Linear quivers and the geometric setting of quantum $GL_n$

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ABSTRACT

This paper presents a connection between the defining basis presented by Beilinson-Lusztig-MacPherson [1] in their geometric setting for quantum $GL_n$ and the isomorphism classes of linear quiver representations. More precisely, the positive part of the basis in [1] identifies with the defining basis for the relevant Ringel-Hall algebra; hence, it is a PBW basis in the sense of quantum groups. This approach extends to $q$-Schur algebras, yielding a monomial basis property with respect to the Drinfeld-Jimbo type presentation for the positive (or negative) part of the $q$-Schur algebra. Finally, the paper establishes an explicit connection between the canonical basis for the positive part of quantum $GL_n$ and the Kazhdan-Lusztig basis for $q$-Schur algebras.

1. INTRODUCTION

The quantized enveloping algebra $U$ associated to a Cartan datum $(I, \cdot)$ is defined by means of a well-known presentation, originally due to Drinfeld and Jimbo. An important question centers on obtaining concrete realizations of $U$. Beilinson, Lusztig and MacPherson solved this problem in [1] in type $A_n$ by providing a realization $V$ of $U$ in terms of the geometry of flags on a finite dimensional vector space. At about the same time, Ringel [18] obtained a realization, not of $U$, but only of its positive part $U^+$. In Ringel's approach, the realization $H$ was given in terms of the representation theory of finite dimensional hereditary algebras; explicitly, Ringel extended the usual theory of Hall

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algebras (built on the representation theory of a discrete valuation ring with finite residue field [14]) to include the representation theory of finite algebras.

Here, we present a new investigation into the relationship between the Ringel-Hall algebra $H$ and the Beilinson-Lusztig-MacPherson algebra $V$. Thus, the first main result, given in Theorem 4.4, provides a direct proof of the isomorphism between the positive part $V^+$ of $V$ and the Ringel–Hall algebra $H$ of a linear quiver. The argument makes no use of any connection of these algebras to quantum groups. As a consequence, we obtain an identification of the positive part of the basis in [1] with the basis of isoclasses of quiver representations.

Our result has several applications: First, it provides an extension of the Ringel–Hall algebra approach to the theory of $q$-Schur algebras and a systematic description, given in Theorem 7.7, of many monomial and integral monomial bases for $q$-Schur algebras (suggested by recent work given in [5] and [4]). Second, it leads to a direct and explicit relation between the canonical basis for the positive part of a quantum $GL_n$ and the Kazhdan-Lusztig bases for $q$-Schur algebras; see Theorem 8.3. The theory of $q$-Schur algebras plays a central role in the so-called non-defining representation theory of finite general linear groups; one hopes that a better understanding of $q$-Schur algebras may eventually lead to an extension of that theory to include the other finite groups of Lie type.

The paper is organized as follows. Section 2 briefly reviews the basic theory of Ringel–Hall algebras and introduces the key notion of generic extensions and the associated degeneration order $\preceq$. Section 3 focuses on the linear quiver case and obtains some important multiplication formulas in the corresponding Ringel-Hall algebra. Section 4 establishes the main result, providing the isomorphism $V^+ \cong H$, together with its integral version. Section 5 shows that the degeneration order $\leq$ agrees with the partial order $\leq$ used in [1]; this is essential for the identification of the canonical bases.

The next three sections are devoted to a discussion of $q$-Schur algebras. After a brief introduction to the Drinfeld-Jimbo presentation in Section 6, Section 7 establishes that $q$-Schur algebras inherit the strong monomial basis property of quantum groups. Section 8 gives a comparison between the canonical bases for the positive parts of a quantum $GL_n$ and $q$-Schur algebras.

While the representation theory of hereditary algebras (and the associated theory of Ringel-Hall algebras) provides an attractive approach to the positive part of the quantum enveloping algebras, that theory does not appear to be rich enough to solve satisfactorily the realization problem for the entire algebra $U$. In recent work [15], Peng and Xiao have solved the realization problem for all symmetrizable Kac-Moody algebras by working with a kind of Ringel-Hall algebra based on the derived category associated to a finite dimensional hereditary algebra. It would be interesting to know in the quantized enveloping algebra case if such ideas, perhaps combined with those in [1], might also solve the realization problem.
Some notation and conventions. Throughout $Z$ will denote the ring $Z[v, v^{-1}]$ of Laurent polynomials in a variable $v$. Put $Z^+ = Z[v]$ and $Z^- = Z[v^{-1}]$. Let $\cdot : Z \to Z$ be the $Z$-linear map satisfying $v^i \mapsto v^{-i}$. Often we let $q = v^2$. Set

$$[m] = \frac{v^m - v^{-m}}{v - v^{-1}} \quad \text{and} \quad [m] = \frac{q^m - 1}{q^2 - 1}.$$

Also, put

$$[m]! = \prod_{i=1}^{m} \frac{v^i - v^{-i}}{v - v^{-1}}.$$

If $n, r$ are positive integers, we let $\Lambda(n, r)$ be the set of all compositions $\lambda = (\lambda_1, \cdots, \lambda_n)$ of $r$ of length at most $n$: thus, $\lambda_i \geq 0$, $\forall i$, and $|\lambda| := \lambda_1 + \cdots + \lambda_n = r$.

We briefly indicate an abstract setting (see [13, 7.10]) for the notion of a canonical basis that will be used in the paper (see §8). Suppose $F$ is a free $Z$-module with a fixed basis $\{\tau_i\}_{i \in A}$. We assume that the indexing set $A$ is an interval finite poset in the following strong sense: for $j \in A$, the “interval” $(-\infty, j] := \{i \in A | i \leq j\}$ is finite. Assume that $\iota : F \to F$ is an involutory $Z$-linear map which is semilinear for $\cdot$: $\iota(af) = a\iota(f)$ for all $a \in Z, f \in F$. Let $F^-$ be the $Z^-$-free submodule of $F$ with basis $\{\tau_i\}_{i \in A}$. Let $\pi : F^- \to F^- / v^{-1}F^-$ be the quotient morphism. A canonical basis (at $\infty$) of $F$ (with respect to $\{\tau_i\}_{i \in A}$) is a $Z$-basis $\{\tau_i\}_{i \in A}$ for $F$ which satisfies the following three conditions:

(a) Each $\tau_i$ is $\iota$-symmetric: $\iota(\tau_i) = \tau_i$, $\forall i \in A$;
(b) For $i \in A$, $\tau_i \in \sum_{j \leq i} Z^- \tau_j$ and $\pi(\tau_i) = \pi(\tau_j)$;
(c) The basis $\{\tau_i\}_{i \in A}$ is the unique $Z$-basis of $F$ satisfying conditions (a) and (b).

A $\Lambda \times \Lambda$ matrix $A = [a_{i,j}]$ with coefficients in $Z$ is called good provided that $a_{i,j} \neq 0 \Rightarrow i < j$. Thus, a good matrix $A$ is “upper triangular”; if $A$ and $B$ are two good matrices, their matrix product $AB$ is defined formally (since $A$ is interval finite) and is again a good matrix. Let $I = I_A = [\delta_{i,j}]$ be the $\Lambda \times \Lambda$ identity matrix.

Write $\iota(\tau_j) = \sum_i r_{i,j} \tau_i$. Form the $\Lambda \times \Lambda$ matrix $R = [r_{i,j}]$.

Theorem 1.1. Suppose the matrix $R$ is a good unipotent matrix, i. e.,

$$\iota^2 = 1, \quad R^2 = I.$$  

Then there exists a canonical basis $\{\tau_i\}_{i \in A}$ of $F$ with respect to $\{\tau_i\}_{i \in A}$.

Since $i^2 = 1$, $R^2 = I$. The Theorem is proved (see, e. g., [9]) by recursively solving for a good unipotent matrix $P$ satisfying $P = R\hat{P}$. The matrix $P$ is unique if we require it to have off diagonal terms in $v^{-1}Z^-$. Then if $T$ denotes the $1 \times \Lambda$ row matrix $[\tau_i]$, the basis $\{\tau_i\}_{i \in A}$ is defined by the product $C = TP$ since $\iota(C) = \iota(T)P = TR\hat{P} = TP = C$. We leave further details to the reader.

For a familiar example, consider a Coxeter group $W$ and let $H$ be the corresponding Hecke algebra over $Z$ with basis $\bar{T}_w = q^{-\ell(w)}T_w$, $w \in W$. The map
Let $\Delta = (\Delta_0, \Delta_1)$ be a quiver with $\Delta_0 = \{1, 2, \cdots, n\}$ the set of vertices and $\Delta_1$ the set of arrows. For any arrow $\rho \in \Delta_1$, let $h(\rho)$ and $t(\rho)$ denote the head and tail of $\rho$, respectively:

$$
\begin{array}{c}
\bullet \overrightarrow{\rho} \\
\downarrow t(\rho) \\
\bullet
\end{array}
$$

We further assume that the underlying graph of $\Delta$ is a simply-laced (connected) Dynkin graph. In this way, we can identify $\Delta_0$ with the set $\{\alpha_1, \cdots, \alpha_n\}$ of simple roots in an indecomposable root system $\Phi$; let $\Phi^+$ be the corresponding set of positive roots. Let $\mathbb{Z}\Phi^+$ (resp., $\mathbb{Z}\Delta_0$) be the free abelian group on the set $\Phi^+$ (resp., $\Delta_0$), and let $\partial : \mathbb{Z}\Phi^+ \to \mathbb{Z}\Delta_0$ be the linear map sending $\beta \in \Phi^+$ to its linear combination in terms of simple roots. We often use $\partial$ to identify $\Phi^+$ as a subset of $\mathbb{Z}\Delta_0$.

For a field $k$, let $k\Delta\text{-mod}$ be the category of finite dimensional modules for the path algebra $k\Delta$. We will often identify $k\Delta\text{-mod}$ with the category of finite dimensional quiver representations of $\Delta$ over $k$. A quiver representation $V = (V_i)_{i \in \Delta_0}$ assigns to each $i \in \Delta_0$ a finite dimensional vector space $V_i$ and to each arrow $\rho \in \Delta_1$, a linear transformation $V_{t(\rho)} \to V_{h(\rho)}$. Also, $\dim V = (\dim V_1, \cdots, \dim V_n)$ is the dimension vector of $V$. As an object in $k\Delta\text{-mod}$, $\dim V = [\dim V] := \sum \dim V_i$.

