Three-dimensional dynamic Green's functions in transversely isotropic bi-materials

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By virtue of a complete representation using two displacement potentials, an analytical derivation of the elastodynamic Green's functions for a linear elastic transversely isotropic bi-material full-space is presented. Three-dimensional point-load Green's functions for stresses and displacements are given in complex-plane line-integral representations. The formulation includes a complete set of transformed stress–potential and displacement–potential relations, within the framework of Fourier expansions and Hankel integral transforms, that is useful in a variety of elastodynamic as well as elastostatic problems. For numerical computation of the integrals, a robust and effective methodology is laid out which gives the necessary account of the presence of singularities including branch points and pole on the path of integration. As illustrations, the present Green's functions are analytically degenerated to the special cases such as half-space, surface and full-space Green's functions. Some typical numerical examples are also given to show the general features of the bi-material Green's functions.

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1. Introduction

The increased use of composite materials in engineering applications in recent decades has become an incentive for extensive research, both basic and applied, into various failure modes of such materials. It was also recognized that the performance of composite materials was closely related to the effects occurring at the interface between the different components of the composite. Issues such as interfacial fracture and crack problems in bi-material systems are at the forefront of many investigations (Lambros and Rosakis, 1995). The reader is referred to Wu et al. (2003) and Prasad et al. (2005) for an extensive list of work in this area. The problem of accurate prediction of interfacial stresses is also a major concern for researchers working on composite laminates (Noor and Burton, 1990). In the field of geomechanics and foundation engineering, a thin embedded inclusion can serve as a basic model for an anchoring region, which can be created by the injection of a cementitious material. The evaluation of the elastic stiffness of these embedded anchoring devices is of particular interest to predict the failure at either the interface or adjacent regions (Selvadurai, 2003). As an idealized model of the anchoring region, Selvadurai (2003) studied the mechanics of a loaded rigid disc that is embedded in bonded contact at the interface between two dissimilar isotropic elastic media. Such a configuration is also a useful idealization for the inhomogeneity or embedded footing problem.
A powerful approach for the analysis of the foregoing mechanics problems is the integral equation or boundary element methods. Central to their success is the availability of suitable Green's functions. Exemplified by the work of Kelvin (Love, 1944), Boussinesq (1885), and Lamb (1904), the determination of the static and dynamic Green's functions for isotropic elastic solids have been the subject of many investigations. By virtue of a method of displacement potentials, for example, Pak (1987) studied an isotropic half-space subjected to buried time-harmonic load analytically. This work is extended by Pak and Guzina (2002) who obtained explicit and effective expressions for the dynamic Green's functions for an isotropic multilayered half-space. To realize a rigorous treatment of the singularities of their dynamic fundamental solutions, static point-load Green's functions for an elastic bi-material full-space was utilized (Guzina and Pak, 1999). For many modern technological applications, however, the isotropic material model can sometimes be only a crude approximation. Numerous innovative, smart, and intelligent materials such as composites, piezoelectrics are anisotropic, and in application should be modeled at least as a transversely isotropic or orthotropic material. The static Green's function problem for the anisotropic and transversely isotropic materials has been studied. Pan and Chou (1976, 1979) for instance derived the static point-load Green's functions for a transversely isotropic full-space. The approach which was introduced by Buchwald (1961), has been used, with small changes, by Mirsky (1965) and Rajapakse and Wang (1993). Mirsky (1965) studied the problem of propagation of longitudinal waves in transversely isotropic cylinders based on an extension of Buchwald's work. Rajapakse and Wang (1993) presented three-dimensional dynamic Green's functions for a transversely isotropic elastic half-space subjected to interior time-harmonic loading. While the issue of completeness of their representation was not addressed, they were able to reduce the equations of motion by means of three potential functions proposed by Buchwald (1961) to a set of two coupled partial differential equations and one separated partial differential equation. Burridge (1971) investigated a half-space of anisotropic elastic material subjected to an impulsive line of traction. Tewary (1995) studied the transient displacement field due to a point source in infinite and semi-infinite anisotropic cubic solids. Wang and Achenbach (1996), based on representing the wave field by a superposition of time-transient plane waves, also introduced a method to construct solutions for elastic waves generated in a half-space and solved Lamb's problem for a solid of general anisotropy. Every et al. (1997) derived the integral expressions for the displacement response tensor of a semi-infinite anisotropic elastic continuum of unrestricted symmetry to a concentrated force suddenly applied to its surface. Yang et al. (2004) studied three-dimensional Green's function for an anisotropic bi-material full-space, and pointed out that the determination of the Green's function pertinent to the time-harmonic loadings is very complex and presented only the solution for the steady state case under moving load with velocity v. By means of a complete representation using two displacement potentials, Rahimian et al. (2007) presented a new and efficient analytical formulation to obtain the response of a three-dimensional transversely isotropic half-space to general time-harmonic loading.

The objective of the present work is to utilize the elastodynamic potential method presented by Rahimian et al. (2007) to derive explicit expressions for the three-dimensional dynamic Green's functions in transversely isotropic bi-materials. The complete set of point-load Green's functions of displacements and stresses are given in terms of complex-plane line-integral representations. For their evaluation, a numerical scheme, which gives the necessary account of the presence of singularities including branch points and pole on the path of integration is implemented. The formulation and numerical implementation are examined and verified by a comparison of their degenerate forms to some benchmark solutions such as the static Green's functions for a homogeneous transversely isotropic full-space, and dynamic Green's functions for a homogeneous transversely isotropic half-space. With the aid of the Green's functions presented herein, treatments by boundary-integral-equation formulations for the dynamic analysis of interfacial inclusions and cracks in transversely isotropic bi-materials should be facilitated. They should also be useful in a number of foundation–soil interaction and earthquake engineering problems.

2. Governing equations in displacement potentials

The equations of time-harmonic motion for a homogeneous transversely isotropic elastic solid in terms of displacements and in the absence of body forces can be expressed as (Lekhnit'ski, 1963)
Here may express (Sneddon, 1951)

\[ \nabla_i^2 F(r, \theta, z) + \frac{\partial^2 F(r, \theta, z)}{\partial z^2} \] \nabla^2 \chi = 0. \tag{5}

\[ \nabla_i^2 = \nabla^2_{r \theta} + \frac{1}{\mu_i c_{66}} \rho \omega^2 \frac{\partial^2}{\partial z^2}, \quad i = 0, 1, 2, \tag{7} \]

and

\[ \delta = \frac{-1}{\mu_0 c_{44}} - \left( \frac{1}{\mu_1 s_1^2} \right) + \frac{1}{\mu_1 c_{11}} \left( 1 + \frac{c_{33} c_{44}}{c_{11} c_{13}} \right). \tag{8} \]

Here \( s_0 = 1/\sqrt{2} \); \( s_1 \) and \( s_2 \) are the roots of following equation, which in view of the positive-definiteness of the strain energy are not zero or pure imaginary numbers (Lekhnitskii, 1963):

\[ c_{33} c_{44} s_2^4 + (c_{11}^2 + 2c_{13} c_{44} - c_{11} c_{13}) s_1^2 + c_{11} c_{44} = 0. \tag{9} \]

