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Convoluted C-cosine functions and semigroups. Relations with ultradistribution and hyperfunction sines

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Abstract

Convoluted C-cosine functions and semigroups in a Banach space setting extending the classes of fractionally integrated C-cosine functions and semigroups are systematically analyzed. Structural properties of such operator families are obtained. Relations between convoluted C-cosine functions and analytic convoluted C-semigroups, introduced and investigated in this paper are given through the convoluted version of the abstract Weierstrass formula which is also proved in the paper. Ultradistribution and hyperfunction sines are connected with analytic convoluted semigroups and ultradistribution semigroups. Several examples of operators generating convoluted cosine functions, (analytic) convoluted semigroups as well as hyperfunction and ultradistribution sines illustrate the abstract approach of the authors. As an application, it is proved that the polyharmonic operator Δ^{2^n}, $n \in \mathbb{N}$, acting on $L^2[0,\pi]$ with appropriate boundary conditions, generates an exponentially bounded K_n-convoluted cosine function, and consequently, an exponentially bounded analytic K_{n+1}-convoluted semigroup of angle $\frac{\pi}{2}$, for suitable exponentially bounded kernels K_n and K_{n+1}.

Keywords: Convoluted C-cosine functions; Convoluted C-semigroups; Ultradistribution sines; Hyperfunction sines

1. Introduction and preliminaries

We study a class of convoluted C-cosine functions extending the class of α-times integrated C-cosine functions, $\alpha > 0$ and continue our researches in [34–37] where we investigate different kinds of convoluted operator type families and their relations with (tempered) ultradistribution semigroups and (Fourier) hyperfunction semigroups.

Local convoluted C-semigroups were introduced and studied in the papers of I. Cioră̆nescu and G. Lumer [9,10] who related them to ultradistribution semigroups, in the particular case $C = I$. We refer to [6–14] (see also [15,43]), [20,25,31,33,37,40] and [46] for further information concerning ultradistribution semigroups. We analyze in this paper ultradistribution and hyperfunction sines continuing the researches of H. Komatsu [31] and P.C. Kunstmann [39,40].

A class of exponentially bounded convoluted semigroups is introduced and studied in [27] via the operator valued Laplace transform while global convoluted semigroups which are not necessarily exponentially bounded have

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been recently analyzed in [35] and [36] (see also [22]). We also refer to investigations of B. Bäumer, G. Lumer and F. Neubrander [4] and [45], for the use of the asymptotic Laplace transform in the theory of convoluted semigroups, as well as to the paper [48] of C. Müler for the approximations of local convoluted semigroups. In this paper, we further study convoluted C-cosine functions introduced in [35] and obtain several generalizations of results known for integrated C-cosine functions (cf. [2,19,25,26,28,32,33,47,51,52,54–57] and [58]). We analyze in Section 2 K-convoluted C-cosine functions by a trustworthy passing to the theory of K-convoluted C-semigroups on product spaces and we compare corresponding integral generators of such operator families. Such an approach enables one to obtain several properties of subgenerators of convoluted C-cosine functions. We also focus our attention to the case $C = I$ and continue the analysis of P.C. Kunstmann [38] concerning stationary dense operators in Banach spaces. We prove that every generator A of a (local) α-times integrated cosine function is stationary dense and satisfies $n(A) \leqslant \lfloor \frac{\lceil\alpha\rceil+1}{2} \rfloor$. It seems to be an open problem to improve this inequality; nevertheless, the concept of stationarity, whose application in the problems of maximal regularity of abstract Cauchy problems is not clearly understandable, makes a difference between integrated operator type families and convoluted operator type families. We generalize in Section 3 results of [35] which are related to the Laplace transform of exponentially bounded K-convoluted C-cosine functions in order to use them in the later analysis of the polyharmonic operator Δ^{2^n} on $L^2[0,\pi]$.

Our main results are given in Sections 4, 5 and 6. In Section 4, we obtain the Hille–Yosida type theorems for generators of analytic convoluted C-semigroups introduced in this paper (see also [36]) and prove the convoluted version of the abstract Weierstrass formula connecting analytic convoluted C-semigroups and convoluted C-cosine functions. We relate in Section 5 ultradistribution and hyperfunction sines to analytic convoluted semigroups and note, in the ultradistribution case, some differences between Beurling and Roumieu-type ultradistribution sines. Theorem 15 connects ultradistribution sines of (M_p)-class, respectively, $\{M_p\}$-class, with ultradistribution semigroups of (M_p^2)-class, respectively, $\{M_p^2\}$-class. In the rest of Section 5, we analyze relations between (local) integrated cosine functions as well as convoluted cosine functions with ultradistribution semigroups. Such results were first obtained by V. Keyantuo in [25, Theorem 3.1] and this theorem has been recently generalized and analyzed in [33, Theorem 4.3, Example 4.4]. Our results can be used in the analysis of abstract Cauchy problems in the framework of various vector-valued generalized function spaces.

We discuss in examples of Section 6 the polyharmonic operator acting on $L^2[0,\pi]$ and point out, motivated by [4], situations when the theory of convoluted cosine functions and semigroups ($C = I$) cannot be used in the analysis of a wide class of elliptic differential operators acting on L^p-type spaces (cf. E.B. Davies [16,17]). In order to prove that the polyharmonic operator Δ^{2^n} on $L^2[0,\pi]$ generates a convoluted cosine function, we essentially use the fact that $-\Delta^{2n}$ generates an analytic C_0-semigroup of angle $\frac{\pi}{2}$ proved by J.A. Goldstein in [21], see also [18, Example 24.11]. Still, it is an open problem to characterize polynomials of $-\Delta$ in the framework of the theory of convoluted cosine functions and semigroups. We refer to [18, Sections VIII, XXIV] for the application of entire regularized groups in the analysis of such problems. Following R. Beals [5,6], we construct an illustrative example of an operator A acting on the Hardy space $H^p(\mathbb{C}_+)$, $1 \leqslant p < \infty$ which generates a hyperfunction sine, but not an ultradistribution sine. Local integrated semigroups generated by multiplication operators were explicitly constructed by W. Arendt, O. El-Mennaoui and V. Keyantuo in [1] (cf. [32] for integrated cosine functions). We construct convoluted cosine functions generated by multiplication operators in Example 3 where we also discuss the maximal interval of existence of a convoluted cosine function and present an example of a global non-exponentially bounded convoluted cosine function.

In order to concentrate the exposition on our main results, several structural properties of K-convoluted C-semigroups and cosine functions are excluded, see [35] for more details. Also, because of that, we do not analyze composition properties, perturbations and approximation type results for convoluted C-cosine functions as well as the corresponding abstract Cauchy problems. These themes will be treated in a separate paper.

Notation. By E and $L(E)$ are denoted a complex Banach space and the Banach algebra of bounded linear operators on E. For a closed linear operator A on E, $D(A)$, $\mathrm{Kern}(A)$, $R(A)$, $\rho(A)$ denote respectively its domain, kernel, range and resolvent set. Put $D_\infty(A) := \bigcap_{n\in\mathbb{N}_0} D(A^n)$. We denote by $[D(A)]$ the Banach space $D(A)$ endowed with the graph norm. In this paper, $C \in L(E)$ is an injective operator satisfying $CA \subset AC$.

We recall the basic facts from the Denjoy–Carleman–Komatsu theory of ultradistributions although a great part of our results can be transferred to the case of ω-type ultradistributions. In the sequel, $(M_p)_p$ is a sequence of positive numbers, $M_0 = 1$, such that the following conditions are satisfied:

(M1) $M_p^2 \leqslant M_{p+1}M_{p-1}$, $p \in \mathbb{N}$,
(M2) $M_n \leqslant AH^n \min_{p+q=n} M_p M_q$, $n \in \mathbb{N}$, for some $A, H > 0$,
(M3)′ $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$.

If (M_p) is such a sequence, then as a matter of routine, one can check that (M_p^2) also satisfies (M1), (M2) and (M3)′.

If $s > 1$ then the Gevrey sequences $(p!^s)_p$, $(p^{ps})_p$ or $(\Gamma(1+ps))_p$ satisfy the above conditions. The associated function is defined by $M(\rho) := \sup_{p\in\mathbb{N}} \ln \frac{\rho^p}{M_p}$, $\rho > 0$; $M(0) := 0$. If $\lambda \in \mathbb{C}$, then $M(\lambda) := M(|\lambda|)$.

We refer to [29] and [30] for the basic properties of locally convex space-valued ultradifferentiable functions defined on $\mathbb{R}$ and corresponding ultradistributions of the Beurling, respectively, Roumieu type. The classes of Beurling, respectively, Roumieu ultradistributions with values in a Banach space E are denoted by $\mathcal{D}^{'(M_p)}(E)$, respectively, $\mathcal{D}^{'\{M_p\}}(E)$ or simply $\mathcal{D}^{'(M_p)}$, respectively, $\mathcal{D}^{'\{M_p\}}$ in the case $E = \mathbb{R}$. We denote by $*$ either (M_p) or $\{M_p\}$. A similar terminology is used for the spaces of Beurling and Roumieu type ultradifferentiable functions. The space of all scalar-valued ultradistributions of $*$-class with the support contained in $[0,\infty)$ is denoted by $\mathcal{D}_0'^*$ ($\mathcal{D}_0'^*(E)$ in the case of E-valued ultradistributions).

The spaces of tempered ultradistributions of Beurling and Roumieu type (cf. [37] and [50]) are defined as duals of $\mathcal{S}^{(M_p)}(\mathbb{R}) := \operatorname{proj}\lim_{k\to\infty} \mathcal{S}^{M_p,k}(\mathbb{R})$, respectively, $\mathcal{S}^{\{M_p\}}(\mathbb{R}) := \operatorname{ind}\lim_{k\to 0} \mathcal{S}^{M_p,k}(\mathbb{R})$, where $\mathcal{S}^{M_p,k}(\mathbb{R}) := \{\phi \in C^\infty(\mathbb{R})\colon \|\phi\|_k < \infty\}$, $k > 0$ and $\|\phi\|_k := \sup\{\frac{k^{\alpha+\beta}}{M_\alpha M_\beta}(1+|t|^2)^{\beta/2}|\phi^{(\alpha)}(t)|\colon t \in \mathbb{R},\ \alpha,\beta \in \mathbb{N}_0\}$. We refer to the book of A. Kaneko [24] for the basic facts about hyperfunctions and Fourier hyperfunctions.

1.1. Terminology used in the paper

1. $LT(\mathbb{C})$ denotes the space of all Laplace transforms of locally integrable, exponentially bounded functions.
2. If $\omega > 0$, put $\Pi_\omega := \{z \in \mathbb{C}\colon \operatorname{Re} z > \omega^2 - \frac{(\operatorname{Im} z)^2}{4\omega^2}\}$. Note that $\Pi_\omega = \{z^2\colon z \in \mathbb{C},\ \operatorname{Re} z > \omega\}$.
3. Let $\varepsilon > 0$ and $C_\varepsilon > 0$. The next region was introduced by S. Ōuchi in [49]: $\Omega_{\varepsilon,C_\varepsilon} := \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant \varepsilon|\lambda| + C_\varepsilon\}$. We will use the notation $\Omega^2_{\varepsilon,C_\varepsilon} := \{\lambda^2\colon \lambda \in \Omega_{\varepsilon,C_\varepsilon}\}$.
4. As in [31], we define $\Omega^{(M_p)}$ as a subset of $\mathbb{C}$ which contains a domain of the form

$$\Omega^{M_p}_{k,C} := \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant M(k|\lambda|) + C\},$$

for some $k > 0$ and $C > 0$, in the Beurling case, respectively, $\Omega^{\{M_p\}}$ as a subset of $\mathbb{C}$ which contains a domain of the form

$$\Omega^{M_p}_{k,C_k} := \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant M(k|\lambda|) + C_k\},$$

for every $k > 0$ and the corresponding $C_k > 0$, in the Roumieu case.