The simple modules $S_i$ in $k\Delta\text{-mod}$ are naturally indexed by elements $i \in \Delta_0$. The dimension vector $\dim M$ of a $k\Delta$-module $M$ is the image of $M$ in the Grothendieck group $K_0(k\Delta) \cong \mathbb{Z}\Delta_0$. Gabriel's Theorem states that the indecomposable objects $M$ in $k\Delta\text{-mod}$ are indexed, up to isomorphism, by $\Phi^+$. Thus, $M$ is uniquely identified among indecomposable modules by its dimension vector $\dim M$, and the set $\Theta := \{f : \Phi^+ \to \mathbb{N}\}$ identifies with the set of isomorphism classes of $k\Delta$-modules. We often identify $\Theta$ with the set of strictly upper triangular $n \times n$-matrices $(f_{i,j})_{i < j}$ with non-negative integer coefficients. Let $M_k(f)$ denote a representative of the class $f$, so $\dim M_k(f) = \partial(f)$. In particular, $S_i := M_k(f_i) (i \in \Delta_0)$, where $f_i \in \Theta$ maps the simple root $\alpha_i$ to $1$ and all other roots to $0$.

For a finite field $k$ and objects $M, N_1, \cdots, N_t$ in $k\Delta\text{-mod}$, let $F_{N_1, \cdots, N_t}^M$ denote the number of filtrations $M = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$.
such that $M_{i-1}/M_i \cong N_i$ for all $1 \leq i \leq t$. Because $M$ is finite (as a set), $F_{N_i...N_1}$ is a non-negative integer. By [17, Theorem 1], for any $f, g_1, \cdots, g_t \in \Theta$, there exists a polynomial $\varphi_{g_1, \cdots, g_t} \in \mathbb{Z}[q]$ such that

$$\varphi_{g_1, \cdots, g_t}(\langle k \rangle) = F_{N_k(g_1) \cdots N_k(g_t)}$$

for all finite fields $k$. These polynomials are called Hall polynomials.

Following [19], we define $H(\Delta)$ to be the associated (twisted) generic Hall algebra over $\mathbb{Z}$. Thus, $H(\Delta)$ is $\mathbb{Z}$-free with basis $\{u_f\}_{f \in \Theta}$ and multiplication

$$u_f u_g = v^{(\langle \partial(f), \partial(g) \rangle)} \sum_{h \in \Theta} \varphi_{f, g, h}^h(v^2) u_h,$$

where $\langle \cdot, \cdot \rangle : \mathbb{Z}\Delta_0 \times \mathbb{Z}\Delta_0 \to \mathbb{Z}$ is the Euler form defined by

$$\langle a, b \rangle = \sum_{i \in \Delta_0} a_i b_i - \sum_{\rho \in \Delta_1} a_{(\rho)} b_{h(\rho)},$$

for $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$. Let $H(\Delta) = H(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}(v)$.

Let $g$ be a semisimple (complex) Lie algebra of type $\Delta$, and let $U = U_v(g)$ be its associated quantized enveloping algebra over $\mathbb{Q}(v)$. The algebra $U$ is usually defined by means of a well-known (Drinfeld-Jimbo) presentation in the generators $E_i, F_i, K_i^{\pm 1} (i \in \Delta_0)$. Let $U^+$ be the positive part of $U$, i.e., the $\mathbb{Q}(v)$-subalgebra generated by the $E_i$. We denote by $U^+$ the Lusztig form of $U^+$, i.e., $U^+$ is the $\mathbb{Z}$-subalgebra of $U$ generated by all the divided powers $E_i^{(m)} = E_i^m / [m]_v$.

For each $i \in \Delta_0$, let $f_i \in \Theta$ be as above and set $u_i = u_{f_i} \in H(\Delta)$. Because there are no self-extensions of simple modules in $k\Delta$-mod, it is easy to see from the definitions that $u_i^{(m)} := u_i^m / [m]_v \in H(\Delta)$. The following result is due to Ringel [19, §7].

**Proposition 2.1.** The algebra $H(\Delta)$ is generated by all $u_i^{(m)}$ $(i \in \Delta_0, m \geq 1)$. Moreover, there is a natural isomorphism

$$\psi : U^+ \cong H(\Delta); E_i^{(m)} \mapsto u_i^{(m)} (i \in \Delta_0, m \geq 1).$$

In the rest of the section, we assume that $k = \bar{k}$ is a fixed algebraically closed field. We shall write $M(f)$ for $M_{\bar{k}}(f)$. Fix $d = (d_i) \in \mathbb{N}^\theta$, and consider the affine space

$$R(d) := \prod_{\rho \in \Delta_1} \text{Hom}_K(k^{d_{\rho}}, k^{d_{h(\rho)}}) \cong \prod_{\rho \in \Delta_1} k^{d_{h(\rho)} \times d_{\rho}}.$$ 

Thus, $x = (x_{\rho})_{\rho} \in R(d)$ determines a $k\Delta$-module $M(f)$ for some $f \in \Theta$ satisfying $\partial(f) = d$. The algebraic group $GL_d(k) := \prod_{i=1}^{\theta} GL_{d_i}(k)$ acts on $R(d)$ by conjugation

$$(g_i)_i \cdot (x_{\rho})_{\rho} = (g_{h(\rho)} x_{\rho} g_{h(\rho)}^{-1})_{\rho},$$

1 We sometimes use without mention the elementary fact that given $M, N \in k\Delta$-mod, we have $\langle \dim M, \dim N \rangle = \dim \text{Hom}_{k\Delta}(M, N) - \dim \text{Ext}_{k\Delta}^1(M, N)$.
and the $GL_d(k)$-orbits $O_f$ in $R(d)$ correspond bijectively to the elements $f$ in $\Theta_d := \vartheta^{-1}(d)$. 

With this correspondence, we define a poset structure on $\Theta$ by setting

\[(2.1.1) \quad f \leq g \iff \vartheta(f) = \vartheta(g) \text{ and } O_f \subset \bar{O}_g,\]

where $\bar{O}_g$ is the Zariski closure of $O_g$ in $R(d)$. Following [2], $\leq$ is called the degeneration order\(^2\); it is proved there that $\leq$ is independent of the field $k$.

Define a multiplication $\ast$ on $\Theta$ as follows: for $f, g \in \Theta$, let $f \ast g \in \Theta$ be the element defined by the two conditions: (a) $M(f \ast g)$ is an extension of $M(f)$ by $M(g)$; (b) if $M(h)$ is an extension of $M(f)$ by $M(g)$, then $h \leq f \ast g$. It is proved in [16, 2.3] that these conditions uniquely determine the element $f \ast g$. In addition, the definition of $f \ast g$ is independent of the field $k$.

For notational convenience, write $M(f), M(g) = M(f \ast g)$, so that $M(f \ast g)$ is unique up to isomorphism; $M(f \ast g)$ is called the generic extension of $M(f)$ by $M(g)$, following [16]. The following result is also due to Reineke in [16, Proposition 3.3].

**Proposition 2.2.** $(\Theta, \ast)$ is a monoid. It is generated by the functions $f_1, \ldots, f_n$.

Let $\Omega$ be the set of all words on the alphabet $\{1, \ldots, n\}$. For $w = i_1i_2\cdots i_m \in \Omega$, let $\varphi(w) \in \Theta$ be the element of $\Omega$ defined by

\[(2.2.1) \quad [S_{i_1}] \ast \cdots \ast [S_{i_m}] = [M(\varphi(w))].\]

The above proposition implies that $\varphi : \Omega \to \Theta$ is surjective, so $\Omega = \cup_{f \in \Theta} \Omega_f$ with $\Omega_f = \varphi^{-1}(f)$. Define a word $w = j_1^i \cdots j_m^i$ with $j_{i-1} \neq j_i, \forall i$, to be distinguished if the Hall polynomial $\varphi_{e_i j_1, \ldots, e_i j_m} = 1$.

