Journal of Algebra **240**, 680–704 (2001)
doi:10.1006/jabr.2000.8693, available online at http://www.idealibrary.com on IDEAL®

Local and Subquotient Inheritance of Simplicity in Jordan Systems[1]

José A. Anquela and Teresa Cortés

Communicated by Georgia Benkart

Received March 16, 2000

DEDICATED TO PROFESSOR DR. OTTMAR LOOS ON THE OCCASION OF HIS 60TH BIRTHDAY

In this paper we prove that the local algebras of a simple Jordan pair are simple. Jordan pairs all of which local algebras are simple are also studied, showing that they have a nonzero simple heart, which is described in terms of powers of the original pair. Similar results are given for Jordan triple systems and algebras. Finally, we characterize the inner ideals of a simple pair which determine simple subquotients, answering the question posed by O. Loos and E. Neher (1994, *J. Algebra* **166**, 255–295). © 2001 Academic Press

INTRODUCTION

Local algebras are introduced by Meyberg [21] and are one of the key tools in the recent works on the structure theory of Jordan pairs and triple systems [2–4, 24]. For any Jordan pair or triple system J, one can consider the set of local algebras J_a, for $0 \neq a \in J$, which seem to keep most of the structural information of the system J, with the advantage of being algebras. Among other things, local algebras of Jordan pairs inherit primitivity [2] and strong primeness [6] and, conversely, Jordan systems inherit primitivity [4] and strong primeness [6] from their local algebras. This two-way flow of regularity conditions between Jordan systems and their local algebras can be used to extend to Jordan pairs and triple

[1] Partially supported by the DGES, PB97-1069-C02-02, and the II Plan Regional de Investigación del Principado de Asturias, FICYT, PB-PGI98-05.

0021-8693/01 $35.00

systems results of Jordan algebras concerning primitivity or strong primeness. As an example, the description in [4] of primitive pairs and triple systems is a suitable translation of the description of primitive Jordan algebras [8] using local algebras. On the other hand, local algebras seem to provide the right language to extend to Jordan systems the associative theory of generalized polynomial identities. The reader can see the recent works of Montaner [22, 23] on the local PI theory of Jordan systems or papers like [7, 9] in which the work with primitive Jordan algebras with PI primitizers is just dealing unexplicitly with generalized polynomial identities.

Our aim is to obtain results on local and local-to-global inheritance of simplicity in Jordan systems.

In particular we prove in the second section that local algebras of a simple Jordan pair are necessarily simple. The proof is based on a combinatorial result due to Zelmanov [24] and D'Amour and McCrimmon [2] on local nilpotency of certain ideals. We have included a first section devoted to study free Jordan algebras and triple systems where we introduce the basic results and terminology needed when dealing with local nilpotency. In this first section, we also study the relationship between systems of generators of a Jordan algebra and its underlying triple system, which will be used in the third section to extend to algebras the results on local inheritance of simplicity for pairs.

Local-to-global inheritance of simplicity is studied in the fourth section. Though, in general, it cannot be said that a Jordan system having simple local algebras is simple, we will show that it is not far from that: it has a "big" simple heart, so that these results are close, in the formal sense, to those concerning strong primeness [6] and primitivity [4, 5].

Finally, the fifth section is devoted to the study of the inheritance of simplicity by subquotients. Subquotients are introduced by Loos and Neher [15]. The work done in [6, 22] reveals the deep relationship between subquotients and local algebras, which shows up again when studying simplicity. Indeed, the same argument used to prove local inheritance of simplicity in Jordan pairs applies to characterize inner ideals of a simple Jordan pair which determine a simple subquotient, answering [15, 2.8(i)].

0. PRELIMINARIES

0.1. We will deal with associative and Jordan algebras, pairs and triple systems over an arbitrary ring of scalars Φ. The reader is referred to [4, 8, 11, 14, 20] for basic results, notation, and terminology, though we will stress some definitions and basic notions. The identities JPx listed in [14] will be quoted with their original numbering JPx without explicit reference to [14].

—Given a Jordan algebra J, its products will be denoted x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively.

—For a Jordan pair $V = (V^+, V^-)$ we will denote the products by $Q_x y$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations denoted by $Q_{x,z}(y) = \{x, y, z\} = D_{x,y} z$.

—A Jordan triple system T is given by its products $P_x y$, for any $x, y \in T$, with linearizations denoted by $P_{x,z} y = \{x, y, z\} = L_{x,y} z$.

Recall the most important example of a Jordan system (algebra, triple, or pair), obtained as the *symmetrization* $R^{(+)}$ of an associative system R: Jordan products $U_x y$, $P_x y$, or $Q_x y$ are given by the associative products xyx and Jordan squares x^2 are just the associative squares xx in the algebra case.

0.2. A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P = U$. By doubling any Jordan triple system T one obtains the *double Jordan pair* $V(T) = (T, T)$ with products $Q_x y = P_x y$, for any $x, y \in T$. From a Jordan pair $V = (V^+, V^-)$ one can get a (*polarized*) Jordan triple system $T(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+} y^- \oplus Q_{x^-} y^+$ [14, 1.13, 1.14].

Similarly, one can consider the underlying triple system of an associative algebra, as well as functors $V(\ )$ and $T(\ )$ between the categories of associative pairs and associative triple systems.

A consequence of the above comments is that we can use the identities JPx of [14] in Jordan algebras (or Jordan triple systems), by simply making notational changes: replacing the products $Q_x y$ by $U_x y$ (or $P_x y$), and $D_{x,y} z$ by $V_{x,y} z$ (or $L_{x,y} z$).

0.3. *Regularity Conditions through the Functors V and T.*

(i) Let T be a Jordan (resp. associative) triple system.

(a) T is nondegenerate (resp. semiprime) if and only if $V(T)$ is nondegenerate (resp. semiprime),

(b) If $V(T)$ is strongly prime (resp. prime), then T is strongly prime (resp. prime),

(c) If $V(T)$ is simple, then T is simple.

(ii) Let V be a Jordan (resp. associative) pair.

(a) V is nondegenerate (resp. semiprime) if and only if $T(V)$ is nondegenerate (resp. semiprime),

(b) V is strongly prime (resp. prime) if and only if $T(V)$ is strongly prime (resp. prime),

(c) V is simple if and only if $T(V)$ is simple.

(iii) Let J be a Jordan (resp. associative) algebra.

(a) J is nondegenerate (resp. semiprime) if and only if $V(J)$ is nondegenerate (resp. semiprime),

(b) J is strongly prime (resp. prime) if and only if $V(J)$ is strongly prime (resp. prime).

Parts (i)(a), (ii)(a) and (iii)(a) are straightforward; (i)(b)(c) follow from the fact that any nonzero ideal I of T gives rise to the nonzero ideal $V(I) = (I, I)$ of $V(T)$; (ii)(b)(c) are proved in [1, p. 230] in the Jordan case and can be easily extended to the associative case; (iii)(b) is just [6, 1.12] in the Jordan case, but the proof can be easily adapted to associative algebras.

0.4. Recall the notion of *McCrimmon radical* $\mathscr{M}c(V)$ of a Jordan pair V (cf. [14, Sect. 4]): the smallest ideal of V which provides a nondegenerate quotient, so that V is nondegenerate if and only if $\mathscr{M}c(V) = 0$. Similarly one can consider the McCrimmon radical $\mathscr{M}c(J)$ of a Jordan triple system or algebra J.

0.5. Local algebras of Jordan and associative systems, introduced in [21], are one of the most useful tools to study primitivity of Jordan pairs and triple systems. They are the way to connect the categories of algebras and pairs and triple systems:

—Given an associative pair $R = (R^+, R^-)$ and $a \in R^{-\sigma}$, the Φ-module R^σ becomes an associative algebra, denoted $R^{\sigma(a)}$ and called the *a-homotope of R*, with product

$$x \cdot_a y = xay,$$

for any $x, y \in R^\sigma$. The set

$$\operatorname{Ker}_R a = \operatorname{Ker} a = \{x \in R^\sigma \mid axa = 0\}$$

is an ideal of $R^{\sigma(a)}$ and the quotient

$$R_a^\sigma = R^{\sigma(a)}/\operatorname{Ker} a$$

is an associative algebra called the *local algebra of R at a*.

—Given a Jordan pair $V = (V^+, V^-)$ and $a \in V^{-\sigma}$, the Φ-module V^σ becomes a Jordan algebra, denoted $V^{\sigma(a)}$ and called the *a-homotope of V*, with products

$$x^{(2,a)} = Q_x a, \qquad U_x^{(a)} y = Q_x Q_a y$$

for any $x, y \in V^\sigma$. The set

$$\mathrm{Ker}_V\, a = \mathrm{Ker}\, a = \{x \in V^\sigma \mid Q_a x = Q_a Q_x a = 0\}$$

is an ideal of $V^{\sigma(a)}$ and the quotient

$$V_a^\sigma = V^{\sigma(a)}/\mathrm{Ker}\, a$$

is a Jordan algebra called the *local algebra of V at a*. When V is nondegenerate or special, $\mathrm{Ker}_V\, a = \{x \in V^\sigma \mid Q_a x = 0\}$.