We use the notation Ω^* for the common case and put $(\Omega^*)^2 := \{\lambda^2\colon \lambda \in \Omega^*\}$. We define $(\Omega^{M_p}_{k,C})^2$ and $(\Omega^{M_p}_{k,C_k})^2$ in a similar way.

5. As in [46] (cf. also J. Chazarain [7]), we use the ultra-logarithmic regions

$$\Lambda_{\alpha,\beta,\gamma} := \left\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant \frac{M(\alpha\lambda)}{\gamma} + \beta\right\}, \quad \alpha,\beta,\gamma > 0$$

and define $\Lambda^2_{\alpha,\beta,\gamma} := \{\lambda^2\colon \lambda \in \Lambda_{\alpha,\beta,\gamma}\}$. Note that (M2) implies that, for every $\alpha,\beta,\gamma > 0$, there exist $\alpha' > 0$ and $\beta' > 0$ so that $\Lambda_{\alpha',\beta',1} \subset \Lambda_{\alpha,\beta,\gamma}$.

6. Let $\alpha,\beta > 0$. The exponential region $E(\alpha,\beta)$ is defined in [1] by

$$E(\alpha,\beta) := \{\lambda\colon \operatorname{Re}\lambda \geqslant \beta,\ |\operatorname{Im}\lambda| \leqslant e^{\alpha \operatorname{Re}\lambda}\}; \qquad E^2(\alpha,\beta) := \{\lambda^2\colon \lambda \in E(\alpha,\beta)\}.$$

7. Let $0 < \alpha \leqslant \pi$. Then $\Sigma_\alpha := \{re^{i\theta}\colon r > 0,\ |\theta| < \alpha\}$.
8. We use occasionally the following condition for K:

(P1) $K \in L^1_{\mathrm{loc}}([0,\infty))$ is Laplace transformable, i.e., there exists $\beta \in \mathbb{R}$ so that $\widetilde{K}(\lambda) = \mathcal{L}(K)(\lambda) := \int_0^\infty e^{-\lambda t}K(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta$.

Put $\operatorname{abs}(K) := \inf\{\operatorname{Re}\lambda\colon \widetilde{K}(\lambda) \text{ exists}\}$. In some statements, the following additional condition is required for K satisfying (P1):

(P2) $\widetilde{K}(\lambda) \neq 0$, $\operatorname{Re}\lambda > \beta$, where $\beta \geqslant \operatorname{abs}(K)$.

In general, (P2) does not hold for exponentially bounded functions, cf. [3, Theorem 1.11.1].

9. A function $K \in L^1_{\mathrm{loc}}([0,\tau))$, $\tau > 0$, is called a kernel if for every $\phi \in C([0,\tau))$, the assumption

$$\int_0^t K(t-s)\phi(s)\,ds = 0, \quad t \in [0,\tau),$$

implies $\phi \equiv 0$. According to Titchmarsh's theorem, K is a kernel if $0 \in \operatorname{supp} K$.

For the later use we recall a family of kernels, see [3, p. 107]:

$$K_\delta(t) := \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{\lambda t-\lambda^\delta}\,d\lambda, \quad t \geqslant 0,\ 0<\delta<1,\ r>0,\ \text{where } 1^\delta = 1.$$

Note, $K_{1/2}(t) = \frac{1}{2\sqrt{\pi t^3}}e^{-\frac{1}{4t}}$, $t > 0$ ($K_{1/2}(0) = 0$).

2. K-convoluted C-cosine functions

We assume in the sequel that K is not identical to zero and put, in the sequel, $\Theta(t) = \int_0^t K(s)\,ds$, $t \geqslant 0$. For the later use, first we recall the definitions of (local) K-convoluted C-semigroups and exponentially bounded, K-convoluted C-semigroups.

Definition 1. (See [36].) Let A be a closed operator and K be a locally integrable function on $[0,\tau)$, $0 < \tau \leqslant \infty$. If there exists a strongly continuous operator family $(S_K(t))_{t\in[0,\tau)}$ such that $S_K(t)C = CS_K(t)$, $S_K(t)A \subset AS_K(t)$, $\int_0^t S_K(s)x\,ds \in D(A)$, $t \in [0,\tau)$, $x \in E$ and

$$A\int_0^t S_K(s)x\,ds = S_K(t)x - \Theta(t)Cx, \quad x \in E, \tag{1}$$

then $(S_K(t))_{t\in[0,\tau)}$ is called a (local) K-convoluted C-semigroup having A as a subgenerator. If $\tau = \infty$, then it we say that $(S_K(t))_{t\geqslant 0}$ is an exponentially bounded, K-convoluted C-semigroup with a subgenerator A if, in addition, there are constants $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$.

The integral generator of $(S_K(t))_{t\in[0,\tau)}$ is defined by

$$\left\{(x,y) \in E^2\colon\ S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)y\,ds,\ t \in [0,\tau)\right\}.$$

It is straightforward to see that the integral generator of $(S_K(t))_{t\in[0,\tau)}$ is an extension of any subgenerator of $(S_K(t))_{t\in[0,\tau)}$.

Definition 2. Let A be a closed operator and $K \in L^1_{\mathrm{loc}}([0,\tau))$, $0 < \tau \leqslant \infty$. If there exists a strongly continuous operator family $(C_K(t))_{t\in[0,\tau)}$ such that:

(i) $C_K(t)A \subset AC_K(t)$, $t \in [0,\tau)$,
(ii) $C_K(t)C = CC_K(t)$, $t \in [0,\tau)$, and

(iii) $\int_0^t (t-s)C_K(s)x\,ds \in D(A)$, $x \in E$, $t \in [0,\tau)$, and

$$A\int_0^t (t-s)C_K(s)x\,ds = C_K(t)x - \Theta(t)Cx, \tag{2}$$

then it is said that A is a subgenerator of a K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$. If $\tau=\infty$, then we say that $(C_K(t))_{t\geqslant 0}$ is an exponentially bounded, K-convoluted C-cosine function with a subgenerator A if, additionally, there exist $M>0$ and $\omega\in\mathbb{R}$ such that $\|C_K(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$.

As a consequence of (i) and (iii), we have $CA\subset AC$. Indeed, if $x\in D(A)$, choose a $t\in[0,\tau)$ with $\Theta(t)\neq 0$. Then (i) and (iii) implies $C_K(t)Ax-\Theta(t)CAx = A\int_0^t(t-s)C_K(s)Ax\,ds = A^2\int_0^t(t-s)C_K(s)x\,ds = A[C_K(t)x-\Theta(t)Cx]$. Since $C_K(t)x\in D(A)$, we obtain $Cx\in D(A)$ and $CAx=ACx$.

Put in Definition 2, $K(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t\in[0,\tau)$, $\alpha>0$. Then $(C_K(t))_{t\in[0,\tau)}$ is an α-times integrated C-cosine function. We point out that C. Lizama used in [44] a slight modification of (2) and (1) in the case of α-times integrated cosine functions and semigroups.

The integral generator of $(C_K(t))_{t\in[0,\tau)}$ is defined by $\{(x,y)\in E^2\colon\ C_K(t)x-\Theta(t)Cx=\int_0^t(t-s)C_K(s)y\,ds,\ t\in[0,\tau)\}$. The integral generator of $(C_K(t))_{t\in[0,\tau)}$ is a closed linear operator which is an extension of any subgenerator of $(C_K(t))_{t\in[0,\tau)}$. Even if $(C(t))_{t\geqslant 0}$ is a global, exponentially bounded C-cosine function, the set of all subgenerators of $(C(t))_{t\geqslant 0}$ need not be a singleton. This can be viewed by transferring [53, Example 2.13] to cosine functions. Moreover, the set of all subgenerators of a K-convoluted C-cosine function can have infinitely many elements. In order to illustrate this fact, choose an arbitrary $K\in L^1_{\mathrm{loc}}([0,\infty))$. Put $E:=l_\infty$, $C\langle x_n\rangle := \langle 0,x_1,0,x_2,0,x_3,\ldots\rangle$ and $C_K(t)\langle x_n\rangle := \Theta(t)C\langle x_n\rangle$, $t\geqslant 0$, $\langle x_n\rangle\in E$. If $I\subset 2\mathbb{N}+1$, define $E_I:=\{\langle x_n\rangle\in E\colon\ x_i=0,\ \text{for all } i\in(2\mathbb{N}+1)\setminus I\}$. Then E_I is a closed subspace of E which contains $R(C)$. Clearly, $E_{I_1}\neq E_{I_2}$, if $I_1\neq I_2$. Define a closed linear operator A_I on E by: $D(A_I)=E_I$ and $A_I\langle x_n\rangle=0$, $\langle x_n\rangle\in D(A_I)$. It is straightforward to see that every subgenerator of $(C_K(t))_{t\geqslant 0}$ is of the form A_I, for some $I\subset 2\mathbb{N}+1$. Hence, in this example, there exist a continuum of subgenerators of $(C_K(t))_{t\geqslant 0}$. See also [53, Example 2.14] for a more complicated construction in the case of global C-semigroups.

If $C=I$, then the proof of [36, Proposition 2.2], with slight modifications, shows that every subgenerator of a (local) K-convoluted cosine function $(C_K(t))_{t\in[0,\tau)}$ coincides with the integral generator of $(C_K(t))_{t\in[0,\tau)}$.

Remark 1. The authors do not know whether the set of all subgenerators $(C_K(t))_{t\in[0,\tau)}$ must be a singleton if $C\neq I$ and $\overline{R(C)}=E$.

We need the following useful extension of [32, Proposition 1.3].

Proposition 3. *Let A be a closed operator and let $K\in L^1_{\mathrm{loc}}([0,\tau))$, $0<\tau\leqslant\infty$. Then the following assertions are equivalent*:

(a) *A is a subgenerator of a K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$ in E.*
(b) *The operator $\mathcal{A}\equiv\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}$ is a subgenerator of a Θ-convoluted $\mathcal{C}$-semigroup $(S_\Theta(t))_{t\in[0,\tau)}$ in E^2, where $\mathcal{C}:=\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}$.*

In this case:

$$S_\Theta(t)=\begin{pmatrix}\int_0^t C_K(s)\,ds & \int_0^t(t-s)C_K(s)\,ds\\ C_K(t)-\Theta(t)C & \int_0^t C_K(s)\,ds\end{pmatrix},\quad 0\leqslant t<\tau,$$

and the integral generators of $(C_K(t))_{t\in[0,\tau)}$ and $(S_\Theta(t))_{t\in[0,\tau)}$, denoted respectively by B and $\mathcal{B}$, satisfy $\mathcal{B}=\begin{pmatrix}0 & I\\ B & 0\end{pmatrix}$. Furthermore, if K is a kernel, then the integral generator of $(C_K(t))_{t\in[0,\tau)}$, respectively, $(S_\Theta(t))_{t\in[0,\tau)}$ is $C^{-1}AC$, respectively, $\mathcal{C}^{-1}\mathcal{A}\mathcal{C}\equiv\begin{pmatrix}0 & I\\ C^{-1}AC & 0\end{pmatrix}$.