3. THE LINEAR QUIVER CASE

From now on, $\Delta = \Delta_{n-1}$ is the following linear quiver:

```
1 --2 -- \cdots -- n-2 -- n-1
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of type $A_{n-1}$. For any field $k$, the path algebra $k\Delta$ has a particularly simple description: it is isomorphic to the algebra of $(n-1) \times (n-1)$ upper triangular matrices over $k$. Given $f \in \Theta$ and $i_0 \in \Delta_0$, the goal of this section is to give, in Theorem 3.3, an explicit expression for the product $u_{i_0}u_f$ in the (generic) Hall algebra $H := H(\Delta_{n-1})$. Actually, we work with a slightly different basis $\{u_f | f \in \Theta\}$ for $H$. (From now on, we abbreviate $H(\Delta_{n-1})$ to $H$, etc.)

For $1 \leq i < j \leq n$, there is a unique (up to isomorphism) indecomposable $k\Delta_{n-1}$-module $M_{i,j}^k$ with top $S_i$ and of length $j - i$. The $M_{i,j}^k, 1 \leq i < j \leq n$, form a complete set of non-isomorphic indecomposable $k\Delta_{n-1}$-modules. In the discussion above, $M_{i,j}^k$ corresponds to the positive root $\alpha_{i,j} := \alpha_i + \cdots + \alpha_{j-1}$. In the language of quiver representations, $M_{i,j}^k = (V_{i,j}^k)_{i \in \Delta_0},$ where $V_{i,j}^k = k$ if

\(^2\) More precisely, the degeneration order in [2], is opposite to the order $\leq$ defined here; cf. the proof of Proposition 5.4 below.
i \leq l < j and V^{i,j}_l = 0 otherwise. If i \leq l < j - 1, V^{i,j}_l \to V^{i,j}_{l+1} is the identity map. For any f = (f_{i,j})_{i<j} \in \Theta, M_k(f) \cong \oplus_{1 \leq i < j \leq n} f_{i,j} M^i,j_k.

The following result can be drawn from [20, Appendix 2].

**Lemma 3.1.** Let i < j and r < s. Then
\[
\text{Hom}(M^i,j_k, M^r,s_k) \cong \begin{cases} k, & \text{if } r \leq i < s \leq j; \\ 0, & \text{otherwise.} \end{cases}
\]

**Proof.** For i < j, let (V^{i,j}_l) be the representation of the quiver \( \Delta_{n-1} \) corresponding to \( M^i,j_k \). Any morphism \( \phi : M^i,j_k \to M^r,s_k \) is completely determined by a linear map \( V^{i,j}_l \to V^{r,s}_l \) and \( \phi \) is nonzero only if \( r \leq i < s \). Further, if \( \phi \neq 0 \), we must have \( s \leq j \). On the other hand, if \( r \leq i < s \leq j \), then \( \text{Hom}(M^i,j_k, M^r,s_k) \cong k \). \( \square \)

Fix \( f = (f_{i,j})_{i<j} \in \Theta \) and \( i_0 < n - 1 \). Suppose \( j_1 < \cdots < j_a \) are all the column indices \( j \) for which \( f_{i_0+1,j} \neq 0 \). Define, for each \( 1 \leq t \leq a, f^{(t)} \in \Theta \) by
\[
(3.1.1) \quad f^{(t)}_{i,j} = \begin{cases} f_{i_0,j_t} + 1, & \text{if } (i,j) = (i_0,j_t); \\ f_{i_0+1,j_t} - 1, & \text{if } (i,j) = (i_0 + 1,j_t); \\ f_{i,j}, & \text{otherwise.} \end{cases}
\]

Define \( f^{(0)} \in \Theta \) by
\[
f^{(0)}_{i,j} = \begin{cases} f_{i_0,j_0+1} + 1, & \text{if } (i,j) = (i_0, i_0 + 1); \\ f_{i,j}, & \text{otherwise.} \end{cases}
\]

**Lemma 3.2.** (1) If \( M_k \) is an extension of \( S_{i_0} \) by \( M_k(f) \), then \( M_k \cong M_k(f^{(t)}) \) for some \( t \in [0, a] \). Moreover, we have
\[
M_k(f^{(0)}) \leq M_k(f^{(1)}) \leq \cdots \leq M_k(f^{(a)}).
\]

(2) The following identity holds in \( H \):
\[
\mu_{i_0} u_f = v^{\sigma(i_0,f)} \sum_{t=0}^a v^2 \sum_{i < j, f_{i_0,j_t} \neq f_{i_0+1,j_t}} [f_{i_0,j_t} + 1] u_{f^{(t)}}
\]
where \( \sigma(i_0,f) = \sum_{i < i_0} f_{i_0,j_t} + 1 - \sum_{i_0+1 < j} f_{i_0+1,j_t} \).

**Proof.** The first assertion of (1) follows since \( \dim \text{Ext}^1_k(S_i, S_j) = \delta_{j,j+1} \). For the last assertion of (1), consider the following (non-split) exact sequence
\[
0 \to M^{i+1,j+1}_k \to M^{i_0,j+1}_k \oplus M^{i+1,j}_k \to M^{i_0+1,j}_k \to 0.
\]
By [2, Theorem 4.5(a)] (or [3, Lemma 2, p. 13]), we have \( M^{i,j}_k \oplus M^{i+1,j+1}_k \leq M^{i_0,j+1}_k \oplus M^{i_0+1,j}_k, \) and so \( M_k(f^{(t)}) \leq M_k(f^{(t+1)}). \)

We now prove (2). For any \( h \in \Theta \) and \( 1 \leq i < j \leq n \), let
\[
(3.2.1) \quad \sigma_{i,j}(h) = \sum_{r \leq i, j \leq s} h_{r,s}.
\]
We first observe that, for $h \in \Theta$, $\partial(h) = \sigma_{1,2}(h)\alpha_1 + \ldots + \sigma_{n-1,n}(h)\alpha_{n-1}$. In particular, $\dim M(h) = \sum_{i=1}^{n-1} \sigma_{i,i+1}(h)$. So we obtain
\[
\sigma(i_0,f) = \sigma_{i_0,i_0+1}(f) - \sigma_{i_0+1,i_0+2}(f) = \langle \partial(f_{i_0}), \partial(f) \rangle.
\]
On the other hand, it is clear from the proof of [4, 9.1] that
\[
\varphi_{i_0,f}^{(i_0,f)} = v^2 \sum_{h < j} f_{i_0,j} + 1 = v^2 \sum_{h < j} [f_{i_0,j} + 1].
\]
Now the result follows from the definition and part (1). \(\square\)

Let $\bar{u}_f = v^{-\dim M(f)} \dim \text{End}(M(f)) u_f$. Then, by Proposition 2.1, the set $\{\bar{u}_f\}_{f \in \Theta}$ forms a $\mathbb{Z}$-basis for $H$. Note that $u_i = \bar{u}_i$ for all $i \in \Delta_0$.

**Theorem 3.3.** Maintain the notation above and let $\varepsilon(i_0,t) = \sum_{j < i} f_{i_0,j} - \sum_{j < i} f_{i_0+1,j}$. The identity
\[
\bar{u}_f \bar{u}_f = \sum_{t=0}^{a} v^{\varepsilon(i_0,t)} [f_{i_0,j} + 1] \bar{u}_f(t)
\]
holds in the generic Hall algebra $H$.

**Proof.** By Lemma 3.2, we only show that, for $0 \leq t \leq a$,
\[
(3.3.1) \quad \varepsilon(i_0,t) = \sigma(i_0,t) + 2 \sum_{j < i} f_{i_0,j} - \ell(f^{(i)} - \ell(f))
\]
where $\ell(g) = -\dim M(g) + \dim \text{End}(M(g))$. Consider the linear order $\leq$ on $\{(i,j) | 1 \leq i < j \leq n\}$ defined by (cf. [20, p. 85])
\[
(3.3.2) \quad (i,j) < (i',j') \iff \begin{cases} \text{either } j > j'; \\ \text{or } j = j', i > i'; \end{cases}
\]
Then one sees easily from Lemma 3.1 that $\text{Hom}(M_{i,j}, M_{r,s}) = 0 \iff (i,j) \leq (r,s)$. Thus, for any $g \in \Theta$, we have
\[
\dim \text{End}(M(g)) = \sum_{i < j} \dim \text{End}(g_{i,j}M_{i,j}) + \sum_{(i,j) < (r,s)} \dim \text{Hom}(g_{i,j}M_{i,j}, g_{r,s}M_{r,s}).
\]
In general, denote the first summand above to the right of the equality sign by $\ell'(g)$ and the second summand by $\ell''(g)$ By (3.1.1),
\[
\ell'(f^{(i)}) - \ell'(f) = (2f_{i_0,j} + 1) + \delta_i(-2f_{i_0+1,j} + 1),
\]
where $\delta_0 = 0$ and $\delta_i = 1$ for $1 \leq t \leq a$.

Consider the induced ordering on the summands $f_{i,j}^{(i)M_{i,j}}$ and $f_{i,j}M_{i,j}$ of $M(f^{(i)})$ and $M(f)$, respectively:

\[
\begin{array}{cccccccc}
\cdots & f_{i,j}M_{i,j} & \cdots & \delta_i(f_{i_0+1,j} - 1)M_{i_0+1,j} & (f_{i_0,j} + 1)M_{i_0,j} & \cdots & f_{r,s}M_{r,s} & \cdots \\
\cdots & f_{i,j}M_{i,j} & \cdots & \delta_i(f_{i_0+1,j} + 1)M_{i_0,j} & f_{i_0,j}M_{i_0,j} & \cdots & f_{r,s}M_{r,s} & \cdots \\
\end{array}
\]

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By Lemma 3.1, we obtain, for $1 \leq t \leq a$,

$$
\ell''(f(t)) - \ell''(f) = - \sum_{i_0+1 < i < j} f_{i,j} + \sum_{i_0 < i < j} f_{i,j} \\
+ (f_{i_0+1,j} - 1) - f_{i_0,j_i} \\
- \sum_{r < i_0+1 < s < j_t} f_{r,s} + \sum_{r < i_0 < s < j_t} f_{r,s} \\
= \sum_{j_t < j} f_{i_0,j} + (f_{i_0+1,j_t} - 1) - f_{i_0,j_t} \\
- \sum_{i_0+1 < s < j_t} f_{i_0+1,s} + \sum_{r < i_0} f_{r,i_0} + 1 \\
= 2f_{i_0+1,j_t} - 1 - 2f_{i_0,j_t} + \sum_{j_t < j} f_{i_0,j} \\
- \sum_{i_0+1 < s < j_t} f_{i_0+1,s} + \sum_{r < i_0} f_{r,i_0} + 1.
$$

If $t = 0$, then $j_0 = i_0 + 1$ and

$$
\ell''(f(0)) - \ell''(f) = \sum_{i_0+1 < j} f_{i_0,j} + \sum_{r < i_0} f_{r,i_0} + 1.
$$

Thus, for $1 \leq t \leq a$,

$$
\ell(f(t)) - \ell(f) = [\ell'(f(t)) - \ell'(f)] + \left[\ell''(f(t)) - \ell''(f)\right] - 1 \\
= \sum_{j_t < j} f_{i_0,j} - \sum_{i_0+1 < s < j_t} f_{i_0+1,s} + \sum_{r < i_0} f_{r,i_0} + 1
$$

and

$$
\ell(f(0)) - \ell(f) = 2f_{i_0,i_0+1} + \sum_{i_0+1 < j} f_{i_0,j} + \sum_{r < i_0} f_{r,i_0} + 1 \\
= \sum_{i_0+1 < j} f_{i_0,j} + \sum_{r < i_0} f_{r,i_0} + 1.
$$

Therefore, we finally obtain for all $0 \leq t \leq a$

$$
\sigma(i_0, f) + 2 \sum_{j_t < j} f_{i_0,j} - (\ell(f(t)) - \ell(f)) = \sum_{j_t < j} f_{i_0,j} - \sum_{j_t < j} f_{i_0+1,j} = \varepsilon(i_0, t)
$$

as desired. \qed

**Remark 3.4.** Put $\tilde{u}_{i,j} = \tilde{u}_{i_0,j}$. Here we identify a positive root $\alpha_{i,j}$ $(i < j)$ with a function in $\Theta$ in an obvious way. We have the following identity due to Ringel [20, Proposition 2]:

$$
\tilde{u}_f = \prod_{i < j} \tilde{u}_{i,j}^{(f(i,j))}
$$