—Homotopes and local algebras of a Jordan algebra or a Jordan triple system J at an element a are simply those of the Jordan pair $V(J)$.

The above notions are compatible with the functors $V(\ )$ and $T(\ )$ (cf. [4, 0.5]).

0.6. Let V be a Jordan pair, $M \subseteq V^+$ be an inner ideal of V. The set

$$\mathrm{Ker}_V\, M = \mathrm{Ker}\, M = \{x \in V^- \mid Q_M x = Q_M Q_x M = 0\}$$

gives rise to the ideal $(0, \mathrm{Ker}\, M)$ of the subpair (M, V^-) of V. The quotient

$$(M, V^-)/(0, \mathrm{Ker}\, M) = (M, V^-/\mathrm{Ker}\, M)$$

is called the *subquotient* of V *determined by M*. Similarly, one can consider subquotients of V determined by an inner ideal $M \subseteq V^-$ of V (see [15]).

0.7. *Local and Subquotient Inheritance of Strong Primeness* [6, 3.2]. Let V be a Jordan pair. Then the following are equivalent:

(i) V is strongly prime,

(ii) all local algebras of V at nonzero elements are strongly prime,

(iii) V is nondegenerate and V_a^+ is strongly prime for all $0 \neq a \in V^-$,

(iv) V is nondegenerate and V_a^- is strongly prime for all $0 \neq a \in V^+$,

(v) all nonzero subquotients of V are strongly prime,

(vi) V is nondegenerate and all subquotients determined by nonzero inner ideals in V^+ are strongly prime,

(vii) V is nondegenerate and all subquotients determined by nonzero inner ideals in V^- are strongly prime.

1. JORDAN TRIPLE AND ALGEBRA MONOMIALS

1.1. We first recall the notions of *free Jordan algebra* $\mathrm{JAlg}(X)$ and *free Jordan triple system* $\mathrm{JTS}(X)$ on the set of variables X.

The free Jordan algebra on X is spanned over Φ by the so-called *Jordan algebra monomials* (on X), defined inductively as follows: The elements in X are monomials, and, given monomials a, b, c, the elements a^2, $a \circ b$, $U_a b$, and $U_{a,b} c$ are also Jordan algebra monomials. We will say that the elements in X are Jordan algebra monomials of degree 1, and, in general, we will say that a Jordan algebra monomial a has degree $n > 1$ if one of the following holds:

(i) $a = b^2$, where b is a Jordan algebra monomial of degree $n/2$ (only if n is even),

(ii) $a = b \circ c$, where b, c are Jordan algebra monomials of degree k, l, respectively, and $k + l = n$,

(iii) $a = U_b c$, where b, c are Jordan algebra monomials of degree k, l, respectively, and $2k + l = n$,

(iv) $a = U_{b,c} d$, where b, c, d are Jordan algebra monomials of degree k, l, m, respectively, and $k + l + m = n$.

We will just need the fact that every monomial "admits" a degree in the variables of X (by the construction). Indeed, it can be shown that $\mathrm{JAlg}(X)$ is graded by the degree in the variables of X, so that every monomial has a unique, well-defined degree. Recall that, however, the set of Jordan algebra monomials is not a basis of $\mathrm{JAlg}(X)$ over Φ (for example, $U_{x,x} y = 2U_x y$ for any $x, y \in X$).

The free Jordan triple system on X is spanned over Φ by the so-called *Jordan triple monomials* (on X), defined inductively as follows: The elements in X are Jordan triple monomials, and, given Jordan triple monomials a, b, c, the elements $P_a b$ and $P_{a,b} c$ are also Jordan triple monomials. We will say that the elements in X are Jordan triple monomials of degree 1, and, in general, we will say that a Jordan triple monomial a has degree $n > 1$ if one of the following holds:

(i) $a = P_b c$, where b, c are Jordan triple monomials of degree k, l, respectively, and $2k + l = n$,

(ii) $a = P_{b,c} d$, where b, c, d are Jordan triple monomials of degree k, l, m, respectively, and $k + l + m = n$.

We remark that the comments on the degree in the algebra case apply here too. Similarly, the set of Jordan triple monomials is not a basis of $\mathrm{JTS}(X)$ over Φ.

As we have already mentioned (0.2), we can consider the underlying triple system of $\mathrm{JAlg}(X)$. The results that follow are aimed at studying sets of elements in $\mathrm{JAlg}(X)$ which generate the whole $\mathrm{JAlg}(X)$ as a triple system. Some of these results are part of the Jordan folklore, but they have been mainly used in the linear case in the literature and we need explicit statements concerning the degree. Thus, for the sake of completeness, we will include sketches of their proofs. We begin by listing some identities in Jordan algebras.

1.2. *Identities Reducing Algebra Products to Triple Products.* The following identities are valid for arbitrary elements $a, b, c, d \in J$, where J is a Jordan algebra:

(i) $(a^2)^2 = U_a a^2$,

(ii) $(a \circ b)^2 = U_{a \circ b, a} b - U_a b^2 + U_b a^2$,

(iii) $(U_a b)^2 = U_a U_b a^2$,

(iv) $(U_{a,c} b)^2 = U_a U_b c^2 + U_c U_b a^2 + \{a, b, U_c(b \circ a)\} - U_{a \circ c, a} U_b c + U_a U_{b, b \circ c} c - 2U_a U_b c^2$,

(v) $a^2 \circ b = U_{a,b} a$,

(vi) $(a \circ b) \circ c = \{a, b, c\} + \{b, a, c\}$,

(vii) $U_a b \circ c = U_{a \circ c, a} b - U_a(b \circ c)$,

(viii) $U_{a,b} c \circ d = U_{a \circ d, b} c + U_{a, b \circ d} c - U_{a,b}(c \circ d)$.

Indeed, (i), (ii), (iii), (v), (vi), and (vii) are true by [13] (they involve either less than three variables, or three variables but are linear in one of them), and (viii) is just a linearization of (vii). We will prove (iv) by passing to a unital hull of J and using Jordan pair identities of [14]:

$$
\begin{aligned}
(U_{a,c} b)^2 &= U_{U_{a,c} b} 1 = U_{\{a,b,c\}} 1 \\
&= \left(U_a U_b U_c + U_c U_b U_a + V_{a,b} U_c V_{b,a} - U_{U_a U_b c, c}\right) 1 \qquad \text{(by JP21)} \\
&= U_a U_b c^2 + U_c U_b a^2 + \{a, b, U_c(b \circ a)\} \\
&\quad - V_{a, U_b c} U_{a,c} 1 + U_a V_{U_b c, c} 1 \qquad \text{(by JP11)} \\
&= U_a U_b c^2 + U_c U_b a^2 + \{a, b, U_c(b \circ a)\} \\
&\quad - \{a, U_b c, a \circ c\} + U_a U_{U_b c, 1} c \\
&= U_a U_b c^2 + U_c U_b a^2 + \{a, b, U_c(b \circ a)\} \\
&\quad - U_{a \circ c, a} U_b c + U_a\left(V_{b,c} U_{b,1} c - U_b V_{c,1} c\right) \qquad \text{(by JP11)} \\
&= U_a U_b c^2 + U_c U_b a^2 + \{a, b, U_c(b \circ a)\} \\
&\quad - U_{a \circ c, a} U_b c + U_a\left(\{b, c, b \circ c\} - U_b(c \circ c)\right) \\
&= U_a U_b c^2 + U_c U_b a^2 + \{a, b, U_c(b \circ a)\} \\
&\quad - U_{a \circ c, a} U_b c + U_a U_{b, b \circ c} c - 2U_a U_b c^2.
\end{aligned}
$$

1.3. PROPOSITION. *For any set of variables* X, *the subset* $\tilde{X} = X \cup \{x^2, x \circ y \,|\, x, y \in X\}$ *of* $\mathrm{JAlg}(X)$ *generates* $\mathrm{JAlg}(X)$ *as a Jordan triple system*:

$$\mathrm{JAlg}(X) = \mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X}).$$

Proof. We just need to show that any Jordan algebra monomial p in the variables of X lies in $\mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$, which can be proved by induction on the degree n of p. If $n = 1$ then $p \in X \subseteq \tilde{X} \subseteq \mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$. So, let us assume $n > 1$, and every Jordan algebra monomial of degree less than n lies in $\mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$. Since $n > 1$, there are four possibilities for p: either $p = q^2$, or $p = q_1 \circ q_2$, or $p = U_{q_1} q_2$, or $p = U_{q_1, q_2} q_3$, where q, q_1, q_2, q_3 are Jordan algebra monomials of degree smaller than n.

In the last two cases, we have $p = U_{q_1} q_2$ or $p = U_{q_1, q_2} q_3$, where $q_1, q_2, q_3 \in \mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$ by the induction assumption. Hence, $p \in \mathrm{subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$ since it is obtained by making triple products with elements in $\mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$.