By virtue of Fourier expansion with respect to the angular coordinate \( \theta \) one may express (Sneddon, 1951)

\[ [F(r, \theta, z), \chi(r, \theta, z)] = \sum_{m=-\infty}^{\infty} [F_m(r, z), \chi_m(r, z)] e^{im\theta}, \tag{10} \]

with similar expressions for the displacement and stress components. Moreover, utilizing the \( m \)th order Hankel transform pair for sufficiently regular function \( f(r, z) \) with respect to the radial coordinate as (Sneddon, 1972)
\[ f(r, z) = \int_0^\infty f'(r) \bar{f}_m(r \xi) \, dr, \]
\[ \bar{F}_m(z) = \int_0^\infty \bar{f}(r, z) \bar{f}_m(r \xi) \, d\xi, \]

the following ordinary differential equations for \( F \) and \( \chi \) can be obtained

\[ \left( \nabla^2 - \omega^2 \frac{d^2}{dz^2} \right) \bar{F}_m(z) = 0, \]
\[ \nabla^2 \bar{F}_m = 0, \]

where

\[ \nabla^2 \bar{F}_m = \frac{\rho \omega^2}{\mu c_{66}} \xi^2 + \frac{1}{s_i^2} \frac{d^2}{dz^2}, \quad i = 0, 1, 2. \]

The general solutions of Eqs. (12) and (13) can be written as

\[ \bar{F}_m(\xi, z) = A_m(\xi) e^{i \xi z} + B_m(\xi) e^{-i \xi z} + C_m(\xi) e^{i \xi z} + D_m(\xi) e^{-i \xi z}, \]
\[ \bar{F}_m(\xi, z) = E_m(\xi) e^{i \xi z} + F_m(\xi) e^{-i \xi z}, \]

where

\[ \lambda_1 = \sqrt{a \xi^2 + b + \frac{1}{2} \sqrt{c \xi^4 + d \xi^2 + e}}, \]
\[ \lambda_2 = \sqrt{a \xi^2 + b - \frac{1}{2} \sqrt{c \xi^4 + d \xi^2 + e}}, \]
\[ \lambda_3 = \sqrt{s_0 \xi^2 - \frac{\rho \omega^2}{c_{66}}}, \]
\[ a = \frac{1}{2} (s_1^2 + s_2^2), \quad b = -\frac{1}{2} \rho \omega^2 \left( \frac{1}{c_{33}} + \frac{1}{c_{44}} \right), \quad c = (s_2^2 - s_1^2)^2, \]
\[ d = -2 \rho \omega^2 \left[ \frac{1}{c_{33}} + \frac{1}{c_{44}} \right] \left( s_1^2 + s_2^2 \right) - 2 \frac{c_{11}}{c_{33}} \left( \frac{1}{c_{11}} + \frac{1}{c_{44}} \right), \]
\[ e = \rho^2 \omega^4 \left( \frac{1}{c_{33}} - \frac{1}{c_{44}} \right)^2, \]

and \( A_m, \ldots, F_m \) are constants of integration to be determined from boundary conditions. The radicals \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are made single-valued by specifying the branch cuts emanating from the branch points \( \xi_1 = \pm \omega \sqrt{\rho/c_{11}}, \xi_2 = \pm \omega \sqrt{\rho/c_{44}} \) and \( \xi_3 = \pm \omega \sqrt{\rho/c_{66}} \) on the complex \( \xi \)-plane (see Fig. 1) such that the real parts of \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are always non-negative.

For a transversely isotropic solid, there will be in general three kinds of body waves. The wave numbers pertinent to these waves are associated with the above three branch points at \( \xi_i (i = 1, 2, 3) \), where \( \xi_1 = k_p \equiv \omega \sqrt{\rho/c_{11}}, \xi_2 = k_{SV} \equiv \omega \sqrt{\rho/c_{44}}, \) and \( \xi_3 = k_{SH} \equiv \omega \sqrt{\rho/c_{66}} \) correspond to compressional (P), shear vertical (SV), and shear horizontal (SH) waves, respectively.

\[ \text{Im}(\xi) \]
\[ \text{Re}(\xi) \]

**Fig. 1.** Branch cuts for \( \lambda_1, \lambda_2, \) and \( \lambda_3. \)**
As can be seen, for a transversely isotropic solid SH waves would travel with a different velocity from that of SV (Stoneley, 1949), For the case of an isotropic solid these branch points reduce to \( \xi_{i} = k_{0} = \omega \sqrt{\rho/(\lambda + 2\mu)} \), and \( \xi_{s} = \xi_{i} = k_{0} = \omega \sqrt{\rho/\mu} \), where \( \lambda \) and \( \mu \) are the Lamé constants; \( k_{0} \) and \( k_{s} \) are compressional and shear wave numbers, respectively (e.g., Pak, 1987; Pak and Guzina, 2002). By means of Eq. (2) and the identities involving Hankel wave numbers, the transformed displacement–potential relations may be expressed compactly (Rahimian et al., 2007) as

\[
\tilde{u}^m_{z} = \left[ \frac{d^2}{dz^2} + \frac{\rho\omega^2}{c_{66}} - \xi^2 (1 + \alpha) \right] \tilde{F}^m_{z},
\]

while the transformed stress–potential relationships can be written as

\[
\tilde{\tau}^m_{zz} = \frac{d}{dz} \left[ \frac{\xi_{1}c_{13}z^{2} + c_{33}\left( \frac{\rho\omega^2}{c_{66}} - \xi^2 (1 + \alpha) \right) + c_{33}c_{2}d^2}{dz^2} \right] \tilde{F}^m_{z},
\]

\[
\tilde{\tau}^m_{z} = \frac{d}{dz} \left[ \frac{\xi_{2}c_{13}z^{2} + c_{33}\left( \frac{\rho\omega^2}{c_{66}} - \xi^2 (1 + \alpha) \right) + c_{33}c_{2}d^2}{dz^2} \right] \tilde{F}^m_{z},
\]

\[
\tilde{\tau}^m_{r} = \frac{d}{dz} \left[ \frac{\xi_{3}c_{13}z^{2} + c_{33}\left( \frac{\rho\omega^2}{c_{66}} - \xi^2 (1 + \alpha) \right) + c_{33}c_{2}d^2}{dz^2} \right] \tilde{F}^m_{z},
\]

\[
\tilde{\tau}^m_{rr} = 2c_{66} \left( \frac{\tilde{u}_{m}}{r} + \frac{\tilde{u}_{m}}{r} \right) = \frac{d}{dz} \left[ \frac{\xi_{3}c_{13}z^{2} + c_{33}\left( \frac{\rho\omega^2}{c_{66}} - \xi^2 (1 + \alpha) \right) + c_{33}c_{2}d^2}{dz^2} \right] \tilde{F}^m_{z},
\]

With the aid of Eqs. (15), (16), (19) and (20), the imposition of the loading, interfacial and regularity conditions associated with a bi-material solid is greatly facilitated, as will be illustrated in the ensuing sections.

### 3. Statement of the problem

Consider the physical domain of interest to be composed of two dissimilar transversely isotropic elastic half-spaces which are fully bonded across the plane \( z = 0 \) (see Fig. 2). In addition, the common axis of symmetry of both half-spaces is assumed to be normal to the horizontal interface. Fig. 2 depicts a cylindrical coordinate system \( (r, \theta, z) \) in such a way that \( z \)-axis is normal to the horizontal interface of the domain, and so it is the common axis of symmetry of both media. Let the upper half-space \( (z < 0) \) be occupied by medium I and the lower half-space \( (z > 0) \) be occupied by medium II. The material density and the elastic constants of the upper-half-space (referred to as medium I) will be denoted as \( \rho_{I}, c_{ik}^{I} \) and the ones of the lower-half-space (referred to as medium II) as \( \rho_{II}, c_{ik}^{II} \). Hereafter the superscripts I and II denote the quantities in media I and II, respectively. An arbitrary time-harmonic interfacial traction is assumed to be distributed on a finite region \( \Pi_{0} \) which is located at the interface of the domain. The action of this arbitrarily distributed source can be represented as a set of prescribed stress discontinuities across the interface, i.e.