where the product uses the ordering defined in (3.3.2). By identifying the Hall algebra $H$ with the positive part of the corresponding quantum group $U^+$ (see Proposition 2.1), this relation together with [20, Theorem 7] shows that the basis $\{\bar{u}_f\}_{f \in \mathbb{E}}$ is a PBW basis for $U^+$.

4. THE POSITIVE PART $V^+$ OF THE BLM ALGEBRA

This section begins by reviewing the BLM algebra $V$. Then Theorem 4.4 gives a direct isomorphism between the positive part $V^+$ of $V$ and the Hall algebra $H$ of a linear quiver $A_{n-1}$.

Let $\Xi$ be the set of all $n \times n$ matrices over $\mathbb{Z}$ with all off-diagonal entries in $\mathbb{N}$, and let $\Xi^+ \subset \Xi$ consist of those matrices in $\Xi$ which have non-negative diagonal entries. Let $\tilde{U}$ be the algebra over $Q(v)$, defined in [1, §4], with basis $\{[A]\}_{A \in \Xi}$. The multiplication $\cdot$ in $\tilde{U}$ is defined in [1, 4.4] from another algebra over $Q(v)/v'[v'^{-1}]$ by specializing $v'$ to 1. This multiplication is induced by a stabilization property of the structure constants for $q$-Schur algebras (see [1, 4.2]). In particular, let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ and form the corresponding diagonal matrix $D = \text{diag}(\lambda_1, \cdots, \lambda_n) \in \Xi$. Then

$$[D] \cdot [A] = \begin{cases} [A], & \text{if } \lambda = \text{ro}(A); \\ 0, & \text{otherwise,} \end{cases}$$

and

$$[A] \cdot [D] = \begin{cases} [A], & \text{if } \lambda = \text{co}(A); \\ 0, & \text{otherwise,} \end{cases}$$

where $\text{ro}(A) = \left(\sum_i a_{1,i}, \cdots, \sum_j a_{n,j}\right)$ and $\text{co}(A) = \left(\sum_i a_{i,1}, \cdots, \sum_i a_{i,n}\right)$ denote the sequences of row and column sums, respectively, of the entries of $A = (a_{i,j})$.

Following [1, 5.1], let $U_\infty$ be the vector space of all formal (possibly infinite) $Q(v)$-linear combinations $\sum_{A \in \Xi} f_A [A]$ such that, for any diagonal matrices $D, D' \in \Xi$, the sums $\sum_{A \in \Xi} \beta_A [D] \cdot [A]$ and $\sum_{A \in \Xi} \beta_A [A] \cdot [D']$ are finite. Define the product of two elements $\sum_{A \in \Xi} \beta_A [A], \sum_{B \in \Xi} \gamma_B [B]$ in $U_\infty$ to be $\sum_{A,B} \beta_A \gamma_B [A] \cdot [B] \in U_\infty$. This gives an algebra structure on $U_\infty$. The algebra $U_\infty$ has an identity element (= the sum of all $[D]$ with $D$ a diagonal matrix in $\Xi$) and naturally contains $\tilde{U}$ as a subalgebra (without 1).

Let $\Xi^\pm$ (resp., $\Xi^0$) be the set of all $A \in \Xi$ whose diagonal (resp. off-diagonal) entries are zero. Given $A \in \Xi^\pm$ and $j = (j_1, \cdots, j_n) \in \mathbb{Z}^n$, we define

$$A(j) = \sum_{D \in \Xi^0} v \sum_{i,j} a_{i,j} [A + D] \in \tilde{U}_\infty,$$

where $D = \text{diag}(d_1, \cdots, d_n)$. For $1 \leq i,j \leq n$, let $E_{i,j} \in \Xi$ be the elementary matrix $(a_{k,l})$ with $a_{k,l} = \delta_{i,k} \delta_{j,l}$.

**Lemma 4.1.** ([1, 5.5]) The subspace $V$ of $U_\infty$ spanned by

$$B = \{ A(j) \mid A \in \Xi^\pm, j \in \mathbb{Z}^n \}$$

3 The integral forms (over $\mathbb{Z}$) of this algebra and the algebra $U$, below are denoted by $K$ and $K_r$ respectively in [1].
is a subalgebra which is generated by the $E_{h,h+1}(0)$, $E_{h+1,h}(0)$ and $0(j)$, $1 \leq h < n$ and $j \in \mathbb{Z}^n$. The set $B$ is a $\mathbb{Q}(v)$-basis for $V$. The algebra $V$ is called a BLM algebra.

Let $\Xi^+$ (resp., $\Xi^-$) be the subset of $\Xi$ consisting of those matrices $(a_{i,j})$ with $a_{i,j} = 0$ for all $i \geq j$ (resp., $i \leq j$). The following lemma is a special case of [1, 5.3].

**Lemma 4.2.** For any $A = (a_{i,j}) \in \Xi^+$. For any $j$, $1 \leq j \leq n$, put $\epsilon(j) = \sum_{i \leq j} a_{i,j} - \sum_{i > j} a_{i+1,j}$. If $1 \leq h < n$, we have
\[
E_{h,h+1}(0)A(0) = v^{(h+1)}[a_{h,h+1} + 1](A + E_{h,h+1})(0) + \sum_{h+1 \leq j \leq n} v^{(j)}[a_{h,j} + 1](A + E_{h,j} - E_{h+1,j})(0).
\]

Following [1], define a partial order $\leq$ on $\Xi^+$ by setting
\[
(4.2.1) \quad A \leq B \iff \sum_{s \leq i, i \leq t} a_{s,t} \leq \sum_{s \leq i, i \leq t} b_{s,t} \text{ and } \sum_{s \geq i, i \geq t} a_{s,t} \leq \sum_{s \geq i, i \geq t} b_{s,t}
\]
for all $i < j$ and $i' > j'$. We set $A < B$ if one of the inequalities is strict.

The following result introduces the algebras $V^+$ and $V^-$.  

**Corollary 4.3.** The subspace $V^+$ (resp. $V^-$) of $V$ spanned by
\[
B^+ = \{A(0) \mid A \in \Xi^+\} \quad \text{(resp. } B^- = \{A(0) \mid A \in \Xi^-\})
\]
is a subalgebra which is generated by the $E_{h,h+1}(0)$ (resp. $E_{h+1,h}(0)$), $1 \leq h < n$. The set $B^+$ (resp., $B^-$) is a $\mathbb{Q}(v)$-basis for $V^+$ (resp., $V^-$).

**Proof.** We sketch a proof for $V^+$ along the line of that of [1, Lemma 5.5]. Let $V_1$ be the subalgebra generated by the $E_{h,h+1}(0)$. By Lemma 4.2 above, $V^+$ is stable under left multiplication by the elements $E_{h,h+1}(0)$, so $V_1 \subseteq V^+$. The same lemma implies that, for any $m \geq 1$, $E_{h,h+1}(0)^m = [m]^1(mE_{h,h+1}(0))$. Thus, $(mE_{h,h+1})(0) \in V_1$ for all $m$ and $h$.

For any $A = (a_{i,j}) \in \Xi^+$, let
\[
(4.3.1) \quad E^{(A)} = \prod_{1 \leq i \leq h < j \leq n} (a_{i,j}E_{h,h+1})(0)
\]
where the product is taken over the following linear order$^4$
\[
(4.3.2) \quad (i,h,j) < (i',h',j') \iff \begin{cases} \text{either } j > j'; \\
\quad \text{or } j = j', i > i'; \\
\quad \text{or } j = j', i = i', h < h'.
\end{cases}
\]
Then, $E^{(A)} \in V_1$ for all $A \in \Xi^+$. By [1, 5.5(c)] and Lemma 4.2,
\[
(4.3.3) \quad E^{(A)} = A(0) + \sum_{B,B < A} c_{B,A} \gamma_{B,A} B(0).
\]

---

$^4$ This order is different from the order given in [1, 3.9]. However, the resulting product is the same.
An inductive argument on \( \leq \) gives that \( A(0) \in V_1 \) for all \( A \in \Xi^+ \). Therefore, \( V^+ \subseteq V_1 \). □

In the following main result, we naturally identify \( \Theta \) with the set \( \Xi^+ \). In particular, given \( 1 \leq h < n \), \( \tilde{u}_h \) will be labeled as \( \tilde{u}_A \) for \( A = E_{h,h+1} \).

**Theorem 4.4.** The linear map \( \xi : H \to V^+ \), \( \tilde{u}_A \mapsto A(0) \), is an algebra isomorphism.

**Proof.** Clearly, \( \xi \) is a linear isomorphism. Lemma 4.2 and Theorem 3.3 imply immediately that
\[
\xi(\tilde{u}_h \tilde{u}_A) = E_{h,h+1}(0) A(0) = \xi(\tilde{u}_h) \xi(\tilde{u}_A).
\]
The \( \tilde{u}_h \) generate \( H \), so \( \xi \) is an algebra isomorphism. □

**Corollary 4.5.** The \( \mathbb{Z} \)-submodule \( V^+ \) of \( V^+ \) generated by the \( A(0), A \in \Xi^+ \), is a subalgebra isomorphic to the integral (generic) Hall algebra \( H \).

The isomorphisms described above are independent of the existence of quantum groups. However, using the isomorphism in 2.1 between the Hall algebra and the positive part of the quantum group, we may further identify the basis \( \{ A(0) \}_{A \in \Xi^+} \) as a PBW basis constructed in [12]; see Remark 3.4.

**Corollary 4.6.** The basis \( \{ A(0) \}_{A \in \Xi^+} \) is a PBW basis of \( V^+ \).

5. A COMPARISON OF ORDER RELATIONS

Let \( V \) be a vector space of dimension \( r \) over a field \( k \), and let \( \mathcal{F} \) be the \( n \)-step filtration variety studied in [1]: \( \mathcal{F} \) consists of all \( n \)-step filtrations
\[
\mathfrak{f} = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V).
\]
The group \( G = GL(V) \) acts naturally on \( \mathcal{F} \) with orbits the fibres of the map \( \mathcal{F} \to \Lambda(n,r) \) given by
\[
(V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n) \mapsto (\dim V_1, \dim V_2/V_1, \cdots, \dim V_n/V_{n-1}).
\]
If \( \mathcal{F}_\lambda \) denotes the inverse image of \( \lambda \in \Lambda(n,r) \), then \( \mathcal{F} = \bigcup_{\lambda \in \Lambda(n,r)} \mathcal{F}_\lambda \), a disjoint union of orbits. If \( \mathfrak{f} \in \mathcal{F}_\lambda \) and \( P_\lambda \) is the stabilizer of \( \mathfrak{f} \) in \( G \), then \( \mathcal{F}_\lambda \cong G/P_\lambda \).

Let \( \mathcal{V} = \mathcal{V}(r) = \mathcal{F} \times \mathcal{F} \) and let \( G \) act on \( \mathcal{V} \) diagonally. For \( (\mathfrak{f}, \mathfrak{f}') \in \mathcal{V} \) where \( \mathfrak{f} = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n) \) and \( \mathfrak{f}' = (V'_1 \subseteq V'_2 \subseteq \cdots \subseteq V'_n) \), the subspaces