Assume that $p = q^2$. If the degree of q is 1, then $q \in X$ and $p \in \tilde{X} \subseteq \mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$. If the degree of q is greater than 1, then either $q = a^2$, or $q = a \circ b$, or $q = U_a b$, or $q = U_{a,b} c$. Thus, either $p = (a^2)^2$, or $p = (a \circ b)^2$, or $p = (U_a b)^2$, or $p = (U_{a,b} c)^2$. By (1.2)(i)–(iv), p can be written in terms of triple products of a, b, c, a^2, b^2, c^2, $a \circ b$, $a \circ c$, and $b \circ c$, which lie in $\mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$ by the induction assumption. Thus $p \in \mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$.

Assume $p = q_1 \circ q_2$. If the degrees of both q_1 and q_2 are 1, then $q_1, q_2 \in X$ and $p \in \tilde{X} \subseteq \mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$. Otherwise, we can assume that the degree of q_1 is greater than 1, hence either $q_1 = a^2$, or $q_1 = a \circ b$, or $q_1 = U_a b$, or $q_1 = U_{a,b} c$, and $p = a^2 \circ q_2$, or $p = (a \circ b) \circ q_2$, or $p = U_a b \circ q_2$, or $p = U_{a,b} c \circ q_2$. Similarly, $p \in \mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$ by applying (1.2)(v)–(viii) and the induction assumption. ∎

1.4. COROLLARY. *Let* J *be a Jordan algebra generated by* $A \subseteq J$. *Then the underlying triple system of* J *is generated* (*as a triple system*!) *by the subset* $\tilde{A} = A \cup \{a^2, a \circ b \,|\, a, b \in A\}$.

In the proof of (1.3) there is an implicit (inductive) description on how to express a Jordan algebra monomial in the variables X in terms of Jordan triple monomials in the bigger set of elements $\tilde{X}$. The following discussion is intended to provide precise relations between the degrees of algebra and triple monomials involved in that description.

1.5. Let X be a set of variables and let $\tilde{X}$ be the subset of $\mathrm{JAlg}(X)$ given by $\tilde{X} = X \cup \{x^2, x \circ y \mid x, y \in X\}$ as in (1.3). Let Y be another set of variables such that there exists a bijection

$$\varphi : Y \to \tilde{X},$$

and consider the free Jordan triple system $\mathrm{JTS}(Y)$ on Y. For a positive integer n, denote by $\mathrm{MAlg}_n(X)$ the set of Jordan algebra monomials of degree n in $\mathrm{JAlg}(X)$, and by $\mathrm{MTS}_n(Y)$ the set of Jordan triple monomials of degree n in $\mathrm{JTS}(Y)$. Now, it is immediate that

(i) $\mathrm{JAlg}_n(X) := \Phi(\bigcup_{k \geq n} \mathrm{MAlg}_k(X))$ is an ideal of $\mathrm{JAlg}(X)$,

(ii) $\mathrm{JTS}_n(Y) := \Phi(\bigcup_{k \geq n} \mathrm{MTS}_k(Y))$ is an ideal of $\mathrm{JTS}(Y)$.

1.6. The above ideals are used to define nilpotency of Jordan algebras and triple systems: Fix an infinite set X (resp. Y) of variables. A Jordan algebra (resp. triple system) J is said to be *nilpotent of degree n* if the evaluation $\mathrm{JAlg}_n(J) = 0$ (resp. $\mathrm{JTS}_n(J) = 0$). If A is a generating subset of J, then nilpotency of degree n of J is equivalent to $\mathrm{JAlg}_n(A) = 0$ (resp. $\mathrm{JTS}_n(A) = 0$).

Notice that the set X (resp. Y) is irrelevant in the evaluation $\mathrm{JAlg}_n(J)$ of $\mathrm{JAlg}_n(X)$ (resp. $\mathrm{JTS}_n(J)$ of $\mathrm{JTS}_n(Y)$) on J.

1.7. Keeping the notation of (1.5), let $\mathrm{JAlg}(Y)$ denote the free Jordan algebra on Y. We will consider the following inclusions:

$$j_X : X \to \mathrm{JAlg}(X), \qquad j : \tilde{X} \to \mathrm{JAlg}(X), \qquad j_Y : Y \to \mathrm{JAlg}(Y),$$
$$j'_Y : Y \to \mathrm{JTS}(Y).$$

By using the universal properties of $\mathrm{JTS}(Y)$, $\mathrm{JAlg}(Y)$, and the underlying triple systems of $\mathrm{JAlg}(X)$ and $\mathrm{JAlg}(Y)$, there exist:

(i) a unique Jordan triple system homomorphism $f : \mathrm{JTS}(Y) \to \mathrm{JAlg}(Y)$ such that $fj'_Y = j_Y$. Notice that f is the map that replaces P's by U's in Jordan triple monomials to provide Jordan algebra monomials in which no squarings $(\ )^2$ and circles $\circ$ are involved.

(ii) a unique Jordan algebra homomorphism $\tau : \mathrm{JAlg}(Y) \to \mathrm{JAlg}(X)$ such that $j\varphi = \tau j_Y$. Notice that τ is the map that replaces the variables of Y by their images by φ providing Jordan algebra monomials in $\tilde{X}$, hence in X, out of Jordan algebra monomials in Y.

(iii) a unique Jordan triple system homomorphism $\sigma : \mathrm{JTS}(Y) \to \mathrm{JAlg}(X)$ such that $\sigma j'_Y = j\varphi$.

Now,

$$\tau f j'_Y = \tau j_Y \text{ (by (i))} = j\varphi \text{ (by (ii))} = \sigma j'_Y,$$

by (iii). Hence, by uniqueness in the universal property of JTS(Y), we have

(iv) $\tau f = \sigma$.

The next result and its proof will show how $\tau f = \sigma$ provides a more precise way to establish the link between algebra and triple monomials given in (1.3).

1.8. PROPOSITION. *With the notation of* (1.5), (1.7),

$$\mathrm{JAlg}_{2n}(X) \subseteq \sigma(\mathrm{JTS}_n(Y))$$

for any positive integer n.

Proof. We will first remark that σ is surjective. Indeed, $\tilde{X} = j(\tilde{X}) = j\varphi(Y) = \sigma j'_Y(Y)$ by (1.7)(iii). Hence, $\tilde{X}$ is contained in the image of σ, which is a Jordan subtriple of the underlying triple system of JAlg(X). Thus, the image of σ contains $\mathrm{Subtriple}_{\mathrm{JAlg}(X)}(\tilde{X})$, hence the whole JAlg($X$) by (1.3).

Indeed, notice that the proof of (1.3), just gives an inductive way to show that

(1) any Jordan algebra monomial p of JAlg(X) has the form $p = \sum_i \lambda_i \tau(f(q_i))$, where q_i is a Jordan triple monomial in JTS(Y), $\lambda_i \in \Phi$.

Moreover, in every inductive step of the proof of (1.3), the algebra degree in X is preserved, so that

(2) every $\tau(f(q_i))$ is a Jordan algebra monomial in X of the same degree as p.

In view of (1.5)(i), we just need to show that $p \in \sigma(\mathrm{JTS}_n(Y))$ for any $p \in \mathrm{MAlg}(X)_k$, with $k \geq 2n$. By (1) and (2), we can assume that $p = \tau(f(q)) = \sigma(q)$ (by (1.7)(iv)), where q is a Jordan triple monomial in JTS(Y), so that we just need to prove the following inequality:

(3) $\deg_X(\sigma(q)) \leq 2\deg_Y(q)$, for any Jordan triple monomial q in Y,

where $\deg_Y(\ )$ denotes the (triple system) degree in Y, and $\deg_X(\ )$ denotes the (algebra) degree in X.

We will show (3) by induction on $\deg_Y(q)$. If $\deg_Y(q) = 1$, then $q \in \mathrm{MTS}_1(Y) = Y$, hence $\sigma(q) = \sigma j'_Y(q) = j\varphi(q) =$ (by (1.7)(iii)) $= \varphi(q) \in \tilde{X} = \mathrm{MAlg}_1(X) \cup \mathrm{MAlg}_2(X)$, i.e., $\deg_X(\sigma(q)) \leq 2$, and (3) holds. Let us assume (3) for Jordan triple monomials of degree less than $k > 1$, and let $\deg_Y(q) = k$. Either $q = P_a b$, where $a \in \mathrm{MTS}_r(Y)$, $b \in \mathrm{MTS}_s(Y)$, and $2r + s = k$, or $q = P_{a,b}c$, where $a \in \mathrm{MTS}_r(Y)$, $b \in \mathrm{MTS}_s(Y)$, $c \in \mathrm{MTS}_t(Y)$, and $r + s + t = k$. In the case $q = P_a b$, since σ is a Jordan triple system homomorphism, $\sigma(q) = U_{\sigma(a)}\sigma(b)$ and $\sigma(a) = \tau f(a)$, $\sigma(b) = \tau f(b)$ are Jordan algebra monomials by (2); moreover, by the induction assumption $\deg_X(\sigma(a)) \leq 2r$, $\deg_X(\sigma(b)) \leq 2s$, hence $\deg_X(\sigma(q)) \leq 4r + 2s = 2(2r + s) = 2\deg_y(q)$ and (3) holds. If $q = P_{a,b}c$, (3) holds similarly. ∎

By "evaluating" the previous result on a subset of a Jordan algebra, we obtain the next result which will be used later on to link nilpotency (resp. local nilpotency) of a Jordan algebra and nilpotency (resp. local nilpotency) of its underlying triple system.

