-discrete second countable G and for *any* convex $F \subset P_1(G)$

$$\text{card}\,\{[w^*\text{ cl } F] \cap TIM(\hat{G})\} \geqslant 2^c$$

if $[w^* \text{ cl } F] \cap TIM(\hat{G}) \neq \emptyset$, where card (c) stand for cardinality (of the continuum.) This is done, by noting that $[w^* \text{ cl } F] \cap TIM(\hat{G})$ does not contain $w^* - G_\delta$ points and applying a theorem of Cech-Pospisil. For details see our memoir [5] p. 61. It seems, as is the case for invariant means on σ-compact groups, that the continuum hypothesis is not necessary here (see C. CHOU [1]).

PROPOSITION 2: Let G be compactly generated and nondiscrete. Then $TIM(\hat{G})$ is not norm separable.

PROOF: Let H be a compact normal subgroup of G such that G/H is second countable and of Haar measure zero (see [4] p. 71). It is shown in RENAUD [7] p. 288 that there is a linear *isometry* from the set of topologically invariant elements in $VN(G/H)^*$ into the set of topologically invariant elements in $VN(G)^*$ (which maps $TIM(\hat{G}/H)$ into $TIM(\hat{G})$). If $TIM(\hat{G})$ would be norm separable then $TIM(\hat{G}/H)$ would be such. By the previous theorem G/H would be discrete so H would be open and of Haar measure zero. This cannot be.

NOTATIONS: Let K be an open subgroup of G, $r: A(G) \to A(K)$ the restriction map $ru = u_{|K}$ and $t: A(K) \to A(G)$ the extension map defined by $tv = \mathring{v}$ where $\mathring{v} = v$ on K and 0 outside K. By EYMARD [3] p. 215 r is onto and t is an isometry, $tP_1(K) \subset P_1(G)$ and $r(P_1(G)) = P_1(K)$. Clearly $rt: A(K) \to A(K)$ is the identity.

REMARKS: 1. $t^{**}: VN(K)^* \to VN(G)^*$ satisfies $t^{**}(TIM(\hat{K})) \subset TIM(\hat{G})$. In fact if ψ' is any topologically invariant element of $VN(K)^*$ so is $t^{**}\psi'$, since if $u \in P_1(G)$ and $\phi \in VN(G)$: $(t^{**}\psi')(u \cdot \phi) = \psi'(t^*(u \cdot \phi)) = \psi'[(ru) \cdot t^*\phi)] = \psi' t^* \phi = (t^{**}\psi')(\phi)$, since $t^*(u \cdot \phi) = (ru) \cdot t^* \phi$. [In fact if $v \in A(G)$ then $[t^*(u \cdot \phi)]v = \phi(u\mathring{v}) = \phi(t[(ru)v]) = [(ru) \cdot t^*\phi](v)]$. That $t^{**}\psi' \geqslant 0$ and $t^{**}\psi'(I) = 1$ is immediate since $tP_1(K) \subset P_1(G)$.

2. We show now that $r^{**}[TIM(\hat{G})] = TIM(\hat{K})$. In fact $r^{**}t^{**}$: $VN(K)^* \to VN(K)^*$ is the identity map. Hence $TIM(\hat{K}) = r^{**}t^{**}[TIM(\hat{K})] \subset r^{**}[TIM(\hat{G})]$ by remark (1) above. But $r^{**}[TIM(\hat{G})] \subset TIM(\hat{K})$. Since if $\psi \in TIM(\hat{G})$ and $u_1 \in P_1(K)$, $\phi_1 \in VN(K)$ then $r^{**}\psi(u_1 \cdot \phi_1) = \psi r^*(u_1 \cdot \phi_1) = \psi((tu_1) \cdot r^*\phi_1) = \psi(r^*\phi_1) = (r^{**}\psi)\phi_1$ since if $v \in A(G)$ then $r^*(u_1 \cdot \phi_1)(v) = \phi_1(u_1 rv) = \phi_1(r[(tu_1)v]) = [(tu_1) \cdot r^*\phi](v)$ thus $r^*(u_1 \cdot \phi_1) = tu_1 \cdot r^*\phi$.

3. $t^{**}: VN(K)^* \to VN(G)^*$ is an isometry into. In fact by [3] p. 205 r is the adjoint of an isometry from $C_\varrho(K)$ to $C_\varrho(G)$. Thus $\|r\| = \|r^{**}\| \leqslant 1$. If for some $\psi \in VN(K)^*$ $\|t^{**}\psi\| < \|\psi\|$ then, since $r^{**}t^{**}$ is the identity map one has $\|\psi\| = \|r^{**}t^{**}\psi\| \leqslant \|t^{**}\psi\|$ which cannot be.

THEOREM 3: Let K be an open subgroup of G. Then $t^{**}: VN(K)^* \to \to VN(G)^*$ is an isometry such that

$$t^{**}[TIM(\hat{K})] = TIM(\hat{G}).$$

PROOF: Let $\psi \in TIM(\hat{G})$ and $\psi' = r^{**}\psi \in TIM(\hat{K})$. Let $u_1 \in P_1(K)$, $\phi \in VN(G)$. Then:

$$\psi(\phi) = (\mathring{u}_1 \cdot \phi) = \psi(r^*(u_1 \cdot t^*\phi)) = r^{**}\psi(u_1 \cdot t^*\phi) = \psi' t^*\phi = t^{**}\psi'(\phi)$$

since if $v \in A(G)$ then $[r^*(u_1 \cdot t^*\phi)](v) = t^*\phi(u_1 rv) = \phi(t(u_1 rv)) = \phi((tu_1)v) = = (\mathring{u}_1 \cdot \phi)(v)$. Thus $t^{**}\psi' = \psi$ where $\psi' \in TIM(\hat{K})$.