Proof. (a) $\Rightarrow$ (b) The properties of $(C_K(t))_{t\in[0,\tau)}$ and $CA\subset AC$ imply that $(S_\Theta(t))_{t\in[0,\tau)}$ is a strongly continuous operator family in E^2 which satisfies $S_\Theta(t)\mathcal{A}\subset\mathcal{A}S_\Theta(t)$ and $S_\Theta(t)\mathcal{C}=\mathcal{C}S_\Theta(t)$ for $0\leqslant t<\tau$. Furthermore,

$$\mathcal{A}\int_0^t S_\Theta(s)\binom{x}{y}\,ds = \mathcal{A}\int_0^t \begin{pmatrix} \int_0^s C_K(r)x\,dr + \int_0^s (s-r)C_K(r)y\,dr \\ C_K(s)x - \Theta(s)Cx + \int_0^s C_K(r)y\,dr \end{pmatrix} ds$$
$$= \mathcal{A}\begin{pmatrix} \int_0^t (t-s)C_K(s)x\,ds + \int_0^t \frac{(t-s)^2}{2}C_K(s)y\,ds \\ \int_0^t C_K(s)x\,ds - \int_0^t \Theta(s)Cx\,ds + \int_0^t (t-s)C_K(s)y\,ds \end{pmatrix}$$
$$= \begin{pmatrix} \int_0^t C_K(s)x\,ds - \int_0^t \Theta(s)Cx\,ds + \int_0^t (t-s)C_K(s)y\,ds \\ C_K(t)x - \Theta(t)Cx + \int_0^t C_K(s)y\,ds - \int_0^t \Theta(s)Cy\,ds \end{pmatrix}$$
$$= S_\Theta(t)\binom{x}{y} - \int_0^t \Theta(s)\binom{Cx}{Cy}\,ds, \quad 0 \leqslant t < \tau.$$

(b) $\Rightarrow$ (a) Put $S_\Theta(t) = \left(\begin{smallmatrix} S_\Theta^1(t) & S_\Theta^2(t) \\ S_\Theta^3(t) & S_\Theta^4(t) \end{smallmatrix}\right)$, $t \in [0,\tau)$, where $S_\Theta^i(t) \in L(E)$, $i \in \{1,2,3,4\}$, $0 \leqslant t < \tau$. A simple consequence of $S_\theta(t)\mathcal{C} = \mathcal{C}S_\Theta(t)$, $t \in [0,\tau)$ is: $S_\Theta^i(t)C = CS_\Theta^i(t)$, $t \in [0,\tau)$, $i \in \{1,2,3,4\}$. Since $S_\Theta\mathcal{A} \subset \mathcal{A}S_\Theta$:

$$S_\Theta^1(t)x + S_\Theta^2(t)y \in D(A),$$
$$S_\Theta^1(t)y + S_\Theta^2(t)Ax = S_\Theta^3(t)x + S_\Theta^4(t)y,$$
$$S_\Theta^3(t)y + S_\Theta^4(t)Ax = A\big(S_\Theta^1(t)x + S_\Theta^2(t)y\big), \quad 0 \leqslant t < \tau,\ x \in D(A),\ y \in E.$$

Hence, $S_\Theta^3(t)x = S_\Theta^2(t)Ax$, $x \in D(A)$, and $S_\Theta^3(t)y = AS_\Theta^2(t)y$, $y \in E$, $0 \leqslant t < \tau$. This implies that for every $x \in D(A)$, we have $S_\Theta^3(t)Ax = AS_\Theta^2(t)Ax = AS_\Theta^3(t)x$, $t \in [0,\tau)$. Thus, $S_\Theta^3(t)A \subset AS_\Theta^3(t)$, $t \in [0,\tau)$, and $(S_\Theta^3(t) + \Theta(t)C)_{t\in[0,\tau)}$ is a strongly continuous operator family in E. Now, the simple calculation deduced from $\mathcal{A}\int_0^t S_\Theta(s)\binom{x}{y}\,ds = S_\Theta(t)\binom{x}{y} - \int_0^t \Theta(s)\binom{Cx}{Cy}\,ds$ gives

$$\int_0^t S_\Theta^3(s)x\,ds + \int_0^t S_\Theta^4(s)y\,ds = S_\Theta^1(t)x + S_\Theta^2(t)y - \int_0^t \Theta(s)Cx\,ds, \quad \text{and}$$
$$A\left[\int_0^t S_\Theta^1(s)x\,ds + \int_0^t S_\Theta^2(s)y\,ds\right] = S_\Theta^3(t)x + S_\Theta^4(t)y - \int_0^t \Theta(s)Cy\,ds,$$

for all $0 \leqslant t < \tau$, $x, y \in E$. Hence, $\int_0^t S_\Theta^3(s)x\,ds = S_\Theta^1(t)x - \int_0^t \Theta(s)Cx\,ds$ and $A\int_0^t S_\Theta^1(s)x\,ds = S_\Theta^3(t)x$, $0 \leqslant t < \tau$, $x \in E$. Consequently,

$$A\left[\int_0^t (t-s)\big(S_\Theta^3(s)x + \Theta(s)Cx\big)\,ds\right] = A\left[\int_0^t (t-s)\left(\frac{d}{dv}S_\Theta^1(v)x\right)_{v=s} ds\right]$$
$$= A\int_0^t S_\Theta^1(s)x\,ds$$
$$= \big[S_\Theta^3(t)x + \Theta(t)Cx\big] - \Theta(t)Cx, \quad 0 \leqslant t < \tau,\ x \in E.$$

Thus, we have proved that A is a subgenerator of the K-convoluted C-cosine function $(S_\Theta^3(t) + \Theta(t)C)_{t\in[0,\tau)}$. Clearly, $S_\Theta^1(t) = S_\Theta^4(t)$ and $S_\Theta^2(t) = \int_0^t S_\Theta^1(s)\,ds$, $0 \leqslant t < \tau$. Next, we will prove that $\mathcal{B} = \left(\begin{smallmatrix} 0 & I \\ B & 0 \end{smallmatrix}\right)$. To see this, fix some $x, y, x_1, y_1 \in E$. Then

$$S_\Theta(t)\binom{x}{y} - \int_0^t \Theta(s)\binom{Cx}{Cy}\,ds = \int_0^t S_\Theta(s)\binom{x_1}{y_1}\,ds, \quad t \in [0,\tau),$$

iff

$$C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)y_1\,ds, \quad \text{for all } t \in [0,\tau), \text{ and } y = x_1.$$

Namely, if $S_\Theta(t)\binom{x}{y} - \int_0^t \Theta(s)\binom{Cx}{Cy}\,ds = \int_0^t S_\Theta(s)\binom{x_1}{y_1}\,ds$, $t \in [0,\tau)$, then

$$\int_0^t C_K(s)x\,ds + \int_0^t (t-s)C_K(s)y\,ds - \int_0^t \Theta(s)Cx\,ds = \int_0^t (t-s)C_K(s)x_1\,ds + \int_0^t \frac{(t-s)^2}{2}C_K(s)y_1\,ds, \tag{3}$$

and

$$C_K(t)x - \Theta(t)Cx + \int_0^t C_K(s)y\,ds - \int_0^t \Theta(s)Cy\,ds = \int_0^t C_K(s)x_1\,ds - \int_0^t \Theta(s)Cx_1\,ds + \int_0^t (t-s)C_K(s)y_1\,ds. \tag{4}$$

Differentiating (3) with respect to t, one obtains

$$C_K(t)x + \int_0^t C_K(s)y\,ds - \Theta(t)Cx = \int_0^t C_K(s)x_1\,ds + \int_0^t (t-s)C_K(s)y_1\,ds.$$

The last equality and (4) imply $\int_0^t \Theta(s)Cy\,ds = \int_0^t \Theta(s)Cx_1\,ds$; consequently, $y = x_1$. Then (4) gives $C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)y_1\,ds$, $t \in [0,\tau)$ and $(x, y_1) \in B$. Conversely, suppose that $y = x_1$ and that $(x, y_1) \in B$. Then

$$C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)y_1\,ds, \quad t \in [0,\tau).$$

This implies (4). Integrating (4) with respect to t one obtains (3) and this gives $\binom{x}{y}, \binom{x_1}{y_1} \in \mathcal{B}$. Hence, we have proved $\binom{x}{y}, \binom{x_1}{y_1} \in \mathcal{B}$ iff $y = x_1$ and $(x, y_1) \in B$. Furthermore, our assumption $CA \subset AC$ implies $\mathcal{C}\mathcal{A} \subset \mathcal{A}\mathcal{C}$ and one can employ an adequate assertion for K-convoluted C-semigroups (cf. [35,41]) in order to see that the integral generator of $(S_\Theta(t))_{t\in[0,\tau)}$ is $\mathcal{C}^{-1}\mathcal{A}\mathcal{C}$. As a matter of routine, we obtain $\mathcal{C}^{-1}\mathcal{A}\mathcal{C} = \begin{pmatrix} 0 & I \\ C^{-1}AC & 0 \end{pmatrix}$. By the previously given arguments, we know that this implies that the integral generator of $(C_K(t))_{t\in[0,\tau)}$ is $C^{-1}AC$. □

Remark 4. When $\tau = \infty$ and Θ is an exponentially bounded function, then $(C_K(t))_{t\geqslant 0}$ is exponentially bounded if and only if $(S_\Theta(t))_{t\geqslant 0}$ is exponentially bounded.

Proposition 3 implies the following facts which remain true in the case of convoluted C-semigroups.

Suppose in this paragraph that $0 < \tau \leqslant \infty$ and that $K \in L^1_{\text{loc}}([0,\tau))$ is a kernel. If A and B are subgenerators of $(C_K(t))_{t\in[0,\tau)}$, then: $C^{-1}AC = C^{-1}BC$, $C(D(A)) \subset D(B)$ and $A = B \Leftrightarrow D(A) = D(B)$. Moreover, if A is the integral generator of $(C_K(t))_{t\in[0,\tau)}$, then it can be easily seen that the set of all subgenerators of $(C_K(t))_{t\in[0,\tau)}$ is a singleton if $C(D(A))$ is a core for $D(A)$, cf. also [53, Proposition 2.8] and [18]. It can be proved that all subgenerators of $(C_K(t))_{t\in[0,\tau)}$ form a lattice. Further analysis of such a lattice can be found [53].

If K is a kernel, then by Proposition 3 and the corresponding statement in the case of semigroups that every (local) K-convoluted C-cosine function is uniquely determined by one of its subgenerators. The standard proof is omitted.

Let $(C_K(t))_{t\in[0,\tau)}$ be a (local) K-convoluted C-cosine function whose integral generator is A. Proposition 3 and arguments of [35] and [41, Proposition 1.3] yield $A = C^{-1}AC$.