\[
\tau_{m}(r, \theta, 0^-) - \tau_{m}(r, \theta, 0^+) = \begin{cases} P(r, \theta), & (r, \theta) \in \Pi_{0} \\
0, & (r, \theta) \notin \Pi_{0} \end{cases}
\]

\[
\tau_{m}(r, \theta, 0^-) - \tau_{m}(r, \theta, 0^+) = \begin{cases} Q(r, \theta), & (r, \theta) \in \Pi_{0} \\
0, & (r, \theta) \notin \Pi_{0} \end{cases}
\]

\[
\tau_{m}(r, \theta, 0^-) - \tau_{m}(r, \theta, 0^+) = \begin{cases} R(r, \theta), & (r, \theta) \in \Pi_{0} \\
0, & (r, \theta) \notin \Pi_{0} \end{cases}
\]

where \( P(r, \theta), Q(r, \theta), \) and \( R(r, \theta) \) are the specified interfacial traction distributions in radial, angular, and axial directions, respectively; and the harmonic time factor \( e^{i\omega t} \) is suppressed. Consistent with the regularity condition at infinity the general solutions (15) and (16) for \( F \) and \( \chi \) can be rearranged as

\[
\tilde{F}^m_{z}(\xi, z) = A^m_{z}(\xi) e^{k^{F} z} + B^m_{z}(\xi) e^{k^{F} z},
\]

\[
\tilde{\chi}^m_{z}(\xi, z) = C^m_{z}(\xi) e^{k^{F} z},
\]
in medium I, and
\[ F_m(\xi, z) = A_{m}^{\|}(\xi) e^{-i \xi z} + B_{m}^{\|}(\xi) e^{+i \xi z}, \]
\[ \tilde{F}_m(\xi, z) = C_{m}^{\|}(\xi) e^{-i \xi z}, \]  
(24)

in medium II. In the above, \( A_{m}^{\|}, \ldots, C_{m}^{\|} \) are the integration constants to be determined using the boundary conditions. Under the choices of the branches as shown in Fig. 1, the \( e^{-i \xi z} \) terms in medium I and \( e^{+i \xi z} \) terms in medium II become inadmissible due to radiation conditions and are thus omitted. For the general bi-material full-space problem of interest, an exact solution therefore requires the determination of six coefficients. With the aid of (19) and (20), interfacial traction conditions (21) together with the continuity of displacements across the interface provide six equations required for the solution of the six unknown coefficients \( A_{m}^{\|}, \ldots, C_{m}^{\|} \). Substitution of the result into Eq. (19) gives the transformed Fourier components of the displacement field in the form of
\[ \tilde{u}_m^{m-1} - \tilde{u}_m^{m-1} = \gamma_1(\xi, \zeta) \left( \frac{X_m - Y_m}{2c_{44}} \right) + \gamma_2(\xi, \zeta) \left( \frac{X_m + Y_m}{2c_{44}} \right) + \gamma_3(\xi, \zeta) \left( \frac{Z_m}{c_{44}} \right), \]
\[ \tilde{u}_m^{m+1} + \tilde{u}_m^{m+1} = -\gamma_1(\xi, \zeta) \left( \frac{X_m - Y_m}{2c_{44}} \right) + \gamma_2(\xi, \zeta) \left( \frac{X_m + Y_m}{2c_{44}} \right) - \gamma_3(\xi, \zeta) \left( \frac{Z_m}{c_{44}} \right), \]
\[ \tilde{u}_m^{m+1} = \Omega_1(\xi, \zeta) \left( \frac{X_m - Y_m}{2c_{44}} \right) + \Omega_2(\xi, \zeta) \left( \frac{Z_m}{c_{44}} \right). \]  
(26)

Analogously, Eq. (20) yields the transformed Fourier components of the stress field as
\[ \tilde{\sigma}_{m+1}^{*} + \tilde{\sigma}_{m-1}^{*} = \xi \Omega_1(\xi, \zeta) \left( \frac{X_m - Y_m}{2c_{44}} \right) + \xi \Omega_2(\xi, \zeta) \left( \frac{Z_m}{c_{44}} \right) \]
\[ \tilde{\sigma}_{m+1}^{*} - \tilde{\sigma}_{m-1}^{*} = c_{44} \left( \Omega_1(\xi, \zeta) \frac{X_m - Y_m}{2c_{44}} \right) + c_{44} \left( \Omega_2(\xi, \zeta) \frac{Z_m}{c_{44}} \right) \]
\[ \tilde{\sigma}_{m+1}^{*} + 2c_{66} \left( \frac{u_m}{r} + i \frac{u_m^{*}}{r} \right) = c_{13} \left( \frac{d \Omega_1}{dz} - c_{11} \frac{\zeta \gamma_1}{2} \right) \left( \frac{X_m - Y_m}{2c_{44}} \right) + c_{13} \left( \frac{d \Omega_2}{dz} - c_{11} \frac{\zeta \gamma_2}{2} \right) \left( \frac{Z_m}{c_{44}} \right) \]
\[ \tilde{\sigma}_{m+1}^{*} - 2c_{66} \left( \frac{u_m}{r} + i \frac{u_m^{*}}{r} \right) = c_{13} \left( \frac{d \Omega_1}{dz} - c_{11} \frac{\zeta \gamma_1}{2} \right) \left( \frac{X_m - Y_m}{2c_{44}} \right) + c_{13} \left( \frac{d \Omega_2}{dz} - c_{11} \frac{\zeta \gamma_2}{2} \right) \left( \frac{Z_m}{c_{44}} \right) \]
\[ \tilde{\sigma}_{m+1}^{*} + 2c_{66} \left( \frac{u_m}{r} - i \frac{u_m^{*}}{r} \right) = ic_{66} \frac{\zeta \gamma_2}{2} \left( \frac{X_m + Y_m}{2c_{44}} \right). \]  
(27)
In the above
\[ \gamma_1(z, \zeta) = -\frac{2 \kappa_3}{S(\zeta)} (\lambda_1 t_2 e^{-i|z|} - \lambda_2 t_1 e^{-i|z|}), \]
\[ \gamma_2(z, \zeta) = \frac{C_{44}}{C_{44} \zeta^2 + C_{44} z^2} e^{-i|z|}, \]
\[ \gamma_3(z, \zeta) = -\frac{2 \kappa_3 C_{44} z}{C_{44} \zeta^2 + C_{44} z^2} \left( \lambda_1 \kappa_2 e^{-i|z|} - \lambda_2 \kappa_1 e^{-i|z|} \right), \]
\[ \Omega_1(z, \zeta) = -\frac{1}{\zeta S(\zeta)} (\vartheta_1 t_2 e^{-i|z|} - \vartheta_2 t_1 e^{-i|z|}), \]
\[ \Omega_2(z, \zeta) = \frac{C_{44}}{C_{44} S(\zeta)} (\vartheta_1 \kappa_2 e^{-i|z|} - \vartheta_2 \kappa_1 e^{-i|z|}). \] (28)

Here
\[ X_m = \tilde{p}_m^{-1}(\zeta) - i \tilde{q}_m^{-1}(\zeta); \quad Y_m = \tilde{p}_m^{-1}(\zeta) + i \tilde{q}_m^{-1}(\zeta); \quad Z_m = \tilde{r}_m^{-1}(\zeta), \]
\[ \eta_i = (\alpha_3 - \alpha_2) \lambda_i^2 + \zeta^2 (1 + \alpha_1) - \frac{\rho \rho_0}{C_{66}}; \quad \vartheta_i = \alpha_3 \lambda_i^2 - \eta_i, \]
\[ \nu_i = C_{33} (\eta_i - \alpha_3) \zeta^2 - \alpha_3 \lambda_i^2) \lambda_i, \quad i = 1, 2. \] (29)