1.9. COROLLARY. *Let J be a Jordan algebra, $A \subseteq J$, $\tilde{A} = A \cup \{a^2, a \circ b \mid a, b \in A\}$. Then*

$$\mathrm{JAlg}_{2n}(A) \subseteq \mathrm{JTS}_n(\tilde{A})$$

for any positive integer n.

Proof. Apply (1.8) and the fact (1.7) that $\sigma = \tau f$ applied to a Jordan triple monomial replaces P's by U's, and variables in Y by elements in $\tilde{X}$. ∎

2. LOCAL INHERITANCE OF SIMPLICITY IN JORDAN PAIRS AND TRIPLE SYSTEMS

2.1. *Remark.* It is well known [10, 5.2(iii)] that if R is a simple associative pair or algebra then all local algebras of R at nonzero elements are simple.

In this section we will show that the same holds for Jordan pairs and, in the next section we will take care of Jordan algebras. However, the associative proof cannot be easily generalized to the Jordan case, due to the fact that ideals in Jordan systems cannot be as easily described as in the associative case. To overcome that difficulty we will use the next result, due to Zelmanov [24] in the linear case and to D'Amour and McCrimmon in the general quadratic case [2], which provides the construction of an ideal which can be measured up to a "locally nilpotent" level.

2.2. LEMMA [2, 4.13]. *If K is an inner ideal of a Jordan triple system J, then the ideal of J generated by $P_K K$ is locally nilpotent mod $K + P_J K$.*

The fact that simple systems cannot be locally nilpotent is well known (see, for example [17, Corollary, p. 476] for Jordan algebras). The proof of this result can be strengthened in order to apply (2.2).

2.3. LEMMA. *If a simple Jordan triple system or algebra (resp. pair) J is locally nilpotent mod a Φ-submodule M (resp. a pair of Φ-submodules (M^+, M^-)) of J, then $J = M$.*

Proof. We first consider the case of a triple system J. Let $0 \neq x \in J$, and I be the ideal of J obtained as the span of all monomials of degree bigger than or equal to three evaluated in J in which one of the variables is replaced by x. Clearly $0 \neq I$ (otherwise Φx would be a nonzero ideal of J, hence $J = \Phi x$ by simplicity of J, and $P_J J = 0$, which is a contradiction). By simplicity of J, $J = I$ and $x \in I$. This provides an equality

$$x = \sum_i m_i(\ldots, x, \ldots), \tag{1}$$

where m_i are Jordan triple monomials of degree bigger than or equal to three. Let $A = \{x, \ldots\}$ be the finite set of elements of J involved in (1). We have that $x \in \mathrm{JTS}_3(A)$. By replacing in the right hand side of (1) x by $\sum_i m_i(\ldots, x, \ldots)$, we obtain that

$$x = \sum_i m_i\left(\ldots, \left(\sum_j m_j(\ldots, x, \ldots)\right), \ldots\right) \in \mathrm{JTS}_5(A).$$

The above argument can be iterated to show that $x \in \mathrm{JTS}_n(A)$ for n as big as desired. But $\mathrm{JTS}_n(A) \subseteq M$ for a sufficiently big n, due to local nilpotency of J mod M, and we have shown that $x \in M$.

If J is an algebra, we can proceed as for triple systems, taking I to be the span of all monomials of degree bigger than or equal to two, and replacing P's by U's.

If $J = (J^+, J^-)$ is a Jordan pair, we can consider $T(J)$ which is simple by (0.3)(ii)(c). Clearly, $T(J)$ is locally nilpotent mod $M^+ \oplus M^-$, hence $T(J) = M^+ \oplus M^-$, and $J = M$. ∎

2.4. By Lie sandwich results of Kostrikin and Zelmanov [12] it can be obtained that:

(i) the McCrimmon radical $\mathscr{M}c(J)$ of a Jordan system (algebra, pair, or triple system) is locally nilpotent.

(The Jordan version of [12] has not appeared explicitly. Indeed, the results in [12] imply local nilpotency of the McCrimmon radical for pairs and triple systems. The case when J is a Jordan algebra follows by considering its underlying triple system, whose McCrimmon radical is the same, and using (1.4) and (1.9).) In particular, taking $M = 0$ in (2.3) together with (i) yields:

(ii) Any simple Jordan system (algebra, triple, or pair) is nondegenerate (hence strongly prime).

2.5. THEOREM. *If V is a simple Jordan pair and $0 \neq a \in V^{-\sigma}$, then the Jordan algebra V_a^σ is simple* ($\sigma = \pm$).

Proof. Put, for example, $\sigma = +$. Since every simple Jordan pair is nondegenerate (2.4)(ii), V_a^+ is nontrivial. Let L be a nonzero ideal of V_a^+, i.e., $L = I/\mathrm{Ker}\, a$, where I is an ideal of $V^{+(a)}$ strictly containing $\mathrm{Ker}\, a$. Recall that $\mathrm{Ker}\, a = \{x \in V^+ | Q_a x = 0\}$ by nondegeneracy of V, so that there exists $x \in I$ such that $Q_a x \neq 0$.

Define $K^+ := Q_x Q_a V^+$, $K^- := Q_a I$, which are inner ideals of V:

$$Q_{K^+} V^- = Q_{Q_x Q_a V^+} V^- = Q_x Q_a Q_{V^+} Q_a Q_x V^- \text{ (by JP3) } \subseteq Q_x Q_a V^+ = K^+,$$

$$Q_{K^-} V^+ = Q_{Q_a I} V^+ = Q_a Q_I Q_a V^+ \text{ (by JP3) } = Q_a U_I^{(a)} V^+ \subseteq Q_a I = K^-.$$

Moreover, $Q_{K^+} K^- \neq 0$: indeed, suppose

$$0 = Q_{K^+} K^- = Q_{Q_x Q_a V^+} Q_a I = U^{(a)}_{Q_x Q_a V^+} I. \tag{i}$$

By (2.4)(ii), V is strongly prime since it is simple; hence V_a^+ is strongly prime by [6, 3.2] and $\mathrm{Ann}_{V_a^+}(L) = 0$ by [18, 1.6]. But (i) implies $(Q_x Q_a V^+ + \mathrm{Ker}\, a)/\mathrm{Ker}\, a$ is contained in the annihilator $\mathrm{Ann}_{V_a^+}(L)$ of L in V_a^+ (cf. [18, 1.7]). We then have $(Q_x Q_a V^+ + \mathrm{Ker}\, a)/\mathrm{Ker}\, a = 0$, hence $Q_a Q_x Q_a V^+ = 0$, i.e., $Q_{Q_a x} V^+ = 0$ where $Q_a x \neq 0$, which is impossible by nondegeneracy of V.

Let $K = K^+ \oplus K^-$, which is clearly an inner ideal of $T(V)$. Moreover, $P_K K = Q_{K^+} K^- \oplus Q_{K^-} K^+ \neq 0$. Since $T(V)$ is simple by (0.3)(ii)(c), we have that the ideal of $T(V)$ generated by $P_K K$ is the whole $T(V)$. By (2.2), $T(V)$ is locally nilpotent mod $K + P_{T(V)} K$, hence $T(V) = K + P_{T(V)} K$ using (2.3). But

$$K + P_{T(V)} K = \left(K^+ + Q_{V^+} K^-\right) \oplus \left(K^- + Q_{V^-} K^+\right),$$

hence

$$V^+ = K^+ + Q_{V^+} K^- = Q_x Q_a V^+ + Q_{V^+} Q_a I = U_x^{(a)} V^+ + U_{V^+}^{(a)} I \subseteq I$$

since $x \in I$ and I is an ideal of $V^{+(a)}$. Thus $I = V^+$, hence $L = V_a^+$, and we have shown that V_a^+ is simple. ∎

An analogue of (2.5) for Jordan triple systems is false. The situation is similar to that of local inheritance of strong primeness [6, 3.5] or primitivity [2, 6.2]:

2.6. COROLLARY. *If J is a simple Jordan triple system then some local algebra J_a is simple. More precisely, either:*

(i) *The pair $V(J)$ is simple and hence J_a is simple for any $0 \neq a \in J$, or*

(ii) *The pair $V(J)$ is the direct sum $V(J) = W \boxplus W^{\mathrm{op}}$, where W (and then W^{op}) is a simple Jordan pair and $J = T(W)$ is polarized. In this case, for any $0 \neq a^+ \in W^+$ and $0 \neq a^- \in W^-$, $J_{a^+ \oplus 0} \cong W^-_{a^+}$ and $J_{0 \oplus a^-} \cong W^+_{a^-}$ are simple, while $J_{a^+ \oplus a^-} \cong W^+_{a^-} \boxplus W^-_{a^+}$ is a direct sum of two simple algebras.*

Proof. If $V(J)$ is simple, we can apply (2.5), since $J_a = V(J)^{\sigma}_a$, for any $a \in J$, $\sigma = \pm$.