Let $\lambda \in \mathbb{C}$ and let E^2 be endowed by the norm $\|(x, y)\| = \|x\| + \|y\|$, $x, y \in E$. Then $\lambda^2 \in \rho(A)$ iff $\lambda \in \rho(\mathcal{A})$, and, in this case, we have:

$$\|R(\lambda^2 : A)\| \leqslant \|R(\lambda : \mathcal{A})\|,$$
$$\|R(\lambda : \mathcal{A})\| \leqslant (1 + |\lambda|)\sqrt{1 + |\lambda|^2}\|R(\lambda^2 : A)\| + 1,$$
$$R(\lambda : \mathcal{A})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R(\lambda^2 : A)(\lambda x + y) \\ AR(\lambda^2 : A)x + \lambda R(\lambda^2 : A)y \end{pmatrix}, \quad x, y \in E,$$

see [32, Lemma 1.10] for the proof. Further on, $D(\mathcal{A}^n) = D(A^{\lceil \frac{n}{2} \rceil}) \times D(A^{\lfloor \frac{n}{2} \rfloor})$, $n \in \mathbb{N}_0$ and $D_\infty(\mathcal{A}) = D_\infty(A) \times D_\infty(A)$. Here, $\lfloor t \rfloor := \sup\{k \in \mathbb{Z}\colon k \leqslant t\}$ and $\lceil t \rceil := \inf\{k \in \mathbb{Z}\colon k \geqslant t\}$, $t \in \mathbb{R}$.

Let us recall [38] that a closed linear operator A is stationary dense if $n(A) := \inf\{k \in \mathbb{N}_0\colon (\forall n \geqslant k)\ D(A^n) \subset \overline{D(A^{n+1})}\} < \infty$. We will prove that every generator of an integrated cosine function is stationary dense. In Example 3 we will show that this is not necessarily true if A generates a convoluted cosine function.

Lemma 5. *Let A be a closed operator. Then A is stationary dense if and only if $\mathcal{A}$ is stationary dense. Moreover, $n(\mathcal{A}) = 2n(A)$.*

Proof. Assume that A is stationary dense and that $n(A) = n \in \mathbb{N}_0$. Let us prove that $D(\mathcal{A}^m) \subset \overline{D(\mathcal{A}^{m+1})}$, for all $m \in \mathbb{N}_0$ with $m \geqslant 2n$. Suppose $m = 2i$, for some $i \geqslant n$. We have to prove that $D(A^i) \times D(A^i) \subset \overline{D(A^{i+1}) \times D(A^i)}$. But, this is a consequence of $D(A^i) \subset \overline{D(A^{i+1})}$. If $m = 2i + 1$ for some $i \geqslant n$, then $D(\mathcal{A}^m) \subset \overline{D(\mathcal{A}^{m+1})}$ is equivalent with $D(A^{i+1}) \times D(A^i) \subset \overline{D(A^{i+1}) \times D(A^{i+1})}$, which is valid since $i \geqslant n$. Thus, $\mathcal{A}$ is stationary dense and $n(\mathcal{A}) \leqslant 2n(A)$. Furthermore, $n(\mathcal{A}) = 0$, if $n(A) = 0$. Suppose $n(\mathcal{A}) < 2n(A)$. If $n(\mathcal{A}) = 2i$, for some $i \in \{0, 1, \ldots, n-1\}$, then $D(A^i) \times D(A^i) \subset \overline{D(A^{i+1}) \times D(A^i)}$. It gives $D(A^i) \subset \overline{D(A^{i+1})}$ and the contradiction is obvious. Similarly, if $n(\mathcal{A}) = 2i + 1$, for some $i \in \{0, 1, \ldots, n-1\}$, then $D(A^{i+1}) \times D(A^i) \subset \overline{D(A^{i+1}) \times D(A^{i+1})}$. Again, $D(A^i) \subset \overline{D(A^{i+1})}$ and this is in contradiction with $n(A) = n$. Hence, we have proved that $\mathcal{A}$ is stationary dense and $n(\mathcal{A}) = 2n(A)$. Assume conversely that $\mathcal{A}$ is stationary dense. Similarly as in the first part of the proof, one obtains that A is stationary dense. Then we know that $n(\mathcal{A}) = 2n(A)$. □

Proposition 6. *Let A be the generator of an α-times integrated cosine function $(C_\alpha(t))_{t \in [0,\tau)}$, $0 < \tau \leqslant \infty$, $\alpha > 0$. Then $n(A) \leqslant \lfloor \frac{\lceil \alpha \rceil + 1}{2} \rfloor$.*

Proof. Due to Proposition 3 and [42, Proposition 2.4(a)], the operator $\mathcal{A}$ is the generator of an $(\lceil \alpha \rceil + 1)$-times integrated semigroup $(S_{\lceil \alpha \rceil + 1}(t))_{t \in [0,\tau)}$. Thus, an application of [38, Corollary 1.8] gives $n(\mathcal{A}) \leqslant \lceil \alpha \rceil + 1$. Now the proposition follows from Lemma 5. □

Comment and problem. As it is illustrated in [3, Example 3.15.5, p. 224], the generator B of the standard translation group on $L^1(\mathbb{R})$ fulfills the next statement: $A := (B^*)^2$ (the second derivative) is the non-densely defined generator of a sine function in $L^\infty(\mathbb{R})$. Proposition 6 implies $n(A) = 1$. Hence, in the general situation of the previous proposition, the estimate $n(A) \leqslant \lfloor \frac{\lceil \alpha \rceil + \beta}{2} \rfloor$ cannot be proved for any $\beta \in [0, 1)$ since here $n(A) = 1$ and $\alpha = 1$. The next problem can be posed: Given an arbitrary $\alpha > 0$, is it possible to construct a Banach space E_α, a closed linear operator A_α on E_α which generates a (local) α-times integrated cosine function and satisfies $n(A_\alpha) = \lfloor \frac{\lceil \alpha \rceil + 1}{2} \rfloor$?

3. Global exponentially bounded K-convoluted C-cosine functions

Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C}\colon R(C) \subset R(\lambda - A)$ and $\lambda - A$ is injective$\}$. The proof of the next theorem is the standard one and because of that it is omitted.

Theorem 7. *Assume that K satisfies* (P1) *and that A is a closed linear operator.*

(a) *Assume that A is a subgenerator of an exponentially bounded, K-convoluted C-cosine function $(C_K(t))_{t\geqslant 0}$ and $\|C_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, for some $M > 0$ and $\omega \geqslant 0$. If $\omega_1 = \max(\omega, \beta)$, then*

$$\{\lambda^2\colon \operatorname{Re}\lambda > \omega_1,\ \widetilde{K}(\lambda) \neq 0\} \subset \rho_C(A), \quad \textit{and} \tag{5}$$

$$\lambda(\lambda^2 - A)^{-1}Cx = \frac{1}{\widetilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} C_K(t)x\,dt, \quad x \in E,\ \operatorname{Re}\lambda > \omega_1,\ \widetilde{K}(\lambda) \neq 0. \tag{6}$$

(b) *Suppose that $(C_K(t))_{t\geqslant 0}$ is a strongly continuous operator family satisfying $\|C_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, $\omega \geqslant 0$. Put $\omega_1 = \max(\omega, \beta)$. If* (5) *and* (6) *are fulfilled, then $(C_K(t))_{t\geqslant 0}$ is an exponentially bounded, K-convoluted C-cosine function with a subgenerator A.*

Note that there exist examples of local integrated C-cosine functions and semigroups whose integral generators have the empty C-resolvent sets (cf. [42]).

In the next statement, we relate exponentially bounded, convoluted C-semi-groups to exponentially bounded, convoluted C-cosine functions.

Proposition 8. *Let K satisfy* (P1). *Suppose that A and $-A$ are subgenerators of exponentially bounded, K-convoluted C-semigroups. Then A^2 is a subgenerator of an exponentially bounded, K-convoluted C-cosine function.*

Proof. Suppose that A and $-A$ are subgenerators of exponentially bounded, K-convoluted C-semigroups $(S_K(t))_{t\geqslant 0}$ and $(V_K(t))_{t\geqslant 0}$, respectively. Define

$$C_K(t) := \frac{1}{2}\big(S_K(t) + V_K(t)\big), \quad t \geqslant 0.$$

We will prove that A^2 is a subgenerator of a K-convoluted C-cosine function $(C_K(t))_{t\geqslant 0}$. Clearly, $(C_K(t))_{t\geqslant 0}$ is an exponentially bounded operator family. Arguing similarly as in the proof of Theorem 7, we obtain that there is an $\omega_1 > 0$ such that

$$\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > \omega_1,\ \widetilde{K}(\lambda) \neq 0\} \subset \rho_C(A) \cap \rho_C(-A) \tag{7}$$

and that

$$(\lambda - A)^{-1}Cx = \frac{1}{\widetilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} S_K(t)x\,dt, \quad x \in E,\ \operatorname{Re}\lambda > \omega_1,\ \widetilde{K}(\lambda) \neq 0.$$

The previous equality poses the natural analog for $-A$ and $(V_K(t))_{t\geqslant 0}$. Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_1$ and $\widetilde{K}(\lambda) \neq 0$. Due to (7), we have that $\lambda^2 - A^2$ is injective. Moreover, it is straightforward to see that $R(C) \subset R(\lambda^2 - A^2)$ and that

$$\begin{aligned}(\lambda^2 - A^2)^{-1}Cx &= \frac{1}{2\lambda}\big[(\lambda - A)^{-1}Cx + (\lambda + A)^{-1}Cx\big] = \frac{1}{\lambda\widetilde{K}(\lambda)} \int_0^\infty e^{-\lambda t}\left[\frac{1}{2}\big(S_K(t)x + V_K(t)x\big)\right]dt \\ &= \frac{1}{\lambda\widetilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} C_K(t)x\,dt, \quad x \in E.\end{aligned}$$

The proof ends an application of Theorem 7. $\square$

4. On the abstract Weierstrass formula

We will state the convoluted version of the abstract Weierstrass formula (Theorem 11). First we introduce the class of analytic K-convoluted C-semigroups.

Definition 9. Let $0 < \alpha \leqslant \frac{\pi}{2}$ and let A be a subgenerator of a K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$. Then we say that $(S_K(t))_{t\geqslant 0}$ is an analytic K-convoluted C-semigroup of angle α having A as a subgenerator, if there exists an analytic function $\mathbf{S}_K : \Sigma_\alpha \to L(E)$ which satisfies

(i) $\mathbf{S}_K(t) = S_K(t)$, $t > 0$, and
(ii) $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{S}_K(z)x = 0$, for all $\gamma \in (0,\alpha)$ and $x \in E$.

It is said that A is a subgenerator of an exponentially bounded, analytic K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ of angle α, if for every $\gamma \in (0,\alpha)$, there exist $M_\gamma > 0$ and $\omega_\gamma > 0$ such that $\|\mathbf{S}_K(z)\| \leqslant M_\gamma e^{\omega_\gamma \operatorname{Re} z}$, $z \in \Sigma_\gamma$.

We also write S_K for $\mathbf{S}_K$. If $C = I$, the previous definition has been recently introduced in [36]. Although one can reformulate a great part of facts known for analytic convoluted semigroups in the general case, we focus our attention on the next result which improves [36, Theorem 6.3].

Theorem 10. *Assume* $0 < \alpha \leqslant \frac{\pi}{2}$, K *satisfies* (P1) *and* $\omega \geqslant \max(0, \operatorname{abs}(K))$. *Suppose that A is a closed linear operator with* $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > \omega,\ \widetilde{K}(\lambda) \neq 0\} \subset \rho_C(A)$ *and that the function*

$$\lambda \mapsto \widetilde{K}(\lambda)(\lambda - A)^{-1}C, \quad \operatorname{Re}\lambda > \omega,\ \widetilde{K}(\lambda) \neq 0,$$

can be analytically extended to a function

$$\tilde{q} : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(E)$$

satisfying

$$\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}} \big\|(\lambda-\omega)\tilde{q}(\lambda)\big\| < \infty, \quad \gamma \in (0,\alpha) \quad \text{and} \quad \lim_{\lambda\to+\infty} \lambda\tilde{q}(\lambda) = 0.$$

Then A is a subgenerator of an exponentially bounded, analytic K-convoluted C-semigroup of angle α.