\[
X_{i,j} = X_{i,j}(\mathfrak{f}, \mathfrak{f}') := V_{i-1} + (V_i \cap V'_j) \quad (1 \leq i \leq n, 1 \leq j \leq n)
\]
(where \( V_0 = V'_0 = 0 \)) form an \( n^2 \)-step filtration:
\[
X_{11} \subseteq \cdots \subseteq X_{1n} \subseteq X_{21} \subseteq \cdots \subseteq X_{nn} = V.
\]
Let \( a_{i,j} = \dim X_{i,j}/X_{i,j-1} \). Setting \( \Psi(\mathfrak{f}, \mathfrak{f}') = (a_{i,j}) \) defines a map \( \Psi : \mathcal{V} \to \Xi \). Put \( \Xi_r := \text{im} \Psi = \{ A \in \Xi \mid \sum a_{i,j} = r \} \). The \( G \)-orbits on \( \mathcal{V} \) are the fibres of \( \Psi \).

Now assume that \( k \) is algebraically closed in the rest of the section. Thus, \( \mathcal{V} \) is a projective variety. For \( A \in \Xi_r \), let \( \mathcal{O}_A = \Psi^{-1}(A) \). As explained above, \( \mathcal{O}_A \) is a
G-orbit for the natural action of G on V. The Bruhat order\(^5\) \(\leq B_0\) on \(\Xi_r\) is defined by setting

\[(5.0.1)\quad A \leq B_0 A' \iff O_A \subseteq O_{A'}.
\]

Let \(\mathcal{R}(d)\) be a representation variety of \(\Delta_{n-1}\) (as per §2). For \(x = (x_1, \ldots, x_{n-2}) \in \mathcal{R}(d)\), let \(M_x\) be the corresponding quiver representation of \(\Delta_{n-1}\):

\[M_x : k^{d_1} \xrightarrow{x_1} k^{d_2} \xrightarrow{x_2} \cdots \xrightarrow{x_{n-2}} k^{d_{n-2}} \xrightarrow{x_{n-1}} k^{d_{n-1}}.
\]

We now associate a pair \((\tilde{f}_x, \tilde{f}'_x)\) of \(n\)-step filtrations to \(x\). Let \(V = k^{d_1} \oplus k^{d_2} \oplus \cdots \oplus k^{d_{n-1}}\) and let \(x_{i,j} = x_{j-1} \circ \cdots \circ x_i\) for \(1 \leq i < j \leq n - 1\). We define

\[\tilde{f}_x = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n), \quad \tilde{f}'_x = (V'_1 \subseteq V'_2 \subseteq \cdots \subseteq V'_n),
\]

where

\[
\begin{align*}
V_i &= k^{d_1} \oplus \cdots \oplus k^{d_{i-1}} \oplus \text{im} \ x_i \oplus \cdots \oplus \text{im} \ x_{n-2}, \quad 1 \leq i \leq n - 2; \\
V_n &= V_{n-1} = V,
\end{align*}
\]

and

\[
\begin{align*}
V'_i &= 0 \oplus \text{im} \ x_1 \oplus \cdots \oplus \text{im} \ x_{n-2}, \quad \text{and for } 1 < i \leq n - 1; \\
V'_j &= \ker x_{i,j} \oplus (\ker x_{2,j} + \text{im} \ x_1) \oplus \cdots \oplus (\ker x_{j-2,j} + \text{im} \ x_{j-1}) \oplus \text{im} \ x_{j-1} \oplus \cdots \oplus \text{im} \ x_{n-2}; \\
V'_n &= V.
\end{align*}
\]

The definition of the \(V'_i\) can be easily visualized using the following table, in which \(V'_i\) is the direct sum of the vector spaces in the \(i\)th row.

<table>
<thead>
<tr>
<th>(V'_i)</th>
<th>(V'_2)</th>
<th>(V'_3)</th>
<th>(V'_{n-1})</th>
<th>(V'_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>(\ker x_{1,2})</td>
<td>(\ker x_{1,3})</td>
<td>(\ker x_{1,n-1})</td>
<td>(k^{d_1})</td>
</tr>
<tr>
<td>(\text{im} \ x_1)</td>
<td>(\text{im} \ x_1)</td>
<td>(\text{im} \ x_{2,3})</td>
<td>(\ker x_{2,n-1})</td>
<td>(k^{d_2})</td>
</tr>
<tr>
<td>(\text{im} \ x_2)</td>
<td>(\text{im} \ x_2)</td>
<td>(+ \text{im} \ x_1)</td>
<td>(\ker x_{3,n-1})</td>
<td>(k^{d_3})</td>
</tr>
<tr>
<td>(\cdots)</td>
<td>(\cdots)</td>
<td>(\cdots)</td>
<td>(\cdots)</td>
<td>(\cdots)</td>
</tr>
<tr>
<td>(\text{im} \ x_{n-2})</td>
<td>(\text{im} \ x_{n-2})</td>
<td>(\text{im} \ x_{n-3})</td>
<td>(\ker x_{n-2,n-1})</td>
<td>(k^{d_{n-2}})</td>
</tr>
<tr>
<td>(\text{im} \ x_{n-3})</td>
<td>(\text{im} \ x_{n-3})</td>
<td>(\text{im} \ x_{n-2})</td>
<td>(\ker x_{n-2,n-1})</td>
<td>(k^{d_{n-1}})</td>
</tr>
</tbody>
</table>

For \(0 \leq i, j \leq n\), let \(X_{i,j} = V_{i-1} + (V_i \cap V'_j)\). Then

\[\text{5 This order } \leq B_0 \text{ is denoted by } \leq \text{ in [1].}\]
\[ X_{i,j} = \begin{cases} \oplus_{i=1}^{n-1} \text{im } x_i, & \text{if } i = j = 1; \\ V_{i-1}, & \text{if } i \geq j, (i,j) \neq (1,1); \\ \langle \oplus_{i=1}^{m-1} l^k \rangle \oplus (\ker x_{i,j} + \text{im } x_{i-1}) \oplus (\oplus_{i=1}^{n-1} \text{im } x_i), & \text{if } i < j. \end{cases} \]

Therefore, we have

\[ X_{i,j}/X_{i,j-1} \cong \begin{cases} \oplus_{i=1}^{n-1} \text{im } x_i, & \text{if } (i,j) = (1,1); \\ (\ker x_{i,j} + \text{im } x_{i-1})/(\ker x_{i,j-1} + \text{im } x_{i-1}), & \text{if } i < j; \\ 0, & \text{otherwise.} \end{cases} \]

Let \( A_x : = (f_x, f'_x) \in \Xi \), and let \( A_x^+ \) be the strictly upper triangular part of \( A_x \). We regard \( A_x^+ \) as an element \( \delta \) in the set \( \Theta \) of functions \( \Phi^+ \rightarrow \mathbb{N} \).

**Lemma 5.1.** For any \( x \in \mathcal{R}(d) \), we have \( M_x \cong M(A_x^+) \).

**Proof.** Choose a basis \( u_1, \ldots, u_d \) for \( k^d \) such that the first \( a_{1,2} \) vectors form a basis for \( \ker x_{1,2} \), and the next \( a_{1,3} \) vectors form a basis for \( \ker x_{1,3}/\ker x_{1,2} \), and so on. Each of the basis elements generates an indecomposable \( kA \)-module whose head is isomorphic to \( S_1 \). Clearly, the module generated by these basis elements is isomorphic to \( M_1 = \oplus_{j=2}^n a_{1,j} M_{1,j} \). Now, the assertion follows from induction on the quiver representation \( M_x/M_1 \). \( \Box \)

Let \( r = |d| := d_1 + \cdots + d_n - 1 \). The group \( GL_d(k) \) will be viewed as a Levi subgroup of \( GL_r(k) \cong GL(V) \). Thus, \( GL_d(k) \) acts on \( V = \oplus k^d \): explicitly, given \( g_i \in GL_d(k) \) and \( v_i \in k^d \), we have \( (g_i)\cdot (v_i) = (g_i v_i) \). Therefore, \( GL_d(k) \) also acts naturally on the projective variety \( V \).

**Proposition 5.2.** The map \( \xi : \mathcal{R}(d) \rightarrow V \) defined by \( \xi(x) = (f_x, f'_x) \) is \( GL_d(k) \)-equivariant.

**Proof.** Let \( g = (g_i)_i \) and suppose \( x \in \mathcal{R}(d) \) and \( y = g \cdot x \), that is, \( y_i = g_{i+1} x_i g_i^{-1} \) \( \forall i \). It is clear that \( \text{im } y_i = g_i \text{im } x_i \forall i \), and \( \ker y_{i,j} = g_i \ker x_{i,j} \forall i < j \). Thus, \( (f_y, f'_y) = g (f_x, f'_x) \). \( \Box \)

As a result, \( \xi \) induces a map \( \mathcal{R}(d)/GL_d(k) \rightarrow V/GL(V) \) of orbit spaces, and so it induces a map

\[ (5.2.1) \quad \xi : \Theta \rightarrow \Xi \]

satisfying \( \xi(f) \in \Xi \), provided \( \dim M(f) = r \).

**Remark 5.3.** Recall that \( \Theta \) has a poset structure \( \leq \) defined in (2.1.1), while \( \Xi \) carries the Bruhat order \( \leq^B_0 \) defined in (5.0.1). However, the map \( \xi \) given in (5.2.1) is not a map of posets. For example, let \( n = 3 \), \( M_x \cong 2S_1 \oplus S_2 \oplus 2M^{1,3} \) and \( M_y \cong S_1 \oplus 3M^{1,3} \). Then \( M_x \nleq M_y \) and

\[ \text{dim } x_j = \dim M_x - \sum_{i < j} a_{i,j}. \]

\[ 472 \]
\[ A_x = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_y = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

It is clear that \( A_x \) and \( A_y \) are incomparable under \( \leq B_0 \), since \( ro(A_x) \neq ro(A_y) \) (see [1, 1.4]). In other words, \( A_x^+ \leq A_y^+ \not\implies A_x \leq B_0 A_y. \)

We identify \( \Theta \) with \( \Xi^+ \) and consider the partial orderings \( \leq \) and \( \leq \) on \( \Theta \), defined in (2.1.1) and (4.2.1), respectively. Let

\[ \Theta_d = \{ f \in \Theta \mid \partial(f) = d \}. \]

Given \( f \in \Theta \) and \( d = (d_1, \ldots, d_{n-1}) \), observe that

\[ \partial(f) = d \implies \sigma_{i,i+1}(f) = d_i, \forall i. \]

**Proposition 5.4.** Let \( f, g \in \Theta_d \). Then \( f \leq g \) if and only if \( f \leq g \).

**Proof.** By [2, 3.2], we have \( f \leq g \) if and only if

\[ \dim \text{Hom}(X, M(f)) \geq \dim \text{Hom}(X, M(g)) \quad \forall X. \]

It suffices to assume \( X \) is indecomposable. Thus, by Lemma 3.1, (5.4.1) is equivalent to

\[ \rho_{i,j}(f) \geq \rho_{i,j}(g) \quad \forall i < j, \]

where \( \rho_{i,j}(h) = \sum_{r \leq i, s \leq j} h_{r,s} \), which is obviously equivalent to

\[ \sigma_{i,j}(f) \leq \sigma_{i,j}(g) \quad \forall i < j, \]

using (5.3.1) and the fact that \( \rho_{i,j}(h) + \sigma_{i,j+1}(h) = \sigma_{i,j+1}(h) \). \( \square \)