REMARK: We have incidentally proved that for all topologically invariant $\psi \in VN(G)^*$ one has $t^{**}r^{**}\psi = \psi$.

COROLARY: $TIM(\hat{K})$ and $TIM(\hat{G})$ have the same cardinality and one is norm separable iff the other is so.

REMARKS: At first sight, this result is surprising. Since if G is any discrete group, then $K = \{e\}$ is an open subgroup containing one element. And $t^{**}[TIM(\hat{K})] = TIM(\hat{G})$. We know however that $TIM(\hat{G})$ contains exactly one element namely the one defined by: $\psi_0(\phi) = \phi(1_e)$. (see RENAUD [7] p. 286). The anologous result for topological invariant means on $L^\infty(K)$, $L^\infty(G)$ is certainly incorrect. In fact take G to be abelian, infinite, discrete and K to be the trivial subgroup generated by e. Then $L^\infty(K)$ admits a unique TIM while $L^\infty(G) = l^\infty(G)$ admits many TIM's (see [5]). Furthermore, theorem 3 implies for example that if G contains a compact subgroup H, which is open, then $TIM(\hat{H})$ and $TIM(\hat{G})$ are weak * and norm, linearly (affinely) homeomorphic. If in addition H is abelian, then $\hat{H}$ is discrete and abelian and $TIM(\hat{H})$ is just the set of invariant means on $l^\infty(\hat{H})$. We see hence that the w^* (norm) affine structure of $TIM(\hat{G})$ is unable to reveal too much information about the topological or algebraic structure of G.

We continue to assume that K is an open subgroup of G. It has been proved in the proof of theorem 3 that if $u_1 \in A(K)$ and $\phi \in VN(G)$ then $r^*(u_1 \cdot t^*\phi) = \mathring{u}_1 \cdot \phi$. (Thus $r^*[A(K) \cdot VN(K)] = [A^\circ(K)] \cdot VN(G)$, since $t^*: VN(G) \to VN(K)$ is onto (EYMARD [3] p. 215).

PROPOSITION 4: Let K be an open subgroup of G. If $TIM(\hat{G})$ contains some $w^* - G_\delta$ point so does $TIM(\hat{K})$ i.e. if for some $\{\phi_n\} \subset VN(G)$ and $\psi_0 \in TIM(\hat{G})$ $\{\psi_0\} = \{\psi \in TIM(\hat{G});\ \psi(\phi_n) = 0,\ n \geqslant 1\}$ then for some $\phi_n' \in VN(K)$

$$\{r^{**}\psi_0\} = \{\psi \in TIM(\hat{K});\ \psi(\phi_n') = 0,\ n \geqslant 1\}.$$

Proof: Pick any $u \in P_1(K)$. If $\psi \in TIM(\hat{G})$, $\psi(\mathring{u} \cdot \phi_n) = \psi(\phi_n)$, $n \geqslant 1$. Thus if we denote $\phi_n' = u \cdot t^* \phi_n$, one has $r^* \phi_n' = \mathring{u} \cdot \phi_n$. Hence $\{\psi_0\} = \{\psi \in TIM(\hat{G});\ \psi(r^* \phi_n') = 0, n \geqslant 1\}$, (using the above observation).

We claim now that:

$$(*) \qquad \{r^{**}\psi_0\} = \{\psi' \in TIM(\hat{K});\ \psi'(\phi_n') = 0, n \geqslant 1\}.$$

That $r^{**}\psi_0$ belongs to the right side of (*) is obvious from above.

Let now $\psi_1' \in TIM(\hat{K})$ be such that $\psi_1'(\phi_n') = 0$, all n. Let $t^{**}\psi_1' = \psi_1 \in TIM(\hat{G})$. Then $r^{**}\psi_1 = \psi_1'$ and $r^{**}\psi_1(\phi_n') = \psi_1(r^* \phi_n') = 0$. Thus $\psi_1 = \psi_0$ so $\psi_1' = r^{**}\psi_0$ which finishes this proof.

This proposition reduces the burden of proof of the fact that "for any nondiscrete group G, $TIM(\hat{G})$ does not contain w^* exposed points (or $w^* - G_\delta$-points)" to the case that G is compactly generated, in marked difference from the $L^\infty(G)$ case (see [5] p. 53].

Conjecture. "Let G be *any* locally compact group. If for some convex set $F \subset P_1(G)$ and some $\phi_n \in VN(G)$, $n = 1, 2, \ldots$ the set

$$\{w^* \text{ cl } F\} \cap \{\psi \in TIM(\hat{G});\ \psi(\phi_n) = 0, n \geqslant 1\}$$

has w^* exposed points then G is discrete." This conjecture holds true if G is separable metric, by theorem 1 of the present paper.

The reader should note that if G is even abelian and compact with prescribed discrete noncountable dual group $\hat{G}$ then the truth of our conjecture implies that the set of left invariant means on $L^\infty(\hat{G}) = l^\infty(\hat{G})$ does not posses w^* exposed points. This is part of a conjecture made by C. Chou in [2] p. 301. For *countable* such $\hat{G}$ (i.e. in case G is compact abelian and metric) a weaker conjecture is proved by C. Chou in [2] (no ϕ_n's and no F are present) while the full conjecture (for countable $\hat{G}$) is proved in our memoir [5] p. 40.

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