Proof. The use of [3, Theorem 2.6.1] implies that there exists an analytic function $S_K : \Sigma_\alpha \to L(E)$ so that $\sup_{z\in\Sigma_\gamma} \|e^{-\omega z} S_K(z)\| < \infty$ for all $\gamma \in (0,\alpha)$ and that

$$\tilde{q}(\lambda) = \int_0^\infty e^{-\lambda t} S_K(t)\,dt, \quad \operatorname{Re}\lambda > \omega.$$

Put $S_K(0) := 0$, fix $x \in E$ and $\gamma \in (0,\alpha)$. We will prove that

$$\lim_{z\to 0,\, z\in\Sigma_\gamma} S_K(z)x = 0.$$

Note that $f(z) := e^{-\omega z} S_K(z)x$, $z \in \Sigma_\alpha$, is analytic and $\sup_{z\in\Sigma_\gamma} \|f(z)\| < \infty$. By [3, Proposition 2.6.3], it is enough to show $\lim_{t\downarrow 0} S_K(t)x = 0$. This is a consequence of the assumption $\lim_{\lambda\to+\infty} \lambda\tilde{q}(\lambda) = 0$ and a Tauberian type theorem [3, Theorem 2.6.4]. It follows that $(S_K(t))_{t\geqslant 0}$ is a strongly continuous, exponentially bounded operator family which satisfies

$$\widetilde{K}(\lambda)(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t} S_K(t)x\,dt, \quad \lambda \in \mathbb{C},\ \operatorname{Re}\lambda > \omega,\ \widetilde{K}(\lambda) \neq 0.$$

Similarly as in the proof of Theorem 7 (cf. also [35] and [36]), we have that A is a subgenerator of an exponentially bounded, K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$. Since $(S_K(t))_{t\geqslant 0}$ verifies conditions (i) and (ii), given in Definition 9, $(S_K(t))_{t\geqslant 0}$ is an exponentially bounded analytic K-convoluted C-semigroup of angle α having A as a subgenerator. $\square$

The main result of this section reads as follows.

Theorem 11. *Assume that for some $M > 0$ and $\beta > 0$: $|K(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$. Let A be a subgenerator of an exponentially bounded K-convoluted C-cosine function $(C_K(t))_{t\geqslant 0}$. Then A is a subgenerator of an exponentially bounded analytic K_1-convoluted C-semigroup $(S(t))_{t\geqslant 0}$ of angle $\frac{\pi}{2}$, where*:

$$K_1(t) := \int_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}} K(s)\,ds \quad \text{and} \quad S(t)x := \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C_K(s)x\,ds, \quad t > 0,\ x \in E.$$

Proof. We follow the proof of the abstract Weierstrass formula (cf. [3, p. 220]). Due to [3, Proposition 1.6.8], K_1 fulfills (P1), $\operatorname{abs}(K_1) \geqslant \beta^2$ and $\widetilde{K}_1(\lambda) = \widetilde{K}(\sqrt{\lambda})$, $\operatorname{Re}\lambda > \beta^2$. Let $x \in E$ be fixed. Putting $r = s/\sqrt{t}$, and using the dominated convergence theorem after that, one obtains

$$S(t)x = \int_0^\infty \frac{e^{-r^2/4}}{\sqrt{\pi}} C_K(r\sqrt{t})x\,dr \to 0, \quad t \to 0+. \tag{8}$$

Define $S(0) := 0$. By (8), $(S(t))_{t\geqslant 0}$ is a strongly continuous, exponentially bounded operator family. Furthermore, one can employ Theorem 7 and [3, Proposition 1.6.8] to obtain that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta^2$ and $\widetilde{K}_1(\lambda) \neq 0$, the following holds

$$\begin{aligned}\int_0^\infty e^{-\lambda t} S(t)x\,dt &= \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C_K(s)x\,ds\,dt = \frac{1}{\sqrt{\lambda}} \int_0^\infty e^{-\sqrt{\lambda}s} C_K(s)x\,ds \\ &= \frac{1}{\sqrt{\lambda}} \sqrt{\lambda}\widetilde{K}(\sqrt{\lambda})(\lambda - A)^{-1}Cx = \widetilde{K}_1(\lambda)(\lambda - A)^{-1}Cx.\end{aligned}$$

As above, one concludes that $(S(t))_{t\geqslant 0}$ is an exponentially bounded K_1-convoluted C-semigroup with a subgenerator A. If $\operatorname{Re} z > 0$, we define $S(z)x$ in a natural way: $S(z)x = \frac{1}{\sqrt{\pi z}} \int_0^\infty e^{-s^2/4z} C_K(s)x\,ds$. Then, $S : \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\} \to L(E)$ is analytic. Using the same arguments as in the proof of the Weierstrass formula, see for instance [3], one obtains that for all $\beta \in (0, \frac{\pi}{2})$, there exist M, $\omega_1 > 0$ such that $\|S(z)\| \leqslant Me^{\omega_1|z|}$, $z \in \Sigma_\beta$. It remains to be shown that, for every fixed $\beta \in (0, \frac{\pi}{2})$, $\lim_{z\to 0,\, z\in\Sigma_\beta} S(z)x = 0$. For this, choose an $\omega_2 > \frac{\omega_1}{\cos\beta}$. Then the function $z \mapsto e^{-\omega_2 z}S(z)x$, $z \in \Sigma_\beta$ is analytic and satisfies $\sup_{z\in\Sigma_\beta} \|e^{-\omega_2 z}S(z)\| < \infty$. Since $\lim_{t\to 0+} e^{-\omega_2 t}S(t)x = 0$, [3, Proposition 2.6.3] implies $\lim_{z\to 0,\, z\in\Sigma_\beta} e^{-\omega_2 z}S(z)x = 0$. The proof is now complete. □

5. Relations to ultradistribution and hyperfunction sines

In this section, we assume $C = I$. The next assertion clarifies some properties of generators of local K-convoluted cosine functions in terms of the asymptotic behavior of $\widetilde{K}$.

Theorem 12. (a) *Suppose that Θ fulfills* (P2) *and that $|\Theta(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$, for some $M > 0$ and $\beta > 0$. Let A be the generator of a K-cosine function $(C_K(t))_{t\in[0,\tau)}$, for some $\tau \in (0, \infty)$. Further, suppose that for every $\varepsilon > 0$, there exist $T_\varepsilon > 0$ and $\varepsilon_0 \in (0, \tau\varepsilon)$ such that*

$$\frac{1}{|\widetilde{\Theta}(\lambda)|} \leqslant T_\varepsilon e^{\varepsilon_0|\lambda|}, \quad \lambda \in \Omega_{\varepsilon,C_\varepsilon} \cap \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > \beta\}.$$

Then, for every $\varepsilon > 0$, there exist positive real numbers $\overline{C}_\varepsilon$ and $\overline{K}_\varepsilon$ such that

$$\Omega^2_{\varepsilon,\overline{C}_\varepsilon} \subset \rho(A) \quad \text{and} \quad \|R(\lambda^2 : A)\| \leqslant \overline{K}_\varepsilon e^{\tau\varepsilon|\lambda|}, \quad \lambda \in \Omega_{\varepsilon,\overline{C}_\varepsilon}.$$

(b) *Let $|\Theta(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$, for some $M > 0$ and $\beta > 0$. Let K satisfy* (P2). *Assume that the restriction of K on $[0, \tau)$ (denoted by the same symbol) is $\neq 0$ and that A is the generator of a local K-convoluted cosine function on $[0, \tau)$. If there is an $\alpha > 0$ with $\frac{1}{|\widetilde{\Theta}(\lambda)|} = O(e^{M(\alpha\lambda)})$, $|\lambda| \to \infty$, then, for every $\tau_1 \in (0, \tau)$, there exist $\beta > 0$ and $C > 0$ such that*

$$\Lambda^2_{\alpha,\beta,\tau_1} \subset \rho(A) \quad \text{and that} \quad \|R(\lambda : A)\| \leqslant C\frac{e^{M(\alpha\sqrt{\lambda})}}{1 + \sqrt{|\lambda|}}, \quad \lambda \in \Lambda^2_{\alpha,\beta,\tau_1}.$$

Proof. (a) Since $\mathcal{A}$ generates a Θ-semigroup $(S_\Theta(t))_{t\in[0,\tau)}$ in E^2, then we have proved in [37] that there are an $\overline{C}_\varepsilon > 0$ and an $\overline{K}_\varepsilon > 0$ so that $\Omega_{\varepsilon,\overline{C}_\varepsilon} \subset \rho(\mathcal{A})$ and $\|R(\lambda : \mathcal{A})\| \leqslant \overline{K}_\varepsilon e^{\tau\varepsilon|\lambda|}$, $\lambda \in \Omega_{\varepsilon,\overline{C}_\varepsilon}$. It follows $\Omega^2_{\varepsilon,\overline{C}_\varepsilon} \subset \rho(A)$ and $\|R(\lambda^2 : A)\| \leqslant \|R(\lambda : \mathcal{A})\| \leqslant \overline{K}_\varepsilon e^{\tau\varepsilon|\lambda|}$, $\lambda \in \Omega_{\varepsilon,\overline{C}_\varepsilon}$. This finishes the proof of (a).

(b) We have that $\mathcal{A}$ generates a local Θ-semigroup on $[0, \tau)$. The prescribed assumption on Θ and the arguments of [46, Theorem 1.3.1] (see also [37]) imply that, for every $\tau_1 \in (0, \tau)$, there exist $\beta > 0$ and $C > 0$ such that $\Lambda_{\alpha,\beta,\tau_1} \subset \rho(\mathcal{A})$ and that $\|R(\lambda : \mathcal{A})\| \leqslant Ce^{M(\alpha\lambda)}$, $\lambda \in \Lambda_{\alpha,\beta,\tau_1}$. Now the proof follows by the standard arguments. □

We refer to [37] for the notion of an ultradistribution fundamental solution for a closed linear operator A. The notion of a Fourier hyperfunction fundamental solution for a closed linear operator A was introduced by Y. Ito in [23] while S. Ōuchi was the first who introduced the notion of fundamental solution in the spaces of compactly supported hyperfunctions (cf. [49]).

For the sake of simplicity, we use the next definition of ultradistribution and (Fourier) hyperfunction sines employed by H. Komatsu in [31] in the case of an ultradistribution sine. Similarly, one can introduce and prove the basic characterizations of tempered ultradistribution sines (cf. [37]).

Definition 13. A closed operator A generates an ultradistribution sine of $*$-class if there exists an ultradistribution fundamental solution for the operator $\mathcal{A}$. A closed operator A generates a (Fourier) hyperfunction sine if there exists a (Fourier) hyperfunction fundamental solution for $\mathcal{A}$.

Remark 14. We will not go into details concerning a relationship between ultradistribution (hyperfunction) sines and the solvability of convolution type equations in vector-valued ultradistribution (hyperfunction) spaces. This can be a matter of further investigations. In the case of distribution cosine functions, such an analysis is obtained in [32] by passing to the theory of distribution semigroups (see [32, Theorem 3.10]). It is not so straightforward to link ultradistribution (hyperfunction) sine generated by A, denoted by G, with ultradistribution fundamental solution for $\mathcal{A}$, denoted by $\mathcal{G}$. In the distribution case, we have $\mathcal{G} = \begin{pmatrix} G & G^{-1} \\ G'-\delta & G \end{pmatrix}$ (see [32] for more details). The main problem in transferring [32, Theorem 3.10(i)] to ultradistribution and hyperfunction sines is the presence of G^{-1} in the representation formula for $\mathcal{G}$. Furthermore, relations between (almost-)distribution cosine functions and cosine convolution products have been recently analyzed in [33] and [47]. It is not clear how to obtain the corresponding results in the case of ultradistribution and hyperfunction sines.