If $V(J)$ is not simple, then there exists a nonzero ideal $W = (W^+, W^-)$ of $V(J)$, smaller than the whole $V(J)$. Clearly $W^+ \cap W^-$ and $W^+ + W^-$ are ideals of J. By simplicity of J, $W^+ \cap W^- = 0$ and $W^+ + W^- = J$, hence $J = T(W)$ and $V(J) = W \boxplus W^{\mathrm{op}}$. Moreover, W (hence W^{op}) is simple by simplicity of J, since any ideal $U = (U^+, U^-)$ of W gives rise to the ideal $U^+ + U^-$ of J.

The rest of (ii) follows from (2.5) and the straightforward fact that $T(W)_{a^+ \oplus a^-} \cong W^+_{a^-} \boxplus W^-_{a^+}$. ∎

3. LOCAL INHERITANCE OF SIMPLICITY IN JORDAN ALGEBRAS

The results on local inheritance of simplicity for Jordan algebras can be readily obtained from those for pairs. To do that we need a link between the simplicity of a Jordan algebra J and the Jordan pair $V(J)$ built out of it. The corresponding fact for associative algebras can be easily proved.

3.1. LEMMA. *Let R be an associative algebra. Then the following are equivalent:*

(i) *R is a simple associative algebra,*

(ii) *R is a simple associative triple system,*

(iii) *$V(R)$ is a simple associative pair.*

Proof. (iii) ⇒ (ii). This is (0.3)(i)(c).

(ii) ⇒ (i). This is straightforward since any algebra ideal of R is a triple ideal.

(i) ⇒ (iii). Clearly $V(R)$ is not trivial since $R = RR = RRR$ by simplicity of R. Let $I = (I^+, I^-)$ be a nonzero ideal of $V(R)$. Notice that both I^+ and I^- are nonzero (otherwise, if $I^{\sigma} = 0$, then $RI^{-\sigma}R = 0$, which implies $I^{-\sigma} = 0$ by semiprimeness of R). Let $L = RRI^+RR$. We have $L \neq 0$ by semiprimeness of R and $L \subseteq I^+$. Moreover, L is an ideal of R:

$$RL + LR = RRRI^+RR + RRI^+RRR \subseteq RRI^+RR = L$$

since $RR \subseteq R$. By simplicity of R, $L = R$ which implies $I^+ = R$. Similarly one can prove $I^- = R$, hence $I = V(R)$. ∎

As usual, the difficulty to describe ideals in Jordan systems will make the proof of the Jordan version of (3.1) quite more involved.

3.2. *Remark*. Notice that if J is a Jordan algebra without proper ideals such that $U_J J = 0$ then $J^2 = 0$. Indeed, $U_J J = 0$ implies that J^2 is an ideal of J, hence either $J^2 = 0$ or $J^2 = J$. In the latter case, $J = J^2 = (J^2)^2$ is spanned by elements of the form

$$(x^2)^2 = U_x x^2 \in U_J J = 0$$

and

$$x^2 \circ y^2 = U_{x, y^2} x \text{ (by (1.2)(v))} \in U_J J = 0,$$

i.e., $J = J^2 = 0$.

3.3. *Remark*. If J is a unital Jordan algebra, then every ideal of $V(J)$ has the form $V(I)$ where I is an ideal of J. Indeed, if $L = (L^+, L^-)$ is an ideal of $V(J)$, then

$$L^\sigma = U_1 L^\sigma \in Q_{V(J)^{-\sigma}} L^\sigma \subseteq L^{-\sigma},$$

i.e., $L^+ = L^-$. Moreover $I = L^+ = L^-$ is an ideal of J since

$$I^2 = U_I 1 \subseteq Q_{L^+} V(J)^- \subseteq L^+ = I,$$

$$I \circ J = \{I, 1, J\} \subseteq \{L^+, V(J)^-, V(J)^+\} \subseteq L^+ = I,$$

$$U_I J = Q_{L^+} V(J)^- \subseteq L^+ = I,$$

$$U_J I = Q_{V(J)^-} L^+ \subseteq L^- = I.$$

3.4. THEOREM. *Let J be a Jordan algebra. Then the following are equivalent*:

(i) *J is a simple Jordan algebra*,

(ii) *J is a simple Jordan triple system*,

(iii) *$V(J)$ is a simple Jordan pair*.

Proof. (iii) ⇒ (ii). Apply (0.3)(i)(c) to the underlying triple system of J.

(ii) ⇒ (i). Since every ideal of J is an ideal of the underlying triple of J, the algebra J does not have proper ideals. Nontriviality of the algebra J is an immediate consequence of nontriviality of the triple J.

(i) ⇒ (iii). If $U_J J = 0$, then $J^2 = 0$ by (3.2) and J is trivial, which is impossible since it is simple. Thus $U_J J \neq 0$ and $V(J)$ is not trivial.

Let $0 \neq I = (I^+, I^-)$ be an ideal of $V(J)$. Let $\hat{J} = J \oplus \Phi 1$ be a (free, not necessarily tight) unital hull of J. We know that J is an ideal of $\hat{J}$, hence $V(J)$ is an ideal of $V(\hat{J})$. Let $L = (L^+, L^-)$ be the ideal of $V(\hat{J})$ generated by I, so that $0 \neq L \subseteq V(J)$. By [14, 4.10], the quotient L/I is McCrimmon radical, hence locally nilpotent by (2.4)(i). But L is an ideal of $V(\hat{J})$ where $\hat{J}$ is unital, hence $L = (M, M)$, where M is an ideal of $\hat{J}$ by (3.3). Moreover, since $0 \neq L \subseteq V(J)$, $M \subseteq J$ and M is a nonzero ideal of J. By simplicity of J, $M = J$ and we have that $V(J)/I$ is locally nilpotent.

Let $0 \neq x \in J$, and let K be the ideal of J obtained as the span of all algebra monomials of degree bigger than or equal to two evaluated in J in which one of the variables is replaced by x. Clearly $0 \neq K$ (otherwise Φx would be a nonzero ideal of J, $J = \Phi x$ by simplicity of J, and J is trivial), hence $J = K$ by simplicity of J, and $x \in K$. Now, we can proceed as in the proof of (2.3), using algebra monomials instead of triple monomials to find a finite set A of elements in J such that $x \in \mathrm{JAlg}_n(A)$ for n as big as desired. Let $\tilde{A} = A \cup \{a^2, a \circ b \mid a, b \in A\}$. By putting $n = 2m$ and using (1.9), $x \in \mathrm{JTS}_m(\tilde{A})$, for m as big as desired. Since $V(J)/I$ is locally nilpotent, there exists a nonnegative integer m such that $\mathrm{JTS}_m(\tilde{A}) \subseteq I^+ \cap I^-$ and we have $x \in I^+ \cap I^-$, showing $J = I^+ = I^-$, so that $I = V(J)$. ∎

We remark that the equivalence (i) ⇔ (ii) of (3.4) can also be obtained from [23]: in the proof of [23, 2.4] it is shown that for a nondegenerate Jordan algebra any nonzero ideal of the underlying triple system contains a nonzero algebra ideal.

Now, using the fact that, for a Jordan algebra J and an element $a \in J$, $V(J)_a^\sigma = J_a$, (2.5) and (3.4) yield the following result:

3.5. COROLLARY. *If J is a simple Jordan algebra and $0 \neq a \in J$, then the local algebra J_a is simple.*

4. LOCAL-TO-GLOBAL INHERITANCE OF SIMPLICITY IN JORDAN SYSTEMS

4.1. The following example shows that the converse of (2.5) is false. Let Φ be a field and V a vector space over Φ with an enumerable basis $\{v_i \mid i \in \mathbf{N}\}$. Let e_{ij} $(i, j \in \mathbf{N})$ denote the endomorphism of V given by $e_{ij}(v_k) = \delta_{jk} v_i$, for any $k \in \mathbf{N}$, and let x be the endomorphism of V such that $x(v_{2r}) = 0$, $x(v_{2r+1}) = v_{2r+2}$, for any $r \in \mathbf{N}$. One can readily check for $i, j, k, l \in \mathbf{N}$ that:

(i) $e_{ij}e_{kl} = \delta_{jk}e_{il}$,

(ii) $xe_{ij} = e_{i+1\,j}$ if i is odd, $xe_{ij} = 0$ if i is even,

(iii) $e_{ij}x = 0$ if j is odd, $e_{ij}x = e_{i\,j-1}$ if j is even,

(iv) $x^2 = 0$.

These imply that the linear span R of $\{e_{ij} \mid i, j \in \mathbf{N}\} \cup \{x\}$ is an associative subalgebra of $\mathrm{End}_\Phi V$, and the linear span I of $\{e_{ij} \mid i, j \in \mathbf{N}\}$ is an ideal of R. Moreover, (i) can be used to show that I is a simple algebra. Notice that R is a left primitive algebra (V is a left faithful irreducible R-module).