In expressions (26)–(29), \( c_{ul} \) and \( \rho \) are the piecewise constant elastic moduli and density, respectively; given by
\[ c_{ul} = \begin{cases} c_{ul}^l, & z < 0 \\ c_{ul}^h, & z > 0 \end{cases}, \quad \rho = \begin{cases} \rho^l, & z < 0 \\ \rho^h, & z > 0. \end{cases} \] (30)

Subsequently, the same expressions are valid for \( \alpha_i, \lambda_i, \eta_i, \nu_i, \) and \( \vartheta_i, \) In addition, \( \alpha_i, \lambda_i, \) and \( \kappa_i \) are functions defined as
\[ t_i = \alpha_i^2 (\nu_i^l \nu_i^l - \nu_i^l \nu_i^h) \lambda_i^2 - \alpha_i^2 (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \lambda_i^2 + (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \vartheta_i, \]
\[ \kappa_i = \alpha_i^2 (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \lambda_i^2 + \alpha_i (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \lambda_i^2 - \alpha_i (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \lambda_i^2, \quad i = 1, 2. \] (31)

in the upper medium, and
\[ t_i = \alpha_i^2 (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \lambda_i^2 - \alpha_i^2 (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \lambda_i^2 + (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \vartheta_i, \]
\[ \kappa_i = \alpha_i^2 (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \lambda_i^2 + \alpha_i (\nu_i^l \nu_i^h - \nu_i^h \nu_i^h) \lambda_i^2 - \alpha_i (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) \lambda_i^2, \quad i = 1, 2. \] (32)

in the lower medium. Also
\[ S(\zeta) = \alpha_i^2 (\nu_i^l \nu_i^l - \nu_i^l \nu_i^h) (\eta_i^l \nu_i^h - \eta_i^h \nu_i^h) - \alpha_i (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) (\eta_i^l \nu_i^h - \eta_i^h \nu_i^h) + \alpha_i (\nu_i^l \nu_i^h - \nu_i^h \nu_i^h) (\eta_i^l \nu_i^h - \eta_i^h \nu_i^h) + \alpha_i (\nu_i^l \nu_i^h - \nu_i^h \nu_i^h) (\nu_i^l \nu_i^h - \nu_i^h \nu_i^h), \]
\[ + \alpha_i^2 (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) (\eta_i^l \nu_i^h - \eta_i^h \nu_i^h) - \alpha_i^2 (\nu_i^l \nu_i^h - \nu_i^l \nu_i^h) (\eta_i^l \nu_i^h - \eta_i^h \nu_i^h) + \alpha_i (\nu_i^l \nu_i^h - \nu_i^h \nu_i^h) (\eta_i^l \nu_i^h - \eta_i^h \nu_i^h) + \alpha_i (\nu_i^l \nu_i^h - \nu_i^h \nu_i^h) (\nu_i^l \nu_i^h - \nu_i^h \nu_i^h), \] (33)
is the Stoneley wave function corresponding to the interface. In the case of an isotropic material Eq. (33) reduces to the following expression (see also Rahimian et al., 2007)
\[ \left( \frac{\lambda^l + \mu^l}{\rho} \right)^2 \left( \frac{\lambda^h + \mu^h}{\rho} \right)^2 \rho^h \lambda^h S^{iso}(\zeta). \] (34)

where
\[ S^{iso}(\zeta) = -4 \xi^2 (\mu^l - \mu^h)^2 (\xi^2 - \lambda^l \rho^l) + 4 \xi^2 \alpha^l (\mu^l - \mu^h) [\rho^l (\xi^2 - \lambda^l \rho^l) - \rho^h (\xi^2 - \lambda^h \rho^h)] + \xi^2 \alpha^l (\rho^l)^2 - (\rho^h)^2 \]
\[ - \rho^l \xi^2 \alpha^l (\rho^h) + \rho^h \xi^2 \alpha^h (\rho^l) (\rho^l \alpha^l + \rho^h \alpha^l). \] (35)

Again in the above expressions, superscripts I and II denote the quantities in media I and II, respectively; \( \lambda \) and \( \mu \) are Lamé constants of elasticity; \( \alpha = \sqrt{\xi^2 + \rho^l \rho^h (\lambda + 2 \mu)} \) and \( \beta = \sqrt{\xi^2 + \rho^l \rho^h (\lambda + 2 \mu)}. \) Eq. (35) degenerates exactly to the expression given in Pak and Guzina (2002) for the Stoneley wave function corresponding to the interface between two adjacent isotropic layers. On substituting the inverted Fourier components of the displacements and stresses into the corresponding angular eigenfunction expansions, the desired formal solution to the general bi-material problem under consideration can be obtained.

4. Point-load Green’s functions

In the preceding sections, the general solution has been formulated for an arbitrary source distributed on the plane \( z = 0. \) To obtain the point-load Green’s functions, which are useful for integral formulations of boundary value problems, one may define the distributed traction source as (e.g., see Pak and Guzina, 2002)
orthogonality of the angular eigenfunctions

Upon inverting the transformed expressions (26) and (27) and using (39), the displacement and stress point-load Green's
functions may be written as

\[
\mathbf{f}_h(r, \theta, z) = \mathcal{F}_h \frac{\delta(r)}{2\pi r} \delta(z) \mathbf{e}_h, \\
\mathbf{f}_v(r, \theta, z) = \mathcal{F}_v \frac{\delta(r)}{2\pi r} \delta(z) \mathbf{e}_v
\]

with the harmonic time factor \(e^{i\omega t}\) omitted for brevity. In (36), \(\delta\) is the one-dimensional Dirac delta function; \(\mathbf{e}_h\) is the unit horizontal vector in the \(\theta = \theta_0\) direction given by

\[
\mathbf{e}_h = \mathbf{e}_r \cos(\theta - \theta_0) - \mathbf{e}_z \sin(\theta - \theta_0),
\]

(see also Fig. 3); \(\mathbf{e}_r\), \(\mathbf{e}_\theta\), and \(\mathbf{e}_z\) are the unit vectors in the radial, angular, and vertical directions, respectively; and \(\mathcal{F}_h\) and \(\mathcal{F}_v\) are the point-load magnitudes. By virtue of the angular expansions of the stress discontinuities across the plane \(z = 0\) and the orthogonality of the angular eigenfunctions \(\{\mathbf{e}\}^{\infty}_{m=-\infty}\), one finds

\[
P_{m+1}(r) = \mathcal{F}_h e^{i\theta_0} \frac{\delta(r)}{4\pi r}, \quad P_m(r) = 0, \quad m \neq \pm 1, \\
Q_{m+1}(r) = \pm i\mathcal{F}_h e^{i\theta_0} \frac{\delta(r)}{4\pi r}, \quad Q_m(r) = 0, \quad m \neq \pm 1, \\
R_0(r) = \mathcal{F}_v \frac{\delta(r)}{2\pi r}, \quad R_m(r) = 0, \quad m \neq 0,
\]

for the point-loads given in (36). Subsequently, the transformed loading coefficients \(X_m\), \(Y_m\) and \(Z_m\) can be expressed as

\[
X_1 = \frac{\mathcal{F}_h}{2\pi} e^{-i\theta_0}, \quad X_m = 0, \quad m \neq 1, \\
Y_{-1} = \frac{\mathcal{F}_h}{2\pi} e^{i\theta_0}, \quad Y_m = 0, \quad m \neq -1, \\
Z_0 = \frac{\mathcal{F}_v}{2\pi}, \quad Z_m = 0, \quad m \neq 0.
\]