Spectral properties of operators generating ultradistribution and (Fourier) hyperfunction sines are given in the following remark.

Remark 15. 1. (See [31,40].) A closed linear operator A generates an ultradistribution sine of $*$-class iff there exists a domain of the form Ω^* such that:

$$\Omega^{*,2} \subset \rho(A), \quad \text{and} \tag{9}$$

$$\left\|R\left(\lambda^2 : A\right)\right\| \leqslant Ce^{M(k|\lambda|)}, \quad \lambda \in \Omega^{M_p}_{k,C}, \tag{10}$$

for some $k > 0$ and $C > 0$ in (M_p)-case, respectively,

$$\left\|R\left(\lambda^2 : A\right)\right\| \leqslant C_k e^{M(k|\lambda|)}, \quad \lambda \in \Omega^{M_p}_{k,C_k}, \tag{11}$$

for every $k > 0$ and the corresponding $C_k > 0$ in $\{M_p\}$-case.

2. (See [23].) A closed linear operator A generates a Fourier hyperfunction sine iff

$$\text{for every } \varepsilon > 0 \text{ and } \sigma > 0, \text{ there is } C_{\varepsilon,\sigma} > 0, \quad \text{with}$$
$$\left\|R\left(\lambda^2 : A\right)\right\| \leqslant C_{\varepsilon,\sigma} e^{\varepsilon|\lambda|}, \quad \operatorname{Re}\lambda > \sigma. \tag{12}$$

3. (See [49].) A closed linear operator A generates a hyperfunction sine iff for every $\varepsilon > 0$, there exist constants $C_\varepsilon > 0$ and $K_\varepsilon > 0$ satisfying

$$\Omega^2_{\varepsilon,C_\varepsilon} \subset \rho(A) \quad \text{and} \quad \left\|R\left(\lambda^2 : A\right)\right\| \leqslant K_\varepsilon e^{\varepsilon|\lambda|}, \quad \lambda \in \Omega_{\varepsilon,C_\varepsilon}. \tag{13}$$

We refer to [31] for the spectral properties of operators generating Laplace hyperfunction semigroups and sines.

Theorem 16. (a) *Let A generate an ultradistribution sine of (M_p)-class, respectively, $\{M_p\}$-class. Then, for every $\theta \in [0, \frac{\pi}{2})$, there exists an ultradistribution fundamental solution of (M_p^2)-class, respectively, $\{M_p^2\}$-class for $e^{\pm i\theta}A$.*

(b) *Let A generate a hyperfunction sine. Then, for every $\theta \in [0, \frac{\pi}{2})$, there exists an ultradistribution fundamental solution of $\{p!^2\}$-class for $e^{\pm i\theta}A$.*

Proof. We will prove only (a) since the same arguments work for (b). Fix a $\theta \in [0, \frac{\pi}{2})$. If $\overline{M}$ denotes the associated function of (M_p^2), then, for every $k > 0$, $M(k\sqrt{t}) = \sup\{\ln \frac{M_0 k^p \sqrt{t^p}}{M_p}\colon\ p \in \mathbb{N}_0\} = \frac{1}{2}\sup\{\ln \frac{M_0^2 k^{2p} t^p}{M_p^2}\colon\ p \in \mathbb{N}_0\} = \frac{1}{2}\overline{M}(k^2 t)$, $t \geqslant 0$. We have already noted that A generates an ultradistribution sine of (M_p)-class, respectively, $\{M_p\}$-class if and only if there exists a domain of the form Ω^* such that (9) and (10), respectively, (9) and (11) are fulfilled. Since

$$\lim_{|\lambda|\to\infty,\, \lambda \in \partial(\Omega_{k,C}^{M_p})} \cos\bigl(\arg(\lambda)\bigr) = \lim_{|\lambda|\to\infty} \frac{M(k|\lambda|) + C}{|\lambda|} = 0,$$

we obtain $|\arg\lambda| \to \frac{\pi}{2}$, $|\lambda| \to \infty$, $\lambda \in \partial(\Omega_{k,C}^{M_p})$, and therefore, $|\arg\lambda| \to \pi$, $|\lambda| \to \infty$, $\lambda \in \partial((\Omega_{k,C}^{M_p})^2)$. The same estimate holds in the Roumieu case. Hence, there exists a suitable $\omega > 0$ with $\rho(A) \supset \Omega^{*,2} \supset \omega + \Sigma_{\pi/2+\theta}$. Further, $\|R(\lambda : A)\| \leqslant Ce^{M(k\sqrt{|\lambda|})}$, $\lambda \in (\Omega_{k,C}^{M_p})^2$, for some $k > 0$ and $C > 0$ in (M_p^2)-case, respectively, $\|R(\lambda : A)\| \leqslant C_k e^{M(k\sqrt{|\lambda|})}$, $\lambda \in (\Omega_{k,C_k}^{M_p})^2$, for every $k > 0$ and the corresponding $C_k > 0$, in $\{M_p^2\}$-case. The analysis given in the first part of the proof shows that, for every $\theta \in [0, \frac{\pi}{2})$, we have $\{z \in \mathbb{C}\colon\ \operatorname{Re} z > \omega\} \subset \rho(e^{\pm i\theta}A)$, and that $\|R(\lambda : e^{\pm i\theta}A)\| = \|R(\lambda e^{\mp i\theta} : A)\| \leqslant Ce^{M(k\sqrt{|\lambda|})} = Ce^{\frac{1}{2}\overline{M}(k^2|\lambda|)}$, $\operatorname{Re}\lambda > \omega$, in (M_p^2)-case. The similar estimate holds in the Roumieu case. To end the proof we apply the arguments given in Theorem 2.2 of [37]. □

Next, we relate ultradistribution and hyperfunction sines to analytic convoluted semigroups. Recall, the function $K_{1/2}(t) = \frac{1}{2\sqrt{\pi t^3}} e^{-\frac{1}{4t}}$, $t > 0$ is bounded and smooth. Furthermore, $\widetilde{K}(\lambda) = e^{-\sqrt{\lambda}}$, $\operatorname{Re}\lambda > 0$, where $\sqrt{1} = 1$.

Theorem 17. *Suppose that A generates a hyperfunction sine. Then A generates an exponentially bounded, analytic $K_{1/2}$-semigroup of angle $\frac{\pi}{2}$.*

Proof. Clearly, it is enough to show that, for every $\varepsilon \in (0, \frac{1}{\sqrt{2}})$, A generates an exponentially bounded analytic $K_{1/2}$-semigroup of angle $\alpha := 2\arccos\varepsilon - \frac{\pi}{2}$. So, let $\varepsilon \in (0, \frac{1}{\sqrt{2}})$ be fixed. Then there exist $C_\varepsilon > 0$ and $K_\varepsilon > 0$ such that $\Omega^2_{\varepsilon,C_\varepsilon} \subset \rho(A)$ and that $\|R(\lambda : A)\| \leqslant K_\varepsilon e^{\varepsilon\sqrt{|\lambda|}}$, $\lambda \in \Omega^2_{\varepsilon,C_\varepsilon}$. Since

$$\partial(\Omega_{\varepsilon,C_\varepsilon}) = \left\{re^{i\theta}\colon\ r > 0,\ \theta \in \left(0, \frac{\pi}{2}\right),\ r\cos\theta = \varepsilon r + C_\varepsilon\right\},$$

one can conclude that $|\arg\lambda| \to 2\arccos\varepsilon$, $|\lambda| \to \infty$, $\lambda \in \partial(\Omega^2_{\varepsilon,C_\varepsilon})$. This implies that, for a sufficiently large $\omega \in (0,\infty)$, $\omega + \Sigma_{\frac{\pi}{2}+\alpha} \subset \rho(A)$. Furthermore, $\lim_{\lambda\to+\infty} \lambda\widetilde{K}_{1/2}(\lambda)R(\lambda : A) = 0$ and $\widetilde{K}_{1/2}$ can be analytically extended to the function $g : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to \mathbb{C}$, $g(\lambda) = e^{-\sqrt{\lambda}}$, $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha}$. Fix a $\gamma_1 \in (0, \alpha)$. Then it is straightforward to see that $\cos(\frac{\pi}{4} + \frac{\gamma_1}{2}) > \varepsilon$ and that

$$\begin{aligned}&\bigl\|(\lambda-\omega)g(\lambda)R(\lambda : A)\bigr\| \leqslant K_\varepsilon\bigl(|\lambda|+\omega\bigr)e^{\varepsilon\sqrt{|\lambda|}}e^{-\sqrt{|\lambda|}\cos(\arg(\lambda)/2)} \leqslant K_\varepsilon\bigl(|\lambda|+\omega\bigr)e^{\sqrt{|\lambda|}(\varepsilon-\cos(\frac{\pi}{4}+\frac{\gamma_1}{2}))},\\ &\lambda \in \omega + \Sigma_{\gamma_1+\frac{\pi}{2}}.\end{aligned}$$

Theorem 10 ends the proof. □

The similar assertion holds for ultradistribution sines. For the sake of brevity, in the next theorem, we consider only the case when (M_p) is a Gevrey type sequence: $(p!^s)$, (p^{ps}) or $(\Gamma(1+ps))$, $s > 1$. Then we know that, for every $s > 1$, there exists an appropriate $C_s' > 0$ so that $M(t) \sim C_s' t^{\frac{1}{s}}$, $t \to +\infty$.