In the expression given in Lemma 4.2 for the product \( E_{h,h+1}(0) A(0) \) as a linear combination of certain terms \( B(0) \) for \( B = A + E_{h,h+1} \) or \( B = A + E_{h,h+1,j} \) for some \( j > h + 1 \) and \( a_{h+1,j} \geq 1 \). Clearly, any two such terms \( B(0) \) and \( C(0) \) satisfy \( \partial(B) = \partial(C) \) (identifying \( B, C \) as elements in \( \Theta \)). Therefore, using (4.3.1), it follows that those \( B \) appearing in (4.3.3) all satisfy \( \partial(B) = \partial(A) \). Thus, using 5.4, we can rewrite (4.3.3), as

\[ E(A) = A(0) + \sum_{B, B \leq A} \gamma_{B,A} B(0). \]

Also, Theorem 4.4 implies that, up to some factor of the form \( v^d \), the coefficients \( \gamma_{B,A} \) are actually Hall polynomials (cf. [5, (6)]).

6. THE DRINFELD–JIMBO PRESENTATION OF q-SCHUR ALGEBRAS

In this section, we review some results proved in [11]. For indeterminates \( X \) and \( v \), and \( g \in \mathbb{N} \), let

\[ [X; t] = (X - 1)(X - v) \cdots (X - v^{g-1}). \]

Let \( n, r \) be positive integers, and let \( U \), be the associative algebra over \( \mathbb{Q}(v) \) with generators

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\[ e_i, f_i, k_i \quad (1 \leq i \leq n - 1) \]

and relations: for \(1 \leq i, j \leq n - 1,\)

\[ k_i k_j = k_j k_i, \]
\[ [k_1; t_1] [k_2; t_2] \cdots [k_{n-1}; t_{n-1}] = 0 \quad \forall t_i \in \mathbb{N}, t_1 + \cdots + t_{n-1} = r + 1, \]
\[ e_i e_j = e_j e_i, f_i f_j = f_j f_i \quad (|i - j| > 1), \]
\[ e_i^2 e_j - (v + v^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad (|i - j| = 1), \]
\[ f_i^2 f_j - (v + v^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \quad (|i - j| = 1), \]
\[ k_i e_j = v^{\epsilon(i,j)} e_j k_i, \quad k_i f_j = v^{-\epsilon(i,j)} f_j k_i, \quad e_i f_j - f_j e_i = \delta_{i,j} = 0, \]

where \(\epsilon(i, i) = 1, \epsilon(i + 1, i) = -1\) and \(\epsilon(i, j) = 0,\) otherwise; and \(k_i = k_i^{-1} \) for all \(i = 1, 2, \ldots, n - 1\) with \(k_n = v^{r} k_1^{-1} \cdots k_{n-1}^{-1}.\)

Since \([k_i; r + 1] = 0,\) the relations imply each \(k_i\) is invertible. By [11, §5], \(U_r\) is isomorphic to the \(q\)-Schur algebra \(S_q(n, r) := \text{End}_{\mathcal{H}_Q(v)}(V^{\otimes r}),\) where \(V\) is an \(n\)-dimensional vector space over \(Q(v)\) and \(\mathcal{H}_Q(v) = \mathcal{H}(\mathbb{Z}_r)Q(v)\) is the Hecke algebra over \(Q(v)\) associated with the symmetric group \(\mathbb{S}_r.\)

We use the sets \(\Xi, \Xi^+, \Xi^-\) defined in §4. Also, let \(\Xi^0 \subset \Xi\) consist of the diagonal matrices. For any \(A \in \Xi,\) we write \(A = A^+ + A^0 + A^-\) with \(A^+ \in \Xi^+,\) \(A^- \in \Xi^-\) and \(A^0 \in \Xi^0.\)

For \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n,\) let

\[ k_i = \prod_{i=1}^{n} \left[ k_i; 0 \right] \] 
\[ = \begin{bmatrix} k_i; 0 \end{bmatrix} \]
\[ \lambda_i \]
\[ = \begin{bmatrix} k_i; 0 \end{bmatrix} \frac{k_i v^{-s+1} - k_i^{-1} v^{s-1}}{v^s - v^{-s}}. \]

For \(A \in \Xi,\) define

\[ e^{(A^+)} = \prod_{1 \leq i < h < j \leq n} e_h^{(a_{i,j})}, \]
\[ f^{(A^-)} = \prod_{1 \leq j < h < i \leq n} f_h^{(a_{i,j})} \]
\[ \text{where the product } e^{(A^+)} \text{ is taken over the linear order (4.3.2) on triples } (i, h, j) \]
\[ \text{and the product } f^{(A^-)} \text{ is taken over the reversed order. Here } e_h^{(a_{i,j})} := \frac{1}{[a_{i,j}]} e_h^{a_{i,j}} \text{ and } \]
\[ f_h^{(a_{i,j})} := \frac{1}{[a_{i,j}]} f_h^{a_{i,j}}. \]

For \(A = (a_{i,j}) \in \Xi,\) define

\[ \sigma_i(A) = a_{i,i} + \sum_{1 \leq j < i} (a_{i,j} + a_{j,i}) \quad \text{and } \sigma(A) = \sum_{i,j} a_{i,j} \]

and let \(\Xi_r = \{ A \in \Xi | \sigma(A) = r \}\) as before.

The following result is proved in [11, 4.14, 6.4].

**Theorem 6.1** For \(A \in \Xi_r,\) let \(\lambda = \lambda(A) = (\sigma_1(A), \ldots, \sigma_n(A))\) and put \(m(A) := e^{(A^+)} k_1 f^{(A^-)}.\) Then \(\{m(A)\}_{A \in \Xi} \) is a basis for \(U_r.\) Moreover, the \(\mathbb{Z}\)-span \(U_r := \sum_{A \in \Xi} \mathbb{Z}m(A)\) is a \(\mathbb{Z}\)-subalgebra of \(U_r.\)

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7 A slightly different presentation for \(U_r\) is given in [6], using different methods. See [11, Remark 4.4] for a comparison between the two presentations.
The integral algebra \( U_r \) can be expressed as a product

\[ U_r = U^+_r U^0_r U^-_r \]

where

\[ U^+_r = \sum_{A \in \Xi_+} Z e^{(A^+)} \], \[ U^-_r = \sum_{A \in \Xi_-} Z e^{(A^-)} \] and \( U^0_r = \sum_{A \in \Xi_0} Z k_\lambda(A) \).

The subalgebras \( U^+_r \) and \( U^-_r \) are called the positive and negative parts of \( U_r \). In §§7, 8, we show that \( U^+_r \) and \( U^-_r \) inherit the monomial and canonical basis properties from \( H \).

7. THE MONOMIAL BASIS PROPERTY

In this section, \( k \) will denote an algebraically closed field, and we continue to write \( M(f) \) for \( M_k(f) \). For any \( f \in \Theta \), put \( |f| = \sum i < j f_{ij} \). Clearly, \( |f| \) is the number of simple summands in the head \( \text{hd} M(f) \) of \( M(f) \). Recall that \( \text{hd} M(f) \) is the maximal semisimple quotient of \( M(f) \).

Lemma 7.1. (1) Let \( M(g) \) be an extension of \( S_i = M(f_i) \) by \( M(f) \) (resp. of \( M(f) \) by \( S_i \)). Then \( |g| \geq |f| \) and \( |g| \geq |f_i \ast f| \) (resp. \( |g| \geq |f \ast f_i| \)).

(2) The subspace \( J_r \) of \( H \) spanned by \{\( u_f | f \in \Theta, |f| > r \)\} is an ideal.

Proof. If \( f_{i+1,j} = 0 \) for all \( j > i + 1 \), then \( M(g) = S_i \oplus M(f) = S_i \ast M(f) \), so \( |g| = |f| + 1 = |f_i \ast f| \). If \( f_{i+1,j} \neq 0 \) for some \( j > i + 1 \), then, using Lemma 3.2, \( |g| \geq |f| = |f_i \ast f| \). Dually, we can prove \( |g| \geq |f \ast f_i| \). This proves (1).

Suppose \( f \in \Theta \) with \( |f| > r \). By definition, we have

\[ u_i u_f = \varphi_{f_i,f} \sum_{g \in \Theta} \varphi_{j,f} u_g. \]

If \( \varphi_{f_i,f} \neq 0 \) then \( M(g) \) has a submodule isomorphic to \( M(f) \) and \( M(g)/M(f) \cong S_i \). By (1), \( |g| \geq |f| > r \). Therefore, \( u_i u_f \in J_r \). A similar argument shows that \( u_f u_i \in J_r \). Since \( H \) is generated by \( u_i, i \in \Delta_0 \), it follows that \( J_r \) is an ideal, proving (2). \( \square \)

Corollary 7.2. Let \( J_r \) be the \( Z \)-submodule of \( H \) generated by \{\( u_f | f \in \Theta, |f| > r \)\}. Then \( J_r \) is an ideal of \( H \).

Proof. Since \( H \) is generated by the \( u_i^{(m)} \) (see Proposition 2.1) and \{\( u_f \}_{f \in \Theta} \) is a \( Z \)-basis for \( H \), the argument above shows that \( u_i^{(m)} u_f, u_f u_i^{(m)} \in J_r \). Hence, \( J_r \) is an ideal of \( H \). \( \square \)

Lemma 7.3. For \( f, g \in \Theta \), if \( f \leq g \) then \( |f| \geq |g| \).

Proof. Since \( |f| \) is the number of the irreducible summands in \( \text{hd} M(f) \), it suf-
rices to prove that, if \( f \leq g \), then \( \text{hd} M(g) \mid \text{hd} M(f) \). Here \( X \mid Y \) means that \( X \) is isomorphic to a direct summand of \( Y \).

By (2.2.1), we may choose \( w = i_1 i_2 \cdots i_m \in \Omega \) such that \( \varphi(w) = g \) and \( M(f) \) has a composition series

\[
M(f) = M_0 \supset M_1 \supset \cdots \supset M_m \supset M_{m+1} = 0
\]

with \( M_{j-1}/M_j \cong S_j \) for all \( j = 1, \ldots, m \). Since \( \varphi(w) = g \), we have

\[
M(g) = S_{i_1} \ast S_{i_2} \ast \cdots \ast S_{i_m} = S_{i_1} \ast N
\]

where \( N = S_{i_2} \ast \cdots \ast S_{i_m} \). We apply induction on \( m \). The result clearly holds if \( m = 1 \). Assume \( m > 1 \) and \( \text{hd} N \mid \text{hd} M_1 \). Then \( S_{i_1} \mid \text{hd} M_1 \). Thus, \( \text{hd}(S_{i_1} \ast N) \mid \text{hd}(S_{i_1} \ast M_1) \). By Lemma 7.1(1), we see that \( \text{hd}(S_{i_1} \ast M_1) \mid \text{hd} M_0 \), so \( \text{hd} M(g) \mid \text{hd} M(f) \).

\[ \square \]

**Remark 7.4** The above proof does not use the results of [2], as in the proof of Proposition 5.4. However, [2] easily gives another argument for Lemma 7.3: In the notation of (5.4) \( f \leq g \) if and only if \( \rho_{i,j}(f) \geq \rho_{i,j}(g) \), \( \forall i,j \). Thus, if \( f \leq g \),

\[
|f| = \sum_j \rho_{j,j+1}(f) \geq \sum_j \rho_{j,j+1}(g) = |g|,
\]

as required.

Recall that \( \Omega \) denotes the set of all words in \( \Delta_0 = \{1, \ldots, n - 1\} \). We will use the mapping \( \varphi: \Omega \to \Theta \) defined in (2.2.1). For \( w = i_1 \cdots i_m = j^1_s \cdots j^t_s \in \Omega \) with \( j_{i-1} \neq j_s \), \( \forall s \), let