Upon inverting the transformed expressions (26) and (27) and using (39), the displacement and stress point-load Green's
functions may be written as

\[
\begin{align*}
\hat{u}_r^s(r, \theta, z; s) &= -\frac{1}{4\pi c_4} \left\{ 2\mathcal{F}_v \int_0^\infty \gamma_3 \mathcal{J}_1(r\xi) d\xi - \mathcal{F}_h \cos(\theta - \theta_0) \times \left( \int_0^\infty (\gamma_1 + \gamma_2) \mathcal{J}_0(r\xi) d\xi - \int_0^\infty (\gamma_1 - \gamma_2) \mathcal{J}_2(r\xi) d\xi \right) \right\}, \\
\hat{u}_\theta^s(r, \theta, z; s) &= -\frac{1}{4\pi c_4} \left\{ \mathcal{F}_h \sin(\theta - \theta_0) \left( \int_0^\infty (\gamma_1 + \gamma_2) \mathcal{J}_0(r\xi) d\xi + \int_0^\infty (\gamma_1 - \gamma_2) \mathcal{J}_2(r\xi) d\xi \right) \right\}, \\
\hat{u}_z^s(r, \theta, z; s) &= \frac{1}{2\pi c_4} \left\{ \mathcal{F}_v \int_0^\infty \Omega_2 \mathcal{J}_0(r\xi) d\xi + \mathcal{F}_h \cos(\theta - \theta_0) \int_0^\infty \Omega_1 \mathcal{J}_1(r\xi) d\xi \right\}, \\
\hat{\tau}_{zz}^s(r, \theta, z; s) &= \frac{1}{2\pi c_4} \left\{ \mathcal{F}_v \int_0^\infty \left( c_{33} \frac{d\Omega_2}{dz} - c_{13} \gamma_3 \right) \mathcal{J}_0(r\xi) d\xi + \mathcal{F}_h \cos(\theta - \theta_0) \int_0^\infty \left( c_{33} \frac{d\Omega_2}{dz} - c_{13} \gamma_3 \right) \mathcal{J}_1(r\xi) d\xi \right\},
\end{align*}
\]

Fig. 3. Vertical and horizontal point-load configurations.
Fig. 4. Path of integration for transversely isotropic bi-material Green’s functions.

\[ c_{44}^{II} L u_z^z (0,0,z;\omega) \]

\[ \omega_0 = 0.5 \]

Fig. 5. Real and imaginary parts of the displacement Green’s function \( u_z^z \) along \( z \)-axis (\( \omega_0 = 0.5 \)).
\[
\ddot{x}_{zz}(r, \theta, z; s) = \frac{-1}{4\pi} \left\{ 2 \varphi_v \int_0^\infty \left( \frac{\Omega_2}{\varphi_z} + \frac{d\gamma_2}{dz} \right) J_1(r \xi) d\xi - \varphi_h \cos(\theta - \theta_0) \times \left\{ \int_0^\infty \left( \frac{\Omega_1}{\varphi_z} + \frac{d\gamma_1}{dz} \right) J_0(r \xi) d\xi - \int_0^\infty \left( \frac{\Omega_1}{\varphi_z} - \frac{d\gamma_2}{dz} \right) J_2(r \xi) d\xi \right\} \right\},
\]
\[
\ddot{x}_{z\theta}(r, \theta, z; s) = \frac{-1}{4\pi} \left\{ \varphi_h \sin(\theta - \theta_0) \left\{ \int_0^\infty \left( \frac{\Omega_1}{\varphi_z} + \frac{d\gamma_1}{dz} \right) J_0(r \xi) d\xi + \int_0^\infty \left( \frac{\Omega_1}{\varphi_z} - \frac{d\gamma_2}{dz} \right) J_2(r \xi) d\xi \right\} \right\},
\]
\[
\ddot{x}_{r\theta}(r, \theta, z; s) + \frac{2c_{66}}{r} \left\{ \ddot{u}_r + i \left( \ddot{u}_\theta e^{i\theta} - \ddot{u}_z e^{-i\theta} \right) \right\} = \frac{1}{2\pi c_{44}} \times \left\{ \varphi_v \int_0^\infty \left( c_{13} \frac{d\Omega_2}{dz} - c_{11} \frac{d\Omega_3}{dz} \right) J_0(r \xi) d\xi + \varphi_h \cos(\theta - \theta_0) \int_0^\infty \left( c_{11} \frac{d\Omega_1}{dz} - c_{11} \frac{d\gamma_1}{dz} \right) J_0(r \xi) d\xi \right\},
\]
\[
\ddot{x}_{r\theta}(r, \theta, z; s) + \frac{2c_{66}}{r} \left\{ \ddot{u}_r + i \left( \ddot{u}_\theta e^{i\theta} - \ddot{u}_z e^{-i\theta} \right) \right\} = \frac{1}{2\pi c_{44}} \times \left\{ \varphi_v \int_0^\infty \left( c_{13} \frac{d\Omega_2}{dz} - c_{12} \frac{d\gamma_3}{dz} \right) J_1(r \xi) d\xi \right\},
\]
\[
\ddot{x}_{r\theta}(r, \theta, z; s) + \frac{2c_{66}}{r} \left\{ \ddot{u}_r + i \left( \ddot{u}_\theta e^{i\theta} - \ddot{u}_z e^{-i\theta} \right) \right\} = \frac{1}{2\pi c_{44}} \times \left\{ \varphi_h \sin(\theta - \theta_0) \int_0^\infty \gamma_2 \frac{d\Omega_1}{dz} J_1(r \xi) d\xi \right\}.
\]

In the above, the symbols "\ddot{u}" and "\ddot{\tau}" (i, k = r, 0, z) denote, respectively, the displacement and stress Green's functions, with the superscript "*" denoting the direction of the point-load upon appropriate specifications of \( \varphi_h, \varphi_v, \) and \( \theta_0 \) in Eq. (36).

---

**Fig. 6.** Real part of the displacement Green's function \( \ddot{u}_r \) along z-axis \( (\omega_0 = 3) \).
5. Special cases

In this section, it is relevant to examine two degenerate cases: (i) when the moduli of both media are equal, and (ii) when the modulus of the upper medium \( z < 0 \) is zero. As it is apparent from the physics of the problem, such degenerate forms of the general formulation should correspond to the homogeneous full-space and half-space solutions, respectively.

5.1. Homogeneous transversely isotropic full-space

Upon setting \( c_{ik}^{II} = c_{il}^{II} = c_w \) and \( \rho' = \rho'' = \rho \), kernel functions (28) can be reduced to

\[
\begin{align*}
\gamma_1(z, \xi) &= \frac{1}{2(\eta_1 z_2^2 - \eta_2 z_1^2)} (\lambda_1 \psi_2 e^{-i|z|} - \lambda_2 \psi_1 e^{-i|\xi|}), \\
\gamma_2(z, \xi) &= \frac{1}{2\lambda_3} e^{-i|\xi|}, \\
\gamma_3(z, \xi) &= \frac{\alpha_3 c_{44} z}{2\alpha_2} (\psi_1 e^{-i|z|} + \psi_2 e^{-i|\xi|}), \\
\Omega_1(z, \xi) &= \frac{\psi_1}{2\alpha_3} \left( \frac{z}{|z|} e^{-i|z|} - \frac{z_1}{|z|} e^{-i|\xi|} \right), \\
\Omega_2(z, \xi) &= \frac{\psi_2}{2\alpha_3} \left( \frac{z}{|z|} e^{-i|z|} - \frac{z_2}{|z|} e^{-i|\xi|} \right).
\end{align*}
\]

\( (42) \)

Fig. 7. Imaginary part of the displacement Green's function \( u_z \) along z-axis (\( \omega_0 = 3 \)).
for the homogeneous full-space problem. Substitution of (42) into (40) and (41) results in the dynamic point-load Green's functions for a transversely isotropic homogeneous full-space solid.