Theorem 18. *Suppose that A generates an ultradistribution sine of the Beurling, respectively, Roumieu class. Then A generates an exponentially bounded, analytic K_δ-semigroup of angle $\frac{\pi}{2}$, for all $\delta \in (\frac{1}{2s}, \frac{1}{2})$, respectively, for all $\delta \in [\frac{1}{2s}, \frac{1}{2})$.*

Proof. We prove the assertion in the Roumieu case since the proof in the Beurling case can be derived similarly. Let us fix some $\gamma \in (0, \frac{\pi}{2})$ and $\delta \in [\frac{1}{2s}, \frac{1}{2})$. We know that for every $k > 0$ and a corresponding $C_k > 0$:

$$\{\lambda^2 \colon \lambda \in \Omega^{M_p}_{k,C_k}\} \subset \rho(A) \quad \text{and} \quad \|R(\lambda^2 : A)\| \leqslant C_k e^{M(k|\lambda|)}, \quad \lambda \in \Omega^{M_p}_{k,C_k}.$$

Since $M(|\lambda|) \leqslant C_s|\lambda|^{1/s}$, $|\lambda| \geqslant 0$ for some $C_s > 0$, one obtains

$$\{\lambda^2 \colon \operatorname{Re}\lambda \geqslant C_s(k|\lambda|)^{1/s} + C_k\} \subset \rho(A), \quad \text{i.e.,}$$
$$\left\{r^2 e^{2i\theta} \colon r > 0,\ |\theta| < \frac{\pi}{2},\ r\cos\theta \geqslant C_s k^{1/s} r^{1/s} + C_k\right\} \subset \rho(A).$$

Denote $\Gamma = \{re^{i\theta} \colon r\cos\theta = C_s k^{1/s} r^{1/s} + C_k\}$. Then $\lim_{|\lambda|\to\infty,\, \lambda\in\Gamma} |\arg(\lambda)| = \frac{\pi}{2}$. Therefore, there are an $\omega_\gamma > 0$ and a suitable $\overline{C}_k > 0$ so that $\omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma} \subset \rho(A)$ and that

$$\|R(\lambda : A)\| \leqslant \overline{C}_k e^{M(k\sqrt{|\lambda|})} \leqslant \overline{C}_k e^{C_s k^{1/s}|\lambda|^{1/2s}}, \quad \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}.$$

Clearly, the function $g : \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma} \to \mathbb{C}$, $g(\lambda) = e^{-\lambda^\delta}$, $1^\delta = 1$ is analytic. Furthermore,

$$\begin{aligned}\|g(\lambda)R(\lambda : A)\| &\leqslant \frac{\overline{C}_k}{|e^{\lambda^\delta}|} e^{C_s k^{1/s}|\lambda|^{1/2s}} = \overline{C}_k \exp\bigl(-|\lambda|^\delta \cos(\delta \arg(\lambda)) + C_s k^{1/s}|\lambda|^{1/2s}\bigr) \\ &\leqslant \overline{C}_k e^{C_s k^{1/s}|\lambda|^{1/2s} - \cos(\pi\delta)|\lambda|^\delta}, \quad \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}.\end{aligned}$$

Our choice of δ, the fact that a number $k > 0$ can be chosen arbitrarily in the Roumieu case and Theorem 10, imply that A generates an exponentially bounded, analytic K_δ-semigroup of angle γ. This ends the proof. □

Motivated by [25] and [33], up to the end of this section, we discuss relations between (local) integrated cosine functions and ultradistribution semigroups (sines). Recall, a closed linear operator A generates a local integrated cosine function if and only if there exist $\alpha, \beta, M > 0$ and $n \in \mathbb{N}$ so that $E^2(\alpha, \beta) \subset \rho(A)$ and $\|R(\lambda : A)\| \leqslant M(1 + |\lambda|)^n$, $\lambda \in E^2(\alpha, \beta)$ (see [32]).

Remark 19. We recall that V. Keyantuo proved in [25, Theorem 3.1] that if a densely defined operator A generates an exponentially bounded α-times integrated cosine function for some $\alpha \geqslant 0$ (this means that A is densely defined and that A generates an exponential distribution cosine function of [32]), then $\pm iA$ generate ultradistribution semigroups of [7]. Furthermore, the proof of [25, Theorem 3.1] and the assertion of [8, Proposition 2.6] imply that $\pm iA$ generate regular (M_p)-ultradistribution semigroups in the sense of [8, Definition 2.1], where $M_p = p!^s$, $s \in (1, 2)$; see also Example 3 given below. In general, the assertion of [25, Theorem 3.1] does not hold if $s > 2$ and, in the case $s = 2$, this assertion remains true only in the Beurling case, see [33, Section 4].

In [33], the next extension of [25, Theorem 3.1] has been recently showed: If A is the generator of a (local) α-times integrated cosine function, for an $\alpha > 0$, then $\pm iA$ are generators of ultradistribution semigroups of $*$-class, where (M_p) is a Gevrey type sequence with $s \in (1, 2)$. Then it can be easily seen that $-A^2$ generates an ultradistribution sine of (M_p)-class, respectively, $\{M_p\}$-class.

We want to notice that, in general, $-A^2$ does not generate a local integrated cosine function even if A is the densely defined generator of an exponentially bounded, integrated cosine function. In order to illustrate this fact, we choose $E := L^p(\mathbb{R})$, $1 \leqslant p < \infty$ and put $m(x) := (1 - \frac{x^2}{4}) + ix$, $x \in \mathbb{R}$. Define a closed linear operator A on E by:

$$Af(x) = m(x)f(x), \quad x \in \mathbb{R}, \qquad D(A) := \{f \in E \colon mf \in E\}.$$

Then it is proved in [33, Example 4.4] that A generates a dense, exponential distribution cosine function. We have

$$\sigma(-A^2) = \left\{\left(\frac{x^4}{16} + \frac{3}{2}x^2 - 1\right) + 2ix\left(\frac{x^2}{4} - 1\right) \colon x \in \mathbb{R}\right\}$$

and a simple analysis shows that there do not exist $\alpha > 0$ and $\beta > 0$ with $E^2(\alpha, \beta) \cap \sigma(-A^2) \neq \emptyset$ (see [33, Section 4] for more details). Hence, $-A^2$ does not generate a local α-times integrated cosine function, for any $\alpha > 0$.

The next theorem improves [25, Theorem 3.1] in a different direction.

Theorem 20. *Let K satisfy* (P1) *and* (P2) *and let A be the generator of an exponentially bounded, K-cosine function $(C_K(t))_{t\geqslant 0}$. Suppose $M_p = p!^s$, for some $s \in (1,2)$. If there exist $k > 0$ and $\overline{C} > 0$, in the Beurling case, respectively, if for every $k > 0$ there exists an appropriate $\overline{C_k} > 0$, in the Roumieu case, such that*

$$\frac{1}{|\widetilde{K}(\lambda)|} = O\big(e^{M(k|\lambda|)}\big), \quad \operatorname{Re}\lambda \geqslant \overline{C}, \ \textit{respectively,} \ \operatorname{Re}\lambda \geqslant \overline{C_k}, \tag{14}$$

then there exist ultradistribution fundamental solutions of $*$-class for $\pm iA$.

Proof. We prove the assertion in the Roumieu case. To do this, fix $k > 0$. We know that there exist $a > \overline{C_k}$, $l > 0$ and $L > 0$ with $e^{l|\lambda|^{1/s}} \leqslant e^{M(|\lambda|)} \leqslant e^{L|\lambda|^{1/s}}$, $\lambda \in \mathbb{C}$, $|\lambda| \geqslant a$. Theorem 7 and the assumption (14) imply that there exists an $\omega > \max(\operatorname{abs}(K), a)$ so that $\|R(\lambda : A)\| = O(e^{M(k\sqrt{|\lambda|})})$, $\lambda \in \Pi_\omega$. Since $\partial \Pi_\omega = \{x + iy \in \mathbb{C}\colon x = \omega^2 - \frac{y^2}{4\omega^2}\}$, we have

$$\left\{z \in \mathbb{C}\colon \operatorname{Im} z \geqslant (\omega+1)^2 - \frac{(\operatorname{Re} z)^2}{4(\omega+1)^2}\right\} \subset \rho(iA) \tag{15}$$

and for such λ's: $\|R(\lambda : iA)\| = O(e^{M(k\sqrt{|\lambda|})})$. The choice of s implies that there exists a sufficiently large $\beta > 0$ with

$$\frac{x^2}{4(\omega+1)^2} - \frac{x^s}{kl^s} \geqslant (\omega+1)^2, \quad x \geqslant \beta.$$

Put now $C_k = \max(\frac{a}{k}, \beta)$. Suppose

$$z = x + iy \in \Omega^{M_p}_{k,C_k}, \quad \text{i.e.,} \quad \operatorname{Re} z \geqslant M\big(k|z|\big) + C_k.$$

Then $y^2 \leqslant (\frac{x - C_k}{lk^{1/s}})^{2s} - x^2$. According to the choice of C_k, one obtains

$$\begin{aligned} y + \frac{x^2}{4(\omega+1)^2} &\geqslant \frac{x^2}{4(\omega+1)^2} - \sqrt{\left(\frac{x - C_k}{lk^{1/s}}\right)^{2s} - x^2} \geqslant \frac{x^2}{4(\omega+1)^2} - \left(\frac{x - C_k}{lk^{1/s}}\right)^s \geqslant \frac{x^2}{4(\omega+1)^2} - \left(\frac{x}{lk^{1/s}}\right)^s \\ &\geqslant (\omega+1)^2. \end{aligned}$$

Due to (15), $z \in \rho(iA)$. We know $\|R(\lambda : iA)\| = O(e^{M(k\sqrt{|\lambda|})})$, $\lambda \in \Omega^{M_p}_{k,C_k}$ and this proves the claimed assertion for iA. The same arguments work for $-iA$. $\square$

Suppose $M_p = p!^2$ in the formulation of Theorem 20. As before, this theorem remains true only in the Beurling case.

6. Examples and applications

Example 1. Let $A := -\Delta$ on $E := L^2[0,\pi]$ with the Dirichlet boundary conditions (see, for example, [3, Section 7.2]). Motivated by the paper of B. Bäumer [4] we have proved in [37] that there exists an exponentially bounded kernel $K \in C([0,\infty))$ so that A generates a K-semigroup $(S_K(t))_{t\geqslant 0}$ with $\|S_K(t)\| = O(1 + t^3)$. Suppose that $|K(t)| \leqslant Me^{\beta t}$, $t \geqslant 0$, for some $M > 0$ and $\beta > 0$. Moreover, $-A$ also generates an exponentially bounded K-semigroup $(V_K(t))_{t\geqslant 0}$ in E since it is the generator of an analytic C_0-semigroup of angle $\frac{\pi}{2}$. Then Proposition 8 implies that the biharmonic operator Δ^2, endowed with the corresponding boundary conditions, generates an exponentially bounded, K-cosine function $(C_K(t))_{t\geqslant 0}$, where $C_K(t) := \frac{1}{2}(S_K(t) + V_K(t))$, $t \geqslant 0$. Put $K_1(t) := \int_0^t K(s)\,ds$, $t \geqslant 0$. Clearly, $|K_1(t)| \leqslant Mte^{\beta t}$, $t \geqslant 0$, and Δ^2 generates an exponentially bounded, K_1-cosine function $(C_{K_1}(t))_{t\geqslant 0}$, where $C_{K_1}(t)x=\int_0^t C_K(s)x\,ds,\ x\in E,\ t\geqslant 0$. This implies that Δ^2 generates an exponentially bounded, analytic K_2-semigroup of angle $\frac{\pi}{2}$, where $K_2(t):=\int_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}}K_1(s)\,ds,\ t>0$. Note that we have integrated once the function K in order to prove that K_2 is exponentially bounded. This is valid, since for every $t>0$:

$$\begin{aligned}\left|K_2(t)\right| &\leqslant M\int_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}}se^{\beta s}\,ds=\frac{M}{2\sqrt{\pi}}\int_0^\infty r^2e^{\beta r\sqrt{t}-\frac{r^2}{4}}\,dr=\frac{M}{2\sqrt{\pi}}\int_0^\infty r^2e^{\beta^2 t-(\frac{r}{2}-\beta\sqrt{t})^2}\,dr\\ &=\frac{M}{2\sqrt{\pi}}e^{\beta^2 t}\int_0^\infty r^2e^{-(\frac{r}{2}-\beta\sqrt{t})^2}\,dr=\frac{M}{2\sqrt{\pi}}e^{\beta^2 t}\int_{-\beta\sqrt{t}}^\infty 8\big(v^2+2v\beta\sqrt{t}+\beta^2t\big)e^{-v^2}\,dv\\ &\leqslant\frac{4M}{\sqrt{\pi}}e^{\beta^2 t}\left[\int_{-\infty}^\infty v^2e^{-v^2}\,dv+2\beta\sqrt{t}\int_0^\infty ve^{-v^2}\,dv+\beta^2t\int_{-\infty}^\infty e^{-v^2}\,dv\right]\leqslant\overline{M}e^{(\beta^2+1)t},\end{aligned}$$

for a suitable $\overline{M}>0$. Furthermore, K_2 is a kernel since

$$\limsup_{\lambda\to\infty}\frac{\ln|\widetilde{K}_2(\lambda)|}{\lambda}=\limsup_{\lambda\to\infty}\frac{\ln|\widetilde{K}_1(\sqrt{\lambda})|}{\lambda}=0.$$