\[
m^w = u_{i_1} \cdots u_{i_m} \text{ and } m^{(w)} = u_{j_1}^{(e_1)} \cdots u_{j_t}^{(e_t)} = \frac{1}{\prod_{r=1}^t [e_r]} m^w.
\]

In part (2) of the result below, we will use the fact that, given \( f \in \Theta \), the fibre \( \Omega_f \) contains at least one distinguished word \( w \) (as defined in §3); this result is proved in [5, Lemma 5.2]. Also, if \( \varphi_w = \varphi_{j_1^1 \cdots j_m^t} \neq 0 \), then \( f \leq \varphi(w) \); see [5, Theorem 4.2].

**Proposition 7.5.** For each \( f \in \Theta \), fix any representative \( w_f \in \Omega_f = \varphi^{-1}(w) \).

1. The ideal \( J_f \) is spanned by monomials \( \{ m^w \mid w \in \Omega, |\varphi(w)| > r \} \). Moreover, the set \( \{ m^w \mid |f| > r \} \) forms a basis for \( J_f \).

2. The integral ideal \( J_f \) is spanned by \( \{ m^{(w)} \mid w \in \Omega, |\varphi(w)| > r \} \). Moreover, for each \( f \), we can choose \( w_f \) to be distinguished. In this case, the set \( \{ m^{(w)} \mid w_f \text{ distinguished, } |f| > r \} \) forms a \( \mathcal{Z} \)-basis for \( J_f \).

**Proof.** For any \( w = i_1 \cdots i_m = j^1_s \cdots j^t_s \), we have, by remarks immediately before the statement of the proposition,

\[
(7.5.1) \quad m^w = \sum_{g \leq \varphi(w)} \varphi_{j_1^1 \cdots j_m^t} \sum_{g \leq \varphi(w)} \varphi_{j_1^1 \cdots j_m^t} u_{j_1}^{(e_1)} \cdots u_{j_t}^{(e_t)} = \sum_{g \leq \varphi(w)} \varphi_{j_1^1 \cdots j_m^t} u_{j_1}^{(e_1)} \cdots u_{j_t}^{(e_t)}.
\]

By Lemma 7.3, we see that \( m^w \in J_f \) and \( m^{(w)} \in J_f \) if \( |\varphi(w)| > r \). For \( f \in \Theta \) sat-
isfying $|f| > r$, (7.5.1) defines (using $w = w_f$) a triangular decomposition for the $m^{w_f}$ in terms of the $u_g$, $|g| > r$. This proves (1). If $w_f$ is distinguished, then $\varphi_{e_i f_h, \cdots; e_i f_h} = 1$, so (2) also follows. \qed

Let $U = U_q(\mathfrak{g}_n^r)$ be the quantum enveloping algebra over $\mathbb{Q}(q)$ associated to the Lie algebra $\mathfrak{g}_n^r$. This algebra has a well-known presentation; see, e.g., [21, 3.2] or [10, 1.1]. Comparing this presentation that given for $U_r$ given in §6, there is a natural algebra epimorphism $\zeta : U \to U_r$ satisfying $\zeta(E_i) = e_i$, $\zeta(F_i) = f_i$, and $\zeta(K_j) = k_j$, for $1 \leq i \leq n - 1$, $1 \leq j \leq n$. By [10], $\zeta$ induces an epimorphism $\zeta_r : U \to U_r$. By Proposition 2.1, there is a natural isomorphism $\Psi : U^+ \cong H$; in the following result, we identify $U^+$ and $H$ by means of $\Psi$.

**Corollary 7.6.** The restriction $\zeta^+_{r}$ of $\zeta$ to $U^+$ gives rise to a surjective homomorphism $\zeta^+_{r} : U^+ \to U^+_r$ with $\ker \zeta^+_{r} = J_r$. In particular, the elements $\tilde{v}_f := \zeta^+_{r}(\tilde{w}_f), f \in \Theta_r := \{f \in \Theta ||f| \leq r\}$, form a basis for $U^+_r$.

**Proof.** Let, for each $A \in \Xi^+$, $w_A := w_n w_{n-1} \cdots w_2$ where

$$w_f = (j - 1)^{a_{j-1}^+} \cdots (j - 2)^{a_{j-2}^+} (j - 1)^{a_{j-1}^-} \cdots (j - 1)^{a_{j-1}^-}.$$

Then we see from (4.3.1) $E^{(A)} = m^{(w_A)}$. Clearly, if $f = \varphi(w_A)$ (cf. [5, §9]) then $f_{i,j} = a_{i,j}$. The assertion now follows immediately from Proposition 7.5(2) and [11, 4.11, 8.3]. \qed

For any $w \in \Omega$, let $m^w = \zeta_r(m^{w})$ and $m^w = \zeta_r(m^{w})$. Clearly, $m^w = m^{(w_A)} = 0$ if $|\varphi(w)| > r$ and $e^{(A)} = m^{(w_A)}$. Let $\Omega_r = \{w \in \Omega ||\varphi(w)| \leq r\}$. Then $\varphi$ induces a surjection $\varphi : \Omega_r \to \Theta_r$, where $\Theta_r = \{f \in \Theta ||f| \leq r\}$, as above. Now Proposition 7.5 and Corollary 7.6 imply the following monomial basis property.

**Theorem 7.7.** For any $f \in \Theta_r$, choose a representative $w_f \in \varphi^{-1}(f)$. Then the set $\{m^{w_f} | f \in \Theta_r\}$ forms a basis for $U^+_r$. If all $w_f$ are distinguished, then $\{m^{(w_f)} | f \in \Theta_r\}$ forms a basis for $U^+_r$.

8. **THE CANONICAL BASIS AND ITS IDENTIFICATION**

In this section, we give another application of Theorem 4.4 — a direct connection between the canonical bases for: (1) $U^+$ with respect to the PBW basis $\{u_f\}_{f \in \Theta}$ (see 3.4), using the degeneration order $\leq$ (see (2.1.1)) on $\Theta$; and (2) the $q$-Schur algebra $U_r$ with respect to the standard basis $\{[A]\}_{A \in \Xi}$ defined in [1, 1.4]8, using the Bruhat order $\leq_{B_0}$ (see (5.0.1)) on $\Xi_r$.

The $\Xi$-algebra $U_r$ has a normalized basis $\{[A]\}_{A \in \Xi}$ introduced in [1, (1.1)] which satisfies the formulas given in (4.0.1). Identifying $U$ with the algebra $V$ in §4, we have that $U$ has a basis consisting of elements $A(j)$, where $A \in \Xi^+$ and $j \in \mathbb{Z}^n$, defined in (4.0.2). By [1, (5.7)], the homomorphism $\zeta_r : U \to U_r$ satisfies

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8 This basis is not a subset of the basis $\{[A]\}_{A \in \Xi}$ for $U$.

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\[
\zeta_\tau(A(j)) = A(j, r) := \sum_{D \in \mathfrak{X}_r, \sigma(A + D) = r} v^{\sum_i d_i} [A + D].
\]

Let \( \Xi_{\leq r} \) denote the set of all \( A \in \Xi^+ \) satisfying \( \sigma(A) \leq r \). We can naturally identify \( \Xi_{\leq r} \) with \( \Theta \). Now Theorem 4.4 implies that \( \zeta_\tau \) maps \( \tilde{\nu}_A = A(0) \) to \( A(0, r) \). Hence, we obtain \( \tilde{\nu}_A = \zeta_\tau^+(\tilde{\nu}_A) = A(0, r) \) for all \( A \in \Xi^+_{\leq r} \).

**Lemma 8.1.** Let \( A \in \Xi^+_{\leq r} \) and \( \lambda \in A(n, r) \). If \( \sigma_i(A) \leq \lambda_i, \forall i \), let \( D \in \Xi^+ \) be the unique diagonal matrix such that \( A_\lambda := A + D \) satisfies \( \sigma(A_\lambda) = \lambda \), and set \( \mu = \rho_\lambda(A) \).

Then
\[
\tilde{\nu}_A \lambda = k_\mu \tilde{\nu}_A = \begin{cases} [A_\lambda], & \text{if } \sigma_i(A) \leq \lambda_i, \forall i; \\ 0, & \text{otherwise.} \end{cases}
\]

**Proof.** By [11, Corollary 5.3], \( k_\lambda = [\text{diag}(\lambda_1, \ldots, \lambda_n)] \) for any \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \). The result now follows from the multiplication formulas in \( U_r \) for elements \([E][B] \) and \([E][B], B \in \Xi, E \in \Xi^0 \) mentioned above.

In the discussion below, we will use Theorem 1.1 to construct canonical bases for several of the \( \mathcal{Z} \)-algebras considered in this paper. Each construction will require an “input” basis (which will always be a PBW-type basis), together with an involutory ring automorphism of the algebra, and a poset structure on the indexing set for the PBW-basis. For convenience, the involutions will all be denoted by the same symbol \( \iota \). Locally, this should cause no confusion, and globally the various \( \iota \) are all compatible!

Consider first the algebra \( U_r^+ \). (The case of \( U_r^- \) is left to the reader.) Here we follow [5] in order to obtain a canonical basis \( \{c_f\}_{f \in \Theta} \) of the Hall algebra \( H \). In more detail, there is a unique involutory ring automorphism \( \iota : H \rightarrow H \) which fixes all \( u_i^{(m)} = u_i^{(m)} \) and satisfies \( \iota(v) = v^{-1} \). By [5, §7], the PBW-basis \( \{u_f\}_{f \in \Theta} \) (see Remark 3.4) of \( H \) satisfies the conditions required in Theorem 1.1, relative to \( \iota \) and the degeneration order \( \leq \) on \( \Theta \) defined in (2.1.1). Thus, let \( \{c_f\}_{f \in \Theta} \) be the corresponding canonical basis of \( H \). Since \( \iota(m^{(w)}) = m^{(w)} \), Lemma 7.5 implies that \( \iota \) stabilizes the ideal \( J \), defined in Corollary 7.2. So Theorem 1.1 applies also to \( J \), which has canonical basis \( \{c_f \mid f \in \Theta, |f| > r \} \).

There is an involutory ring automorphism \( \iota : U_r \rightarrow U_r \) satisfying
\[
\iota(e_i) = e_i, \quad \iota(k_i) = k_i^{-1}(1 \leq i \leq n - 1), \quad \iota(f_i) = f_i, \quad \text{and} \quad \iota(v) = \tilde{v} = v^{-1}.
\]
Clearly, we have \( \iota(k_\lambda) = k_\lambda \) and \( \iota(m^{(w)}) = m^{(w)} \) for all \( \lambda \in \mathbb{N}^n \) and \( w \in \Omega \). We consider the restriction of \( \iota \) to \( U_r^+ \).