In the case of static problem, i.e. $\omega \rightarrow 0$, for which closed-form solutions are available, the following relations are valid:

$$k_1 = s_1 n, \quad k_2 = s_2 n, \quad k_3 = s_0 n,$$

(43)

Substituting (43) into (42) leads to static full-space kernel functions as

$$\gamma_1(z, \xi) = \frac{1}{2c_{33} s_1} \left( \frac{c_{44} - c_{33}s_2^2}{s_1} e^{-s_1 \xi z} - \frac{c_{44} - c_{33}s_2^2}{s_2} e^{-s_2 \xi z} \right),$$

$$\gamma_2(z, \xi) = \frac{1}{2s_0} e^{-s_0 \xi z},$$

$$\gamma_3(z, \xi) = \frac{-(c_{33} + c_{44})}{2c_{33}} Z \left( e^{-s_1 \xi z} - e^{-s_2 \xi z} \right),$$

$$\Omega_1(z, \xi) = \frac{c_{11}(c_{44} - c_{33}s_2^2)}{2c_{13} s_1^2 s_2^2 (c_{13} + c_{44}) (s_2^2 - s_1^2)} \frac{Z}{|z|} \left( e^{-s_1 \xi z} - e^{-s_2 \xi z} \right),$$

$$\Omega_2(z, \xi) = \frac{c_{11}}{2c_{13} s_1^2} \left( \frac{c_{44} - c_{33}s_2^2}{s_1^2} e^{-s_1 \xi z} - \frac{c_{44} - c_{33}s_2^2}{s_2^2} e^{-s_2 \xi z} \right),$$

for $s_1 \neq s_2$. If the previous kernel functions are directly reduced to the case $s_1 = s_2$, terms with forms of 0/0 will be encountered. This occurs in transversely isotropic materials when $\sqrt{c_{11}c_{33} - c_{13} - 2c_{44}} = 0$. It is worth mentioning that in this case,
one can obtain $s_1 = s_2 = (c_{11}/c_{33})^{1/4}$ (see Eq. (9)). Therefore, in order to obtain kernel functions of the case of $s_1 = s_2$, it is needed to take the limits of the kernel functions of the case of $s_1 \neq s_2$ (i.e. Eq. (44)) by setting $s_2 \rightarrow s_1$. Subsequently, the kernel functions of the case of $s_1 = s_2$ can be obtained as

\[
\begin{align*}
\gamma_1(z, \xi) &= \frac{1}{4c_{11}s_1} (c_{44}s_1^2 + c_{11} + s_1(c_{44}s_1^2 - c_{11})\xi|z|) e^{-s_1|z|}, \\
\gamma_2(z, \xi) &= \frac{1}{2S_0} e^{-s_0|z|}, \\
\gamma_3(z, \xi) &= \frac{c_{13} + c_{44}}{4c_{33}s_1} z e^{-s_1|z|}, \\
\Omega_1(z, \xi) &= \frac{(c_{44}s_1^2 - c_{11})^2}{4c_{11}(c_{13} + c_{44})s_1} z e^{-s_1|z|}, \\
\Omega_2(z, \xi) &= \frac{1}{4c_{33}s_1} (c_{44}s_1^2 + c_{11} - s_1(c_{44}s_1^2 - c_{11})\xi|z|) e^{-s_1|z|}.
\end{align*}
\]

(45)

In order to evaluate the static Green’s functions in closed form, we utilize the following relations for the integrals involved (Erdelyi, 1954):

\[
\int_0^\infty e^{-\delta_k r} z \bar{n}_j(r\zeta) d\zeta = \begin{cases} 
\left(\frac{-1}{r^\alpha}\right)^\frac{1}{2} \frac{\bar{n}_j}{\delta_k} \left(\frac{\sqrt{(d_k^2 + r^2)}^n}{d_k^2 + r^2}\right), & r > 0 \\
\delta\alpha! d_k^{-\alpha+1}, & r = 0
\end{cases}
\]

(46)
where \( l = 0,1; n, k = 0,1,2; d_{n} = s_{n}z|z|; n > -(l + 1); \) and \( \delta_{0} \) is the Kronecker delta. Substitution of (44) and (45) into (40) and using (46) yields the static point-load displacement Green's functions for a transversely isotropic full-space solid as

\[
\hat{u}_{r}^{*}(r, \theta, z) = \frac{-(c_{13} + c_{44})}{4\pi c_{33}c_{44}(s_{2}^{2} - s_{1}^{2})} \sum_{k=1}^{2} (-1)^{k} \frac{r}{R_{k}R_{k}^{'}} + \frac{\mathcal{F}_{v}}{4\pi c_{44}} \cos(\theta - \theta_{0}) \times \left\{ \frac{1}{s_{0}R_{0}^{'}} - \frac{1}{c_{33}(s_{2}^{2} - s_{1}^{2})} \sum_{k=1}^{2} (-1)^{k} \left( \begin{array}{c}
\frac{c_{44} - c_{33}s_{2}^{2}}{s_{k}} \frac{r^{2}}{R_{k}R_{k}^{'}} \frac{1}{R_{k}R_{k}^{'}} \end{array} \right) \right\},
\]

\[
\hat{u}_{y}^{*}(r, \theta, z) = -\frac{\mathcal{F}_{h}}{4\pi c_{44}} \sin(\theta - \theta_{0}) \left\{ \frac{1}{s_{0}} \left( \frac{1}{s_{0}R_{0}^{'}} \frac{r^{2}}{R_{0}R_{0}^{'}} - \frac{1}{c_{33}(s_{2}^{2} - s_{1}^{2})} \sum_{k=1}^{2} (-1)^{k} \left( \frac{c_{44} - c_{33}s_{2}^{2}}{s_{k}} \right) \right) \right\},
\]

\[
\hat{u}_{z}^{*}(r, \theta, z) = -\frac{\mathcal{F}_{h}}{4\pi c_{33}c_{44}} \cos(\theta - \theta_{0}) \left\{ \frac{r^{2}}{R_{0}R_{0}^{'}} \sum_{k=1}^{2} (-1)^{k} \left( \frac{c_{44} - c_{33}s_{2}^{2}}{s_{k}} \right) \frac{r^{2}}{R_{k}R_{k}^{'}} \right\} + \frac{\mathcal{F}_{v}}{4\pi c_{33}c_{44}(s_{2}^{2} - s_{1}^{2})} \sum_{k=1}^{2} (-1)^{k} \left( \frac{c_{44}s_{2}^{2} - c_{11}}{s_{k}} \right).
\]

for \( s_{1} \neq s_{2} \), and

\[
\hat{u}_{r}^{*}(r, \theta, z) = \frac{\mathcal{F}_{h}}{8\pi c_{44}} \cos(\theta - \theta_{0}) \left\{ \frac{2}{s_{0}R_{0}^{'}} + \frac{c_{44}s_{1}^{2} - c_{11}(1 - \frac{r^{2}}{R_{1}^{2}})}{c_{33}s_{1}R_{1}} \left( \frac{1}{R_{1}^{2}} + \frac{2}{s_{1}R_{1}^{2}} \right) \right\} + \frac{(c_{13} + c_{44})s_{1}^{3}}{8\pi c_{11}c_{44}} \frac{r^{2} R_{1}}{R_{1}^{3}},
\]

\[
\hat{u}_{y}^{*}(r, \theta, z) = \frac{\mathcal{F}_{h}}{8\pi c_{44}} \sin(\theta - \theta_{0}) \left\{ \frac{2}{s_{0}R_{0}^{'}} \left( 1 + \frac{r^{2}}{R_{0}R_{0}^{'}} \right) - \frac{c_{44}s_{1}^{2} - c_{11}(1 + \frac{r^{2}}{R_{1}^{2}})}{c_{33}s_{1}R_{1}} + \frac{2}{s_{1}R_{1}^{2}} \right\},
\]

\[
\hat{u}_{z}^{*}(r, \theta, z) = \frac{\mathcal{F}_{h}(c_{44}s_{1}^{2} - c_{11})^{2}}{8\pi c_{11}c_{44}(c_{13} + c_{44})s_{1}} \cos(\theta - \theta_{0}) \frac{r^{2} R_{1}}{R_{1}^{3}} + \frac{\mathcal{F}_{v}}{8\pi c_{11}} \left( \frac{s_{1}^{2}r^{2}}{c_{11}^{2}} + \frac{s_{1}(R_{1}^{2} + s_{1}^{2}z^{2})}{c_{44}} \right).
\]