Using the fact that I is an ideal of R, for any $0 \neq a \in R$, we can consider the homotope $I^{(a)}$ and the local algebra I_a of I at a, even though a need not be in I. Moreover, I_a is simple since I is simple: If $0 \neq \overline{L}$ is an ideal of I_a, $\overline{L} = L/\mathrm{Ker}_I\, a$, where L is an ideal of $I^{(a)}$. Now, $\overline{L} \neq 0$ implies that there exists $y \in L$ such that $aya \neq 0$. By semiprimeness of I, the ideal $IayaI$ of I is nonzero, hence $I = IayaI$, but $IayaI = I \cdot_a y \cdot_a I \subseteq L$ and $I = L$, i.e., $I_a = \overline{L}$. We have shown that I_a does not have proper ideals. On the other hand, I_a is not trivial since, otherwise, we would have $I_a^{(2,a)} = 0$, i.e., $aIaIa = 0$, which is impossible by primeness of R.

Put $a = b + \lambda x \in R$, where $b \in I$, $\lambda \in \Phi$. We can write $b = \sum_{i,j=1}^{n} \alpha_{ij} e_{ij}$ ($\alpha_{ij} \in \Phi$), for some $n \in \mathbf{N}$. By (i), the idempotent $e = e_{11} + \cdots + e_{nn}$ of R satisfies

$$eb = be = b. \tag{v}$$

Moreover,

$$xex = x(e_{11} + \cdots + e_{nn})x = (e_{21} + 0 + e_{43} + 0 + \cdots)x \text{ (by (ii))} = 0 \tag{vi}$$

by (iii). Thus

$$\begin{aligned} axa &= (b + \lambda x)x(b + \lambda x) = bxb \text{ (by (iv))} = bexeb \text{ (by (v))} \\ &= (b + \lambda x)exe(b + \lambda x) \text{ (by (vi))} = aexea. \end{aligned} \tag{vii}$$

Define $\varphi : I^{(a)} \to R_a$ by $y \mapsto y + \mathrm{Ker}_R\, a$. It is clear that φ is an algebra homomorphism. It is clear that $\mathrm{Ker}\,\varphi = \mathrm{Ker}_I\, a$, so that φ induces $\overline{\varphi} : I_a \to R_a$ given by $y + \mathrm{Ker}_I\, a \mapsto y + \mathrm{Ker}_R\, a$ which is an algebra monomorphism. We will show that $\overline{\varphi}$ is an isomorphism by proving that φ is surjective: For any $z \in R$, we can write $z = y + \mu x$, where $y \in I$, $\mu \in \Phi$. Hence,

$$\begin{aligned} aza &= a(y + \mu x)a = aya + \mu axa = aya + \mu aexea \text{ (by (vii))} \\ &= a(y + \mu exe)a. \end{aligned} \tag{viii}$$

Thus, $z - (y + \mu exe) \in \mathrm{Ker}_R\, a$, i.e., $z + \mathrm{Ker}_R\, a = (y + \mu exe) + \mathrm{Ker}_R\, a = \varphi(y + \mu exe)$ (notice that $y + \mu exe \in I$ since $y, e \in I$).

We have shown R_a is isomorphic to I_a, hence R_a is simple, and this happens for any $0 \neq a \in R$. At the same time, R is not simple since I is a proper ideal of R.

The above associative example can be turned into a Jordan one by symmetrization: Let $J = R^{(+)}$. Now J is a Jordan algebra which is not simple, since I is a proper ideal of J, but, for any $0 \neq a \in J$, $J_a = (R^{(+)})_a = (R_a)^{(+)}$ is simple by using [16, Theorem 4] since R_a is simple.

Similarly $V(J)$ (resp. the underlying triple system of J) is a Jordan pair (resp. a Jordan triple system) which is not simple since $V(I)$ (resp. I) is a proper ideal and, for any $0 \neq a \in V(J)^{-\sigma}$, $\sigma = \pm$ (resp. $0 \neq a \in J$) $V(J)^{\sigma}_a = J_a$ (resp. J_a) is simple.

Notice that the above example satisfies $R^2 = R^3 \neq R$, $J^3 \neq J$. The following results show that this is precisely the obstacle to having converses of (2.1), (2.5), (2.6), and (3.5). Though our purpose is studying Jordan systems, we have included in this section, for the sake of completeness, associative versions of the Jordan results. The associative and Jordan proofs are very similar with the exception of small difficulties concerning hearts, so we will deal with associative and Jordan systems at the same time.

4.2. LEMMA. *Let J be a strongly prime* (*resp. prime*) *Jordan* (*resp. associative*) *triple system such that there exists $0 \neq a \in J$ at which the local algebra J_a is simple. Then the heart* Heart J *of J is not zero. Moreover, $P_aJ \subseteq$* Heart J (*resp. $aJa \subseteq$* Heart J).

Proof. Let $0 \neq I$ be an ideal of J, a Jordan triple system under the hypothesis. Since J is strongly prime, $P_aI \neq 0$ (otherwise $a \in \mathrm{Ann}_J\, I = 0$ [18, 1.6, 1.7]), and $0 \neq I_a = I^{(a)}/\mathrm{Ker}_I\, a$ (which makes sense since I is an ideal of J though $a \notin I$) is isomorphic to the nonzero ideal $(I + \mathrm{Ker}_J\, a)/\mathrm{Ker}_J\, a$ of J_a. By simplicity of J_a, $I + \mathrm{Ker}_J\, a = J$ hence $P_aJ = P_a(I + \mathrm{Ker}_J\, a) = P_aI \subseteq I$, and we have shown that P_aJ is contained in every nonzero ideal of I, i.e., $P_aJ \subseteq \mathrm{Heart}\, J$. But P_aJ is nonzero since J is nondegenerate.

The above proof holds in the associative case, replacing [18] by [1, 1.19], strong primeness by primeness, nondegeneracy by semiprimeness, and P_aJ by aJa. ∎

4.3. *Semi-ideals in Jordan Triple Systems*. Recall the notion of *semi-ideal* I of a Jordan triple system J: I is a Φ-submodule of J such that $P_IJ + \{J, J, I\} + P_JP_JI \subseteq I$.

4.4. LEMMA. *Let J be a semiprime associative or Jordan triple system such that $0 \neq H :=$* Heart J. *Then $H = H^3$* (*recall that in the Jordan case H^3 means P_HH, while in the associative case H^3 is HHH*).

Proof. Let us assume that J is a Jordan triple system. Since H is an ideal of J, the cube H^3 of H is a semi-ideal of J by [19, 6.2], hence $P_H H^3 = P_H P_H H$ is a semi-ideal of J again by [19, 6.2], and the outer hull $P_J P_H P_H H + P_H P_H H$ of $P_H P_H H$ is an ideal of J (see [1, 2.13]). Moreover, $P_J P_H P_H H + P_H P_H H \neq 0$ by semiprimeness of J [1, 2.16], hence $H \subseteq P_J P_H P_H H + P_H P_H H$. But obviously $P_J P_H P_H H + P_H P_H H \subseteq H$ since H is an ideal of J, and we have shown

$$P_J P_H P_H H + P_H P_H H = H. \tag{i}$$

Now $P_J P_H P_H H \subseteq P_J P_J P_H H \subseteq P_H H$ since $P_H H$ is a semi-ideal of J, and also $P_H P_H H \subseteq P_H H$, so that (i) implies that $H \subseteq P_H H$, which yields $H = P_H H$.

If J is an associative system, the Jordan argument applies replacing $P_H H$ by HHH. Here, $JHHHHHJ + HHHHH$ is obviously an ideal of J, and also $JHHHHHJ + HHHHH \neq 0$ by semiprimeness of J. As above, this implies $H \subseteq JHHHHHJ + HHHHH \subseteq HHH \subseteq H$, just using that H is an ideal of J. ∎

4.5. *Remark.* In the associative case, (4.4) can be strengthened by an Andrunakievich like argument to show that if R is a semiprime associative triple system where $H = \text{Heart } R \neq 0$, then H is a simple triple system: Indeed, given a nonzero ideal L of H, the ideal $\tilde{L} = LLLLL + LLLLLRR + RRLLLLL + RLLLLLR + RRLLLLLRR$ of R generated by $LLLLL$ is nonzero by semiprimeness of R and also

$$\begin{aligned}\tilde{L} &= LL(LLL) + LL(LLLRR) + (RRLLL)LL \\ &\quad + (RLL)L(LLR) + (RRL)L(LLLRR) \\ &\subseteq LHH + LHH + HHL + HLH + HLH \subseteq L.\end{aligned}$$

But $H \subseteq \tilde{L}$ since $\tilde{L}$ is a nonzero ideal of R, hence $L = H$, and we have shown that H does not have proper ideals.