On the other side, Δ^2 cannot be the generator of a (local) integrated α-times semigroup, $\alpha\geqslant 0$, since the resolvent set of Δ^2 does not contain any ray (ω,∞). For the same reasons, Δ^2 does not generate a hyperfunction (ultradistribution) sine. Hence, in the analysis of Δ^2 and $-\Delta$, we do not need any C, but the use of regularized operator families enables several advantages which hardly can be considered by the use of asymptotic Laplace transform techniques. More generally, suppose $n\in\mathbb{N}$. Since $\Delta=-A$ generates a cosine function (see [3, Example 7.2.1, p. 418]), one can employ a result of J.A. Goldstein proved in [21] (see also [18, p. 215]), in order to see that $-\Delta^{2n}$ generates an analytic C_0-semigroup of angle $\frac{\pi}{2}$. Hence, an application of [18, Theorem 8.2] shows that there exists an injective operator $C_n\in L(L^2[0,\pi])$ so that Δ^{2n} generates an entire C_n-group. Further on, one can apply Proposition 8 in order to see that the polyharmonic operator Δ^4 generates an exponentially bounded, K_2-convoluted cosine function. Put $\overline{K_3}(t):=\int_0^t K_2(s)\,ds,\ t\geqslant 0$. Then K_3 is a kernel and we have $|\overline{K_3}(t)|\leqslant\overline{M}te^{(\beta^2+1)t},\ t\geqslant 0$. Clearly, Δ^4 generates an exponentially bounded, $\overline{K_3}$-cosine function. Then Theorem 11 can be applied again in order to see that Δ^4 generates an exponentially bounded, analytic K_3-semigroup of angle $\frac{\pi}{2}$, where $K_3(t):=\int_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{3/2}}\overline{K_3}(s)\,ds,\ t>0$. Similarly as above, we have that K_3 is an exponentially bounded kernel. Continuing this procedure leads us to the fact, mentioned already in the abstract and the introduction of the paper, that there exist exponentially bounded kernels K_n and K_{n+1} such that Δ^{2^n} generates an exponentially bounded, K_n-convoluted cosine function, and in the meantime, an exponentially bounded, analytic K_{n+1}-convoluted semigroup of angle $\frac{\pi}{2}$. Note that this procedure can be done only with loss of regularity, since we must apply Theorem 11 (see also [3, Proposition 1.6.8]). At the end of this discussion, note that it is not clear whether there exists a kernel $\overline{K_n}$ such that Δ^{2n} generates an exponentially bounded, $\overline{K_n}$-convoluted cosine function.

Suppose now that A is a self-adjoint operator in a Hilbert space H and that A has a discrete spectrum $(\lambda_n)_{n\in\mathbb{N}}$, where we write the eigenvalues in increasing order and repeat them according to multiplicity. Suppose that $\operatorname{Re}\lambda_n>0$, $n\geqslant n_0$ and m is a natural number which is greater than any multiplicity of λ_n, $n\geqslant n_0$. If

$$\sum_{n\geqslant n_0}\left(1-\frac{|\sqrt{\lambda_n}-1|}{\sqrt{\lambda_n}+1}\right)<\infty,$$

then, according to [3, Theorem 1.11.1], there exists an exponentially bounded function K such that $\widetilde{K}(\sqrt{\lambda_n})=0$, $n\geqslant n_0$. This implies that the function $\lambda\mapsto\widetilde{K^{*m}}(\lambda)R(\lambda^2:A)$ can be analytically extended on a right half plane, where K^{*m} denotes the mth convolution power of K. If, additionally,

$$\big\|\widetilde{K^{*m}}(\lambda)R\big(\lambda^2:A\big)\big\|\leqslant M|\lambda|^{-3},\quad \operatorname{Re}\lambda>\omega\ (\geqslant 0),\ \lambda\neq\sqrt{\lambda_n},\ n\geqslant n_0,$$

for a suitable $M>0$, then A generates an exponentially bounded K^{*m}-cosine function. The main problem is to construct a kernel K which fulfills the previous estimate. It is also evident that this procedure cannot be done if

$(\sqrt{\lambda_n})_{n\geqslant n_0}$ is a uniqueness sequence, see for instance [3] and [4]. Therefore, the theory of convoluted cosine functions cannot be applied if $\lambda_n \sim n^{2s}$, $n \to +\infty$, for some $s \in (0,1]$, and it, in turn, implies that the operator $-\Delta$, considered in the first part of this example, cannot be the generator of any exponentially bounded, convoluted cosine function. Finally, we refer to [46, Chapter 2] for the notion and basic properties of the spaces of "new distributions," for the treatment of such kind of problems within the theory of generalized functions.

Example 2. Let $\mathbb{C}_+ = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ and $1 \leqslant p < \infty$. Suppose that $E := H^p(\mathbb{C}_+)$. Recall that R. Beals constructed in the proof of [5, Theorem 2′] an analytic function $a_1 : \mathbb{C}_+ \to \{z \in \mathbb{C}\colon |z| \geqslant 1\}$ with the property that, for every $\varepsilon > 0$, there exists a region of the form $\Omega_{\varepsilon,C_\varepsilon}$ satisfying $a_1(\mathbb{C}_+) \cap \Omega_{\varepsilon,C_\varepsilon} = \emptyset$. Let $B = a_1^2$. Then B is a holomorphic function on $\mathbb{C}_+$ and for all $\varepsilon > 0$ there exist $C_\varepsilon > 0$ and $K_\varepsilon > 0$ so that $B(\mathbb{C}_+) \subset (\Omega^2_{\varepsilon,C_\varepsilon})^c$. Define

$$(AF)(z) := B(z)F(z), \quad \operatorname{Im} z > 0, \qquad D(A) := \bigl\{F \in H^p(\mathbb{C}_+)\colon\ AF \in H^p(\mathbb{C}_+)\bigr\}.$$

Let $\varepsilon \in (0,1)$ be fixed. Choose an $\varepsilon_1 \in (0,\varepsilon)$ such that $B(\mathbb{C}_+) \subset (\Omega^2_{\varepsilon_1,C_{\varepsilon_1}})^c$. Clearly, $\lim_{|\lambda|\to\infty,\lambda\in\partial(\Omega_{\varepsilon_1,C_{\varepsilon_1}})} |\arg\lambda| = \arccos\varepsilon_1$ and there exists a sufficiently large $\overline{C}_\varepsilon > 0$ such that $\Omega_{\varepsilon,\overline{C}_\varepsilon} = \{\lambda\colon \operatorname{Re}\lambda \geqslant \varepsilon|\lambda| + \overline{C}_\varepsilon\} \subset \Omega_{\varepsilon_1,C_{\varepsilon_1}}$ and that the distance $d := \operatorname{dist}(\partial(\Omega_{\varepsilon_1,C_{\varepsilon_1}}), \partial(\Omega_{\varepsilon,\overline{C}_\varepsilon})) > 0$. This implies: $\Omega^2_{\varepsilon,\overline{C}_\varepsilon} \subset \rho(A)$ and $\|R(\lambda : A)\| \leqslant d^{-2}$, $\lambda \in \Omega^2_{\varepsilon,\overline{C}_\varepsilon}$. Therefore, A generates a hyperfunction sine, and it can be easily seen that A does not generate an ultradistribution sine of $*$-class.

Example 3. Let $E = L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$. Consider the next multiplication operator with the maximal domain in E:

$$Af(x) := \bigl(x + ix^2\bigr)^2 f(x), \quad x \in \mathbb{R},\ f \in E.$$

It is clear that A is dense and stationary dense if $1 \leqslant p < \infty$, but A is not the generator of any (local) integrated cosine function, $1 \leqslant p \leqslant \infty$. Moreover, if $p = \infty$, then A is not stationary dense since, for example, the function $x \mapsto \frac{1}{x^{2n}+1}$ belongs to $D(A^n) \setminus \overline{D(A^{n+1})}$, $n \in \mathbb{N}$. Further, one can easily verify that A generates an ultradistribution sine of $*$-class, if $M_p = p!^s$, $s \in (1,2)$. If $M_p = p!^2$, then an analysis given in [33, Example 4.4] shows that A does not generate an ultradistribution sine of the Roumieu class and that A generates an ultradistribution sine of the Beurling class. Suppose now $M_p = p!^s$, for some $s \in (1,2)$, and put $\delta = \frac{1}{s}$. Then A generates a global (non-exponentially bounded) K_δ-cosine function since, for every $\tau \in (0,\infty)$, A generates a K_δ-cosine function on $[0,\tau)$. Indeed, suppose $M(\lambda) \leqslant C_s|\lambda|^{1/s}$, $\lambda \in \mathbb{C}$. Fix $\tau \in (0,\infty)$ and choose an $\alpha > 0$ with $\tau \leqslant \frac{\cos(\frac{\delta\pi}{2})}{C_s\alpha^{1/s}}$. The choice of α implies that there exists a sufficiently large $\beta > 0$ such that $\Lambda^2_{\alpha,\beta,1} \subset \rho(A)$ and that the resolvent of A is bounded on $\Lambda^2_{\alpha,\beta,1}$. Put $\Gamma := \partial(\Lambda_{\alpha,\beta,1})$. We assume that Γ is upwards oriented. Define

$$C_\delta(t)f(x) := \frac{1}{2\pi i}\int_\Gamma \frac{\lambda e^{\lambda t - \lambda^\delta}}{\lambda^2 - (x+ix^2)^2}\,d\lambda\, f(x), \quad f \in E,\ x \in \mathbb{R},\ t \in \left[0, \frac{\cos(\frac{\delta\pi}{2})}{C_s\alpha^{1/s}}\right).$$

Note that the above integral is convergent since

$$\bigl|e^{-\lambda^\delta}\bigr| \leqslant e^{-\cos(\frac{\delta\pi}{2})|\lambda|^\delta}, \quad \operatorname{Re}\lambda > 0, \quad \text{and}$$
$$\bigl|e^{\lambda t - \lambda^\delta}\bigr| \leqslant e^{\beta t} e^{M(\alpha\lambda)t - \cos(\frac{\delta\pi}{2})|\lambda|^\delta} \leqslant e^{\beta t} e^{C_s\alpha^{1/s}|\lambda|^{1/s}t - \cos(\frac{\delta\pi}{2})|\lambda|^\delta}, \quad \lambda \in \Gamma.$$

It is straightforward to check that $(C_\delta(t))_{t\in[0,\tau)}$ is a K_δ-cosine function generated by A. At the end, we point out that there exists an appropriate $\tau_0 \in (0,\infty)$ such that A generates a local $K_{1/2}$-cosine function on $[0,\tau_0)$.

Many other examples of differential operators, acting on $L^2(\mathbb{R}^n)$, $n \in \mathbb{N}$, which generate ultradistribution and hyperfunction sines (semigroups) can be derived similarly as in the previous example; in this context, we also refer to [7, Remarque 6.4]. It seems to be an interesting problem to consider such kinds of operators in $L^p(\mathbb{R}^n)$ spaces, $p \in [1,\infty)$, $p \neq 2$, by the use of Fourier multiplier type theorems.

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