**Lemma 8.2.** There is a unique basis \( \{c_A\}_{A \in \Xi^+_{\leq r}} \) for \( U_r^+ \) satisfying the following properties:
\[
\iota(c_A) = c_A \text{ and } c_A - \tilde{\nu}_A \in \sum_{B < A} v^{-1} \mathbb{Z}[v^{-1}] \tilde{\nu}_B.
\]

Moreover, if \( \{c_A\}_{A \in \Xi^+} \) is the canonical basis of \( U^+ \cong H \) then
\[
\zeta_\tau^+(c_A) = \begin{cases} c_A, & \text{if } A \in \Xi^+_{\leq r}; \\ 0, & \text{otherwise.} \end{cases}
\]

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Proof. For each \( A \in \Xi^+_{\leq r} \), let \( w(A) \) be defined as in the proof of Corollary 7.6. By [5, 7.2], the exponent \( a_{w(A)} \) in (7.5.1) is given by \( \dim \text{End} (M(A)) - \dim M(A) \). Applying \( \zeta^+ \) to 5.4.2, using Theorem 4.4 and the definition of \( \tilde{u}_r \) given above Theorem 3.3, we obtain the following equation in \( U_r^+ \):

\[
\mathbf{m}^{(w(A))} = \mathbf{e}^{(A)} = \tilde{v}_A + \sum_{B \in \Xi^+_{<r}, B < A} \varphi_{B,A} \tilde{v}_B.
\]

Inverting gives

\[
\tilde{v}_A = \mathbf{e}^{(A)} + \sum_{B \in \Xi^+_{<r}, B < A} \psi_{B,A} \mathbf{e}^{(B)}.
\]

Applying \( \iota \) to (**) and using (*) gives

\[
\iota(\tilde{v}_A) = \tilde{v}_A + \sum_{B \in \Xi^+_{<r}, B < A} \gamma_{B,A} \tilde{v}_B.
\]

Now apply Theorem 1.1 to obtain the canonical basis \( \{ c_A \}_{A \in \Xi^+_{<r}} \) of \( U_r^+ \).

Next, the automorphism \( \iota \) on \( H \) is compatible with the automorphism \( \iota \) on \( U_r^+ \) via the homomorphism \( \zeta^+_r : H \cong U^+ \to U_r^+ \). Since \( \tilde{v}_A = \zeta^+_r(\tilde{u}_A) \), the expression for \( \zeta^+_r(\tilde{v}_A) \) required in the second assertion of the lemma follows from the uniqueness property of Theorem 1.1. \( \Box \)

We now recall the definition of the canonical basis \( \{ \theta_A \}_{\theta \in \Xi} \) for \( U_r \). This basis can be defined geometrically as in [1, 1.4], but [7] provides an elementary treatment which uses the Kazhdan-Lusztig basis \( \{ C_w \}_{w \in W} \) of the Hecke algebra \( \mathcal{H} = \mathcal{H}(\Xi, r) \); see remarks after Theorem 1.1 where the \( C_w \)-basis is noted to be the canonical basis for \( \mathcal{H} \) relative to the PBW-basis \( \{ T_w \}_{w \in \Xi} \), and the bar involution on \( \mathcal{H} \). As shown in [7, (3.1)], the bar involution on \( \mathcal{H} \) naturally induces an involution \( \iota : U_r \to U_r \). When \( n \geq r \), \( \mathcal{H} \) is a subalgebra of \( U_r \), and \( \iota \) agrees with the bar involution on \( \mathcal{H} \). The matrix \( R \) of \( \iota \) with respect to the PBW-basis \( \{ A \}_{A \in \Xi} \) and the Bruhat partial order \( \leq_{B_0} \) on the indexing set \( \Xi \) (see (5.0.1) for the definition of \( \leq_{B_0} \)) is good unipotent matrix as required in (1.1.1). Thus, Theorem 1.1 provides a unique canonical associated basis \( \{ \theta_A \}_{\theta \in \Xi} \).

Define partial orders \( \leq' \) and \( \leq'' \) on \( \Xi \) as follows: put \( A \leq' B \) (resp., \( A \leq'' B \)) if and only if \( \varnothing(A) = \varnothing(B) \) (resp., \( \rho(A) = \rho(B) \)) and \( A \leq B \). Because \( A \leq_{B_0} B \) implies that \( A \prec B \) [1, 3.6(d)] and that \( A \) and \( B \) have the same row and column sums [1, 1.4], both \( \leq' \) and \( \leq'' \) are finer partial orders than \( \leq_{B_0} \). Thus, we can replace \( \leq_{B_0} \) by \( \leq' \) (resp., \( \leq'' \)) in the previous paragraph to conclude that \( \{ \theta_A \}_{\theta \in \Xi} \) is the canonical basis for \( U_r \), determined by the PBW-basis \( \{ A \}_{A \in \Xi} \), the involution \( \iota \), and the partial order \( \leq' \) (resp., \( \leq'' \)) on the indexing set \( \Xi \).

The last canonical bases we consider are for the Borel subalgebras \( U_r \geq 0 \), \( U_r \leq 0 \) of \( U_r \). These subalgebras are discussed in detail in [11, §8]. For example, \( U_r \geq 0 \) is defined as the \( \Xi \)-subalgebra of \( U_r \), generated by the elements \( e_i^{(m)} \) and \( k_\lambda \), \( m \in \mathbb{N}, \lambda \in \Lambda(n, r) \).

\[9 \text{ Hence, the resulting basis for } U_r \text{ was called the Kazhdan-Lusztig basis in [7].} \]
Let $\Xi^\geq 0 = \{ A \in \Xi_r \mid A^- = 0 \}$ (resp., $\Xi^\leq 0 = \{ A \in \Xi_r \mid A^+ = 0 \}$). Then $U^\geq_r = U^+_r \oplus U^0_r$ (resp. $U^\leq_r = U^-_r \oplus U^0_r$) has a PBW-basis $\{ [A] \}_{A \in \Xi^\geq 0}$ (resp., $\{ [A] \}_{A \in \Xi^\leq 0}$); see [11, §8]. The involution $\iota$ on $U_r$ defined above stabilizes both $U^\geq_r$ and $U^\leq_r$.

So, by the previous paragraph, $U^\geq_r$ (resp. $U^\leq_r$) has the canonical basis

$$(8.2.1) \quad C^\geq_0 = \{ \theta_A \}_{A \in \Xi^\geq 0} \quad \text{(resp. } C^\leq_0 = \{ \theta_A \}_{A \in \Xi^\leq 0})$$

defined uniquely with respect to the basis $\{ [A] \}_{A \in \Xi^\geq 0}$ (resp. $\{ [A] \}_{A \in \Xi^\leq 0}$), the involution $\iota$, and the partial order $\preceq$ on $\Xi^\geq 0$ (resp., $\Xi^\leq 0$). For the same reasons, we can replace the partial order $\preceq$ by $\preceq''$.

We can now establish the following important alternate description of the canonical bases $C^\geq_0, C^\leq_0$ for $U^\geq_r, U^\leq_r$.

**Theorem 8.3.** Let $\{ c_A \}_{A \in \Xi^\geq 0}$ (resp. $\{ c_A \}_{A \in \Xi^\leq 0}$) be the canonical basis for $U^+_r$ (resp. $U^-_r$) defined in Lemma 8.2. Then the canonical basis $C^\geq_0$ (resp., $C^\leq_0$) for $U^\geq_r$ (resp., $U^\leq_r$) can also be described as follows:

$$C^\geq_0 = \{ c_A k_\lambda \mid A \in \Xi^+_r, \lambda \in A(n, r), \sigma_i(A) \leq \lambda_i \forall i \}$$

$$C^\leq_0 = \{ c_A k_\lambda \mid A \in \Xi^-_r, \lambda \in A(n, r), \sigma_i(A) \leq \lambda_i \forall i \}.$$ 

More precisely, we have $c_A k_\lambda = k_\mu c_A = \theta_A$, where $\mu = ro(A)$. 

**Proof.** By [1, p.669], $\iota(c_A k_\lambda) = \iota(c_A) \iota(k_\lambda) = c_A k_\lambda$. By Lemma 8.2, $c_A = \bar{v}_A + \sum_{B \prec A} p_{B,A} k_B$, where $p_{B,A} \in v^{-1} Z[v^{-1}]$ and $B \in \Xi^-_r$. Therefore,

$$c_A k_\lambda = \bar{v}_A k_\lambda + \sum_{B \prec A} p_{B,A} \bar{v}_B k_\lambda.$$

Suppose that $A \in \Xi^+_r$, and $\lambda \in A(n, r)$ satisfy $\sigma_i(A) \leq \lambda_i$ for all $i$. By Lemma 8.1, $\bar{v}_A k_\lambda = [A_\lambda]$, while if $\bar{v}_B k_\lambda \neq 0$, then it equals $[B_\lambda]$. Also, $\lambda = co(A_\lambda) = co(B_\lambda)$. Since $B \prec A$, $\partial(A) = \partial(B)$ (identifying $A$ and $B$ as elements of $\Theta$), so Proposition 5.4 implies that $B \leq A$. By definition, $B \leq' A_\lambda$, so:

$$c_A k_\lambda = [A_\lambda] + \sum_{B \prec A_\lambda} p_{B,A_\lambda} [B_\lambda]$$

with $p_{B,A_\lambda} \in v^{-1} Z[v^{-1}]$. Of course, any $C \in \Xi^\geq_0$ can be uniquely expressed as $C = A_\lambda$ for $A \in \Xi^+_r$, $\lambda \in A(n, r)$, and $\sigma_i(A) \leq \lambda_i$ for all $i$. By Theorem 1.1, $c_A k_\lambda = \theta_A$. Similarly, using $\preceq''$, $k_\mu c_A = \theta_A$ (since $\bar{v}_A k_\lambda = k_\mu \bar{v}_A = [A_\lambda]$). 

By [13, §8], the canonical basis $\{ c_A \}_{A \in \Xi^-_r}$ of $U^-_r = H$ has the following remarkable property: if $L(\lambda)$ is the irreducible U-module with highest weight $\lambda$ and if $v_\lambda \in L(\lambda)$ is a nonzero $\lambda$-weight vector, then

$$B[\lambda] := \{ c_A v_\lambda \mid A \in \Xi^-_r \} \setminus \{ 0 \}$$

forms a basis for $L(\lambda)$.

On the other hand, if we assume $\lambda \in A^+(n, r)$ (the set of partitions in $A(n, r)$), then $L(\lambda)$ is an $U_r$-module. By [8], the set

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forms a basis for \( L(\lambda) \).

**Corollary 8.4.** We have \( B[\lambda] = B[\lambda]' \).

**Proof.** The condition \( \lambda \in \Lambda^+(n, r) \) guarantees that, if \( c_\lambda v_\lambda \neq 0 \), then

\[
c_\lambda v_\lambda = c_\lambda v_\lambda = c_\lambda \theta_\lambda v_\lambda = \theta_\lambda v_\lambda
\]

by Theorem 8.3, noting that \( k_\lambda v_\lambda = v_\lambda \). Hence, \( B[\lambda] \subseteq B[\lambda]' \), and consequently, \( B[\lambda] = B[\lambda]' \). \( \square \)

**REFERENCES**


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