![Fig. 10. Imaginary part of the displacement Green's function \( \hat{u}_{r}^{*} \) along z-axis \( (\omega_{0} = 3) \).](image-url)
for \( s_1 = s_2 \). In the above expressions

\[
R_i = \sqrt{r^2 + s_i^2 z^2}; \quad R'_i = R_i + s_i |z|, \quad i = 0, 1, 2.
\]  

(49)
When Cartesian components of the above Green’s functions are employed with \( \cos(\theta_0) = x/r \), and \( \sin(\theta_0) = y/r \), Eqs. (47) and (48) can be degenerated exactly to the solutions given by Pan and Chou (1976) as well as Ding et al. (1997) for the homogeneous full-space problem (see Appendix). However, there is a typographical error in expression given by Pan and Chou (1976) for the \( s_1 = s_2 \) case where the right-hand side of Eq. (18) in Pan and Chou (1976) must be multiplied by \( s_1 \) (or by \( v_1 \) in Pan and Chou’s notation).

### 5.2. Homogeneous transversely isotropic half-space

Taking \( c_{kl}^i \to 0 \), \( \rho \to 0 \), \( c_{kl}^{il} = c_{kl} \), and \( \rho^{il} = \rho \), degenerates the kernel functions (28) to the following expressions for the half-space problem.

#### Table 1
Engineering constants of transversely isotropic materials

<table>
<thead>
<tr>
<th>Material No.</th>
<th>( \frac{E_h}{v_h} )</th>
<th>( \frac{f}{v_h} )</th>
<th>( v_h )</th>
<th>( v_{hv} )</th>
<th>( v_{vh} )</th>
<th>( \frac{E_v}{v_v} )</th>
<th>( \frac{f_v}{v_v} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material 1</td>
<td>1.5</td>
<td>0.9</td>
<td>0.25</td>
<td>0.3</td>
<td>0.2</td>
<td>0.25</td>
<td>0.2</td>
</tr>
<tr>
<td>Material 2</td>
<td>3.0</td>
<td>1.0</td>
<td>0.1</td>
<td>0.9</td>
<td>0.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Table 2
Elastic constants of transversely isotropic materials

<table>
<thead>
<tr>
<th>Material no.</th>
<th>( \rho ) (kg/m(^3))</th>
<th>( c_{11} ) (GPa)</th>
<th>( c_{12} ) (GPa)</th>
<th>( c_{13} ) (GPa)</th>
<th>( c_{33} ) (GPa)</th>
<th>( c_{44} ) (GPa)</th>
<th>( \frac{c_{33} c_{44}}{\rho} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material 1</td>
<td>1000</td>
<td>44.8</td>
<td>14.8</td>
<td>11.9</td>
<td>29.8</td>
<td>11.3</td>
<td>1.006063</td>
</tr>
<tr>
<td>Material 2</td>
<td>5000</td>
<td>553</td>
<td>280</td>
<td>250</td>
<td>250</td>
<td>50</td>
<td>1.035318</td>
</tr>
</tbody>
</table>

![Fig. 13. Real part of the stress Green's function \( ^2 \tilde{\tau}_{zz} \) along z-axis (\( \omega_0 = 0.5 \)).](image-url)
\begin{equation}
\gamma_1 (z, \xi) = \frac{-\alpha_3}{I(\xi)} (\lambda_1 v_2 e^{-i\lambda_1 z} - \lambda_2 v_1 e - \lambda_2 z),
\end{equation}

\begin{equation}
\gamma_2 (z, \xi) = \frac{1}{\lambda_3} e^{-i\lambda_3 z},
\end{equation}

\begin{equation}
\gamma_3 (z, \xi) = \frac{-\alpha_3 \alpha_4 e^{-i\lambda_3 z}}{I(\xi)} (\eta_2 \lambda_1 e^{-i\lambda_2 z} - \eta_1 \lambda_2 e^{-i\lambda_2 z}),
\end{equation}

\begin{equation}
\Omega_1 (z, \xi) = \frac{1 + z_1}{z_1} \left\{ \left( \frac{\rho_0 \omega^2}{c_{11}} \right) (v_2 e^{-i\lambda_2 z} - v_1 e^{-i\lambda_2 z}) - \frac{\alpha_2}{1 + z_1} (v_2 \lambda_1^2 e^{-i\lambda_2 z} - v_1 \lambda_2^2 e^{-i\lambda_2 z}) \right\},
\end{equation}

\begin{equation}
\Omega_2 (z, \xi) = \frac{c_{44} (1 + z_1)}{I(\xi)} \left\{ \left( \frac{\rho_0 \omega^2}{c_{11}} \right) (\eta_2 e^{-i\lambda_2 z} - \eta_1 e^{-i\lambda_2 z}) - \frac{\alpha_2}{1 + z_1} (\eta_2 \lambda_1^2 e^{-i\lambda_2 z} - \eta_1 \lambda_2^2 e^{-i\lambda_2 z}) \right\},
\end{equation}

where

\begin{equation}
I(\xi) = \eta_2 v_1 - \eta_1 v_2,
\end{equation}

is the Rayleigh wave function whose zero at \( \xi_k \) corresponds to the Rayleigh wave number associated with the free surface in the half-space medium. The kernel functions in Eq. (50) is exactly the same as those obtained in Rahimian et al. (2007) for a transversely isotropic half-space under surface excitations. Substituting kernel functions (50) into (40) and (41) yields the dynamic point-load Green’s functions for a transversely isotropic homogeneous half-space under time-harmonic surface excitations.

6. Numerical results and discussion

In the previous section, the point-load Green’s functions were expressed in terms of one-dimensional semi-infinite integrals in the complex-plane. As the integrations generally can not be carried out in exact closed forms, a numerical quadrature
technique usually has to be adopted in such evaluations (see, e.g., Apsel and Luco, 1983; Pak, 1987; Pak and Guzina, 2002; Rajapakse and Wang, 1993; Rahimian et al., 2007). In order to evaluate the integrals accurately, it is very important to pay attention to the oscillatory nature of the integrands because of the presence of Bessel functions. For this reason, the numerical integration schemes such as trapezoidal and Simpson rules which use equal intervals are inefficient. In the present work, an adaptive quadrature rule demonstrated in Rahimian et al. (2007) has been incorporated and used successfully. For numerical evaluation of integrals given in Eqs. (40) and (41), however, some special considerations are needed due to the presence of singularities within the range of integration including branch points and pole. There will be in general six branch points lying on the formal path of integration; one pair for the upper medium and one pair for the lower one. Besides, the function $S(\xi)$ defined in (33) yields pole at $\xi_S$ which corresponds to Stoneley wave number. The pole is obtained by setting $S(\xi) = 0$. Depending on the elasticity constants and mass density of the two bonded half-spaces, a Stoneley wave may or may not exist. When it does exist as in the subsonic regime, only one such interfacial wave is possible and it travels at a speed larger than the smaller of the Rayleigh speeds associated with the two half-spaces (Barnett et al., 1985; Barnett, 2000; Destrade and Fu, 2006). In other words, the Stoneley wave number must be smaller than the larger of the Rayleigh wave numbers associated with the two half-spaces. As a result the path of integration may be free of poles or not. Once the locations of the singular points are determined, the path of integration is deformed by semi-circles of radius $\epsilon$ around them (see Fig. 4).