4.6. THEOREM. (i) *Let J be a Jordan triple system (resp. algebra) such that for any $0 \neq a \in J$, J_a is a simple algebra. Then $H :=$ Heart $J = J^3 = H^3$ is a simple Jordan triple system (resp. algebra).*

(ii) *Let R be an associative triple system (resp. algebra) such that for any $0 \neq a \in R$, R_a is a simple algebra. Then $H :=$ Heart $R = H^3$ is a simple associative triple system containing $P_R R$ (resp. $U_R R$), i.e., the linear span of the elements aba, for $a, b \in R$.*

Proof. (i) Let J be a Jordan triple system under the hypothesis. For any $0 \neq a \in J$, J_a is simple, hence strongly prime (2.4)(ii), thus $V(J)$ is strongly prime by [6, 3.2], and J is strongly prime by (0.3)(i)(b). By (4.2),

$H \neq 0$ and, by (4.4), $H = H^3$. Moreover, $P_a J \subseteq H$ for any $0 \neq a \in J$ by (4.2), i.e., $P_J J \subseteq H$. Obviously, $H = H^3 \subseteq P_J J$, and we just need to show that H is simple.

Now, H is strongly prime by [18, 2.5] since J is strongly prime, hence, for any $0 \neq h \in H$, H_h is nonzero. Moreover, H_h is isomorphic to the ideal $(H + \mathrm{Ker}_J h)/\mathrm{Ker}_J h$ of J_h. By simplicity of J_h, $J_h = (H + \mathrm{Ker}_J h)/\mathrm{Ker}_J h$. Since H_h is isomorphic to J_h, we obtain that H_h is simple for any $0 \neq h \in H$. By (4.2), $P_H H$ is contained in every nonzero ideal of H. Since $P_H H = H$, we have that H does not have proper ideals.

If J is a Jordan algebra, the above applies to the underlying triple system of J to show that the heart $\mathrm{Heart}_{\mathrm{triple}}(J) = U_J J$ is a simple triple system. Since $U_J J$ is an algebra ideal of J, $\mathrm{Heart}_{\mathrm{triple}}(J)$ coincides with the algebra heart Heart J of J. Moreover, Heart J is simple as an algebra by (3.4).

(ii) If R_a is simple for any $0 \neq a \in R$ then $(R^{(+)})_a \cong R_a^{(+)}$ is simple for any $0 \neq a \in R$ by [16, Theorem 4], so that the Jordan triple system $R^{(+)}$ is strongly prime as in (i), and R is prime by [4, 5.8(i)]. Hence, $P_R R \subseteq H \neq 0$ by (4.2), H is simple by (4.5), and $H = HHH$ by (4.4).

In the algebra case we obtain the result for $H = \mathrm{Heart}_{\mathrm{triple}}(R)$. But $RH = RHHH = (RH)HH \subseteq RHH \subseteq H$ and $HR = HHHR = HH(HR) \subseteq HHR \subseteq H$ show that H is also an algebra ideal, so that H coincides with the algebra heart Heart R of R, and it is simple as an algebra by (3.1). ∎

By using the functor $T(\ )$ we obtain the following pair versions of (4.6).

4.7. THEOREM. (i) *Let V be a nondegenerate Jordan pair such that for some $\sigma = +$ or $-$, V_a^σ is a simple algebra, for any $0 \neq a \in V^{-\sigma}$. Then $H :=$ Heart $V = H^3$ is a simple Jordan pair and $H^\sigma = Q_{V^\sigma} Q_{V^{-\sigma}} V^\sigma$, $H^{-\sigma} = Q_{V^{-\sigma}} V^\sigma$.*

(ii) *Let R be a semiprime associative pair such that for some $\sigma = +$ or $-$, R_a^σ is a simple algebra, for any $0 \neq a \in R^{-\sigma}$. Then $H :=$ Heart $R = H^3$ is a simple associative pair and $R^\sigma(Q_{R^{-\sigma}} R^\sigma) R^\sigma \subseteq H^\sigma$, $Q_{R^{-\sigma}} R^\sigma \subseteq H^{-\sigma}$.*

Proof. Put $\sigma = +$, for example.

(i) Notice that, for every $0 \neq a \in V^-$, V_a^+ is strongly prime by (2.4)(ii), hence V is strongly prime by [6, 3.2], and $T(V)$ is strongly prime by (0.3)(ii)(b). Since $T(V)_{0 \oplus a} \cong V_a^+$ is simple for any $0 \neq a \in V^-$, $\mathrm{Heart}(T(V)) \neq 0$ by (4.2).

Since V is semiprime (it is nondegenerate), [1, p. 230] shows that every nonzero ideal of $T(V)$ contains a polarized ideal $T(I)$ for some nonzero ideal I of V. This readily implies $\mathrm{Heart}(T(V)) = T(H)$, where $H = (H^+, H^-) =$ Heart V. In particular, $0 \neq H$. Moreover, by (4.4) applied to $T(V)$, $T(H) = T(H)^3$, i.e., $H = H^3$.

Again, since $T(V)_{0\oplus a} \cong V_a^+$ is simple for any $0 \neq a \in V^-$, $P_{0\oplus a}T(V) \subseteq \text{Heart}(T(V))$ by (4.2), i.e., $Q_{V^-}V^+ \subseteq H^-$. But $H = H^3$ implies $H^- = Q_{H^-}H^+ \subseteq Q_{V^-}V^+$ and we have $H^- = Q_{V^-}V^+$. Thus

$$Q_{V^+}Q_{V^-}V^+ = Q_{V^+}H^- \subseteq H^+ = Q_{H^+}H^- = Q_{H^+}Q_{H^-}H^+ \subseteq Q_{V^+}Q_{V^-}V^+,$$

using $H = H^3$, and we have shown $H^+ = Q_{V^+}Q_{V^-}V^+$.

We just need to prove that H is simple. As for triple systems, H is strongly prime by [18, 2.5] since V is strongly prime, hence, for any $0 \neq h \in H^-$, $H_h^+ \neq 0$. Also H_h^+ is isomorphic to the ideal $(H^+ + \text{Ker}_V\, h)/\text{Ker}_V\, h$ of V_h^+. By simplicity of V_h^+, $(H^+ + \text{Ker}_V\, h)/\text{Ker}_V\, h = V_h^+$ and H_h^+ is simple. Thus, what we have already seen applies to H showing that Heart $H = (Q_{H^+}Q_{H^-}H^+, Q_{H^-}H^+)$. But $(Q_{H^+}Q_{H^-}H^+, Q_{H^-}H^+) = (H^+, H^-)$ using $H = H^3$. We have proved $H = \text{Heart}\, H$, hence H is simple.

(ii) If R_a^+ is simple for any $0 \neq a \in R^-$, then $(R^{(+)})_a^+ \cong (R_a^+)^{(+)}$ is simple by [16, Theorem 4], hence strongly prime by (2.4)(ii), and $R^{(+)}$ is strongly prime by [6, 3.2] since $R^{(+)}$ is nondegenerate by semiprimeness of R. Hence R is prime by [4, 4.5(i)] and the argument of (i) applies verbatim replacing V by R to show that $H = \text{Heart}\, R \neq 0$ and $H = H^3$. Moreover H is simple by (0.3)(ii)(c) since $T(H) = \text{Heart}\, T(R)$ and Heart $T(R)$ is simple by (4.5). From (4.2) applied to $T(R)$ we obtain $Q_{R^-}R^+ \subseteq H^-$ and $R^+(Q_{R^-}R^+)R^+ \subseteq H^+$ since H is an ideal of R. ∎

4.8. COROLLARY. (i) *Let V be a Jordan pair such that V_a^σ is a simple algebra, for any $0 \neq a \in V^{-\sigma}$, $\sigma = \pm$. Then $H :=$ Heart $V = V^3 = H^3$ is a simple Jordan pair (here $V^3 = (Q_{V^+}V^-, Q_{V^-}V^+)$).*

(ii) *Let R be an associative pair such that R_a^σ is a simple algebra, for any $0 \neq a \in R^{-\sigma}$, $\sigma = \pm$. Then $H :=$ Heart $R = H^3$ is a simple associative pair containing $(Q_{R^+}R^-, Q_{R^-}R^+)$.*

4.9. COROLLARY. *Let V be a Jordan pair. Then, the following are equivalent:*

(i) *V is simple,*

(ii) *$V = V^3$ and V_a^σ is simple for any $0 \neq a \in V^{-\sigma}$, $\sigma = \pm$.*

(iii) *$V = V^3$, V is nondegenerate, and V_a^+ is simple for any $0 \neq a \in V^-$,*

(iv) *$V = V^3$, V is nondegenerate, and V_a^- is simple for any $0 \neq a \in V^+$.*

Proof. (i) ⇒ (ii). This is (2.5) plus the fact that V^3 is a nonzero ideal of V.

(ii) ⇒ (iii). This is obvious.

(iii) $\Rightarrow$ (i). By (4.7)(i), the heart H of V is simple and $H = (H^+, H^-) = (Q_{V^+}Q_{V^-}V^+, Q_{V^-}V^+)$. But $V = V^3$ implies

$$(Q_{V^+}Q_{V^-}V^+, Q_{V^-}V^+) = (Q_{V^+}V^-, V^-) = (V^+, V^-) = V,$$

$V = H$, and V is simple.