While the kernel of the inversion integral is weakly singular at the branch points, it is strongly singular at the pole if it exists. Thus the integral over the limiting small semi-circle at the pole should be evaluated using the residual theory of integration (e.g. Churchill and Brown, 1990). Since the pole at $\xi_S$ is an interior singular point of the first order, the integrand may be written in the form of $q(\xi)/S(\xi)$, where $q(\xi)$ is analytic at $\xi_S$ and $S(\xi)$ has been given in (33). Therefore the integral over the limiting small semi-circle at the pole is equal to $-\pi i \text{Res}(\xi_S)$, where $\text{Res}(\xi_S) = \lim_{\xi \to \xi_S} (q(\xi)/dS(\xi)/d\xi)$. It is worth mentioning that for simplification of the method of integration, some investigators choose to add damping terms to the elastic moduli in the case of ideal elastic materials (e.g., see Apsel and Luco, 1983; Rajapakse and Wang, 1993). This assumption shifts the branch points and pole of the integrands off the formal path of integration, which in turn allows the numerical integration...
to be carried out with less effort. Because of the artificially added damping, however, such solutions are strictly speaking not ones of elasticity. To illustrate the results obtained in previous sections, some typical point-load Green’s functions are presented in Figs. 5–16 for two transversely isotropic materials with a total of three characteristic cases. The values of the engineering elastic constants for considered transversely isotropic materials are given in Table 1, where $E_h$ and $E_v$ are the Young’s moduli with respect to directions lying in the plane of isotropy and perpendicular to it; $v_h$ and $v_{hv}$ are Poisson ratios which characterize the effect of the horizontal strain on its orthogonal counterpart and the vertical strain (i.e., the z-direction strain), respectively; $v_{vh}$ is the Poisson ratio which characterizes the effect of the vertical strain on horizontal strains; and $f/2$ is the shear modulus for the planes normal to the plane of isotropy. Upon converting to the elasticity moduli $c_{kl}$ and $c_{44}$ choosing $E_v = 100 \text{ GPa}$ and $\rho_2 = 5000 \text{ kg/m}^3$ (see Pan, 1989), the pertinent elastic constants $c_{kl}$ can be stated as those given in Table 2. Also the zero of Eq. (51) at $\xi_5$ corresponding to the half-spaces associated with each of these two materials is given in Table 2.

The three cases considered here are:

Case 1: Bi-material with material 2 in medium II and $c^l_1, \rho^l \to 0$ in medium I, (no upper half-space); a Stoneley wave exists with $\xi_5 = 1.035318 \sqrt{\rho^l/c^l_4} \omega$.
Case 2: Bi-material with material 2 in both media I and II, (two equal half-spaces); Stoneley wave does not exist.
Case 3: Bi-material with material 1 in medium I and material 2 in medium II, (stiffer lower half-space); a Stoneley wave exists with $\xi_5 = 1.007556 \sqrt{\rho^l/c^l_4} \omega$.

The first two cases are compared to the solution for a homogeneous half-space (Rahimian et al., 2007) and the solution obtained by kernel functions (42) for a homogeneous full-space, respectively. The source point is taken to be the origin with

![Fig. 16. Imaginary part of the stress Green’s function $\tau_{zz}$ along z-axis ($\omega_0 = 3$).](image-url)
coordinates (0,0,0). It needs to be pointed out that all numerical results presented here are dimensionless, with a non-dimensional frequency defined as \( \omega_0 = L \omega_1 / \rho^* \sigma_{0}^{\mu} \) where \( L \) represents the unit of length. Figs. 5–7 depict the displacement Green’s function \( \hat{u}^c \) due to the unit point-load in the \( z \)-direction for \( \omega_0 = 0.5 \). Also the displacement Green’s functions \( \hat{u}^r \) due to the unit point-load in the \( r \)-direction are delineated in Figs. 8–10 for \( \omega_0 = 0.5 \). Furthermore, to provide further insight into the problem, the distributions of \( \hat{u}^c \) at the interface and along \( r \)-axis are shown in Figs. 11 and 12 for \( \omega_0 = 3 \). From the display, one may observe the full agreement of the degenerate bi-material solutions cases 1 and 2 with homogeneous half-space and full-space solutions, respectively. The phenomena of radiating functions are singular at the interface to half-space associated with material 2, as it is apparent from the physics of the problem. As expected, in case 2 when the bi-material is stiffer than the other two cases (see Table 2), the real and imaginary parts of the displacement Green’s functions have the lowest values. While the imaginary parts of the displacement Green’s functions are continuous across the interface \( z = 0 \), their real parts are singular at the interface. Both real and imaginary parts of the displacement Green’s functions tend to zero with increasing depth. As frequency increases, both real and imaginary parts show oscillatory variation with the depth. It is also observed that the real parts of the displacement Green’s functions generally decrease with increasing \( \omega_0 \), whereas their imaginary parts increase with \( \omega_0 \).

The distribution of real and imaginary parts of the stress Green’s functions \( \hat{t}^c \) due to the unit point-load in the \( z \)-direction are plotted in Figs. 13–16 for \( \omega_0 = 0.5 \). Similar to the displacement Green’s functions, the real parts of the stress Green’s functions are singular at the interface \( z = 0 \), whereas their imaginary parts are continuous. Again, the Green’s functions for cases 1 and 2 are in agreement with homogeneous half-space and full-space solutions, respectively. The phenomena of radiation condition is clearly evident in Figs. 13–16. Note that in the determination of \( \text{Re}(\hat{t}^c) \), the elastic constant \( c_{13} \) is the dominant component, where its value for the lower half-space (medium II) is about seven times larger than that for the upper one (medium I) in the bi-material case 3 (see Table 2). For this reason, for a given \( \omega_0 \) in case 3, the value of \( \text{Re}(\hat{t}^c) \) for the lower half-space is generally higher than that for the upper one. This effect of the neighboring medium on the stress distribution in either half-space is most pronounced near the material interface. Consistent with the symmetry of the problem, all Green’s functions for the homogeneous full-space configuration (case 2) are symmetric with respect to the plane \( z = 0 \).

7. Conclusion

The three-dimensional dynamic Green’s functions of a transversely isotropic bi-material elastic full-space due to point loads is derived by means of integral transforms and the method of displacement potentials. They are expressed in the form of explicit line-integral complex-plane representations which are essential for the efficient boundary element formulations of the related elastodynamic problems. It is shown that the present transversely isotropic bi-material Green’s functions can analytically be degenerated to special cases such as the dynamic solution for homogeneous half-space and full-space media as well as Pan and Chou’s static solution for a homogeneous full-space medium. Numerical examples are also presented to elucidate the influences of the neighboring medium on the response under dynamic excitations.

Appendix A

In the Cartesian coordinate system with \( \cos(\theta - \theta_1) = x/r \); \( \sin(\theta - \theta_1) = y/r \); \( u_x = u_x \cos(\theta - \theta_1) - u_y \sin(\theta - \theta_1) \); \( u_y = u_x \sin(\theta - \theta_1) + u_y \cos(\theta - \theta_1) \); and \( r^2 = x^2 + y^2 \), Eqs. (47) and (48) can be expressed as

\[
\hat{u}^c(x,y,z) = \frac{-c_{13} + c_{44}}{4\pi c_{11} c_{44} (s_{2}^2 - s_{1}^2)} \sum_{k=1}^{2} (1)^k \left( \frac{x}{R_1 R_0^*} \right) + \frac{1}{4\pi c_{44}} \left( \frac{1}{S_0} \left( \frac{y^2}{R_0 R_0^*} - \frac{1}{c_{13} (s_{2}^2 - s_{1}^2)} \right) \sum_{k=1}^{2} (1)^