The equivalence between (i), (ii), and (iv) follows as above, exchanging $+$ and $-$. ∎

Putting together (3.5), (4.6)(i), and the fact that for any Jordan algebra $J^3 = U_J J$ is an ideal of J yields the following result.

4.10. COROLLARY. *A Jordan algebra J is simple if and only if $J = J^3 = U_J J$ and J_a is simple for any $0 \neq a \in J$.*

4.11. *Remarks.* (i) We do not know whether the heart of a nondegenerate Jordan triple system is simple if it is nonzero (a Jordan analogue of (4.5)). An affirmative answer to that question would simplify the proofs of the Jordan parts of the above results.

(ii) For an associative algebra, pair or triple system R under the hypothesis of (4.6)(ii), (4.7)(ii), or (4.8)(ii), we do not have a description of Heart R in terms of associative powers of R as in the Jordan case. Such a description would provide an analogue for associative pairs (resp. algebras) of (4.9) (resp. (4.10)).

5. SUBQUOTIENT INHERITANCE OF SIMPLICITY IN JORDAN PAIRS

The proof of (2.5) can be modified to study the inheritance of simplicity by subquotients.

5.1. THEOREM. *Let V be a simple Jordan pair, $0 \neq M \subseteq V^\sigma$ be an inner ideal of V, and let S be the subquotient of V determined by M. Then, the heart $H =$* Heart *S of S is simple. Moreover $H^\sigma = Q_M V^{-\sigma} = Q_{S^\sigma} S^{-\sigma}$ and $H^{-\sigma} = S^{-\sigma}$.*

Proof. Put, for example, $\sigma = +$.

Since V is nondegenerate by (2.4)(ii), $0 \neq Q_M V^-$. Thus

$$(Q_M V^-, V^-/\mathrm{Ker}\, M) = (Q_{S^+}S^-, S^-)$$

is a nonzero ideal of S.

Notice that S is strongly prime by [6, 3.2] since V is strongly prime by (2.4)(ii).

Let $0 \neq I$ be an ideal of S. We have $I^+ \subseteq M$ and $I^- = L/\operatorname{Ker} M$ for some Φ-submodule L of V^- containing $\operatorname{Ker} M$. By nondegeneracy of S, $I^+ \neq 0$ and $I^- \neq 0$, hence $\operatorname{Ker} M$ is strictly contained in L. Let $x \in L \setminus \operatorname{Ker} M$. By nondegeneracy of V, $\operatorname{Ker} M = \{x \in V^- \mid Q_M x = 0\}$ [15, 2.4], and we can find $a \in M$ such that $Q_a x \neq 0$.

Define $K^- := Q_x Q_a V^-$, $K^+ := Q_a L$, which are inner ideals of V;

$$Q_{K^-} V^+ = Q_{Q_x Q_a V^-} V^+ = Q_x Q_a Q_{V^-} Q_a Q_x V^+ \text{ (by JP3) } \subseteq Q_x Q_a V^- = K^-,$$

$$\begin{aligned} Q_{K^+} V^- = Q_{Q_a L} V^- &= Q_a Q_L Q_a V^- \text{ (by JP3) } \subseteq Q_a Q_{L/\operatorname{Ker} M} Q_M (V^-/\operatorname{Ker} M) \\ &= Q_a Q_{I^-} Q_{S^+} S^- \subseteq Q_a I^- = Q_a L = K^+, \end{aligned}$$

since I is an ideal of S.

Moreover $Q_{K^-} K^+ \neq 0$: indeed, suppose

$$0 = Q_{K^-} K^+ = Q_{Q_x Q_a V^-} Q_a L = Q_x Q_a Q_{V^-} Q_a Q_x Q_a L,$$

by JP3, hence,

$$Q_{Q_a Q_x Q_a V^-} I^- = Q_a Q_x Q_a Q_{V^-} Q_a Q_x Q_a L = 0 \tag{i}$$

again by JP3 (notice that $Q_a Q_x Q_a V^- \subseteq Q_M V^- \subseteq M = S^+$). By strong primeness of S, $\operatorname{Ann}_S(I) = 0$ (see [18, 1.6]). But (i) implies $Q_a Q_x Q_a V^-$ is contained in $\operatorname{Ann}_S(I)$ (cf. [18, 1.7]). We then have $Q_a Q_x Q_a V^- = 0$, i.e., $Q_{Q_a x} V^- = 0$ where $Q_a x \neq 0$, which is impossible by nondegeneracy of V.

It is clear that $K = K^+ \oplus K^-$ is an inner ideal of $T(V)$. Moreover, $P_K K = Q_{K^+} K^- \oplus Q_{K^-} K^+ \neq 0$. Since $T(V)$ is simple by (0.3)(ii)(c), we have that the ideal of $T(V)$ generated by $P_K K$ is the whole $T(V)$. By (2.2), $T(V)$ is locally nilpotent $\operatorname{mod} K + P_{T(V)} K$, hence $T(V) = K + P_{T(V)} K$ using (2.3). But

$$K + P_{T(V)} K = (K^+ + Q_{V^+} K^-) \oplus (K^- + Q_{V^-} K^+),$$

hence

$$V^- = K^- + Q_{V^-} K^+ = Q_x Q_a V^- + Q_{V^-} Q_a L \subseteq Q_L Q_M V^- + Q_{V^-} Q_M L,$$

hence

$$V^-/\operatorname{Ker} M \subseteq Q_{I^-} Q_{S^+} S^- + Q_{S^-} Q_{S^+} I^- \subseteq I^-$$

since I is an ideal of S, and we have shown $S^- = I^-$.

Now

$$Q_M V^- = Q_M (V^-/\operatorname{Ker} M) = Q_{S^+} S^- = Q_{S^+} I^- \subseteq I^+.$$

We have shown that every nonzero ideal of S contains the nonzero ideal $(Q_M V^-, S^-)$, i.e., $H = (Q_M V^-, S^-)$.

On the other hand, for any $b \in S^+$, $S_b^- \cong V_b^-$ is simple by (2.5). Hence (4.7)(i) applied to S yields H is simple. ▮

Notice that if we apply (4.7)(i) to describe the heart H of S, we just get $H = (Q_{S^+} S^-, Q_{S^-} Q_{S^+} S^-)$.

The following result answers the question on the inheritance of simplicity by subquotients posed in [15, 2.8(i)].

5.2. COROLLARY. *Let V be a simple Jordan pair, $0 \neq M \subseteq V^\sigma$ be an inner ideal of V, and let S be the subquotient of V determined by M. Then, S is simple if and only if $M = Q_M V^{-\sigma}$.*

Proof. Use (5.1), and the obviously general fact that a simple pair coincides with its heart. ▮

The previous result yields a characterization of simple regular pairs in terms of the simplicity of their subquotients:

5.3. COROLLARY. *A Jordan pair V is simple and regular if and only if all of its subquotients determined by nonzero inner ideals are simple.*

Proof. If V is regular, then $a \in Q_a V^{-\sigma}$ for any $a \in V^\sigma$, $\sigma = \pm$. This readily implies $M = Q_M V^{-\sigma}$, for any inner ideal $M \subseteq V^\sigma$. Hence, by (5.2), the subquotients of V determined by nonzero inner ideals inherit simplicity from V.

Conversely, if all subquotients determined by nonzero inner ideals are simple, then they are strongly prime by (2.4)(ii), and V is strongly prime by [6, 3.2]. Moreover, V is simple since V is the subquotient determined by V^+ or V^- ($\operatorname{Ker} V^\sigma = 0$ [15, 1.4] by nondegeneracy of V), and we just need to show that V is regular. Let $0 \neq a \in V^+$, and let $(a) = \Phi a + Q_a V^-$ be the inner ideal of V generated by a. Since $(a) \neq 0$, the subquotient of V determined by (a) is simple, hence $(a) = Q_{(a)} V^-$ by (5.2) and, in particular,

$$a \in Q_{(a)} V^- = Q_{\Phi a + Q_a V^-} V^- \subseteq Q_a V^- + \{a, V^-, Q_a V^-\} + Q_{Q_a V^-} V^- \subseteq Q_a V^-$$

by JP1 and JP3. Similarly every element in V^- is regular. ▮

5.4. *Remark.* In [22, 0.4], the formal coincidence between the notions of local algebras and subquotients is made explicit in the following terms: Given a Jordan pair V, $a \in V^{-\sigma}$, $[a] = Q_a V^\sigma$, the subquotient S determined by the inner ideal $[a]$ is isomorphic to $V(V_a^\sigma)$. Since, in general, not all subquotients of a simple pair are simple, we cannot get (2.5) as a consequence of (5.2) and (3.4). Instead, we can proceed conversely to exploit the above results in order to obtain information about $[a]$: If V is simple, then V_a^{σ} is simple by (2.5), hence $S \cong V(V_a^{\sigma})$ is simple by (3.4), and $Q_{[a]}V^{\sigma} = [a]$, for any $0 \neq a \in V^{-\sigma}$, by (5.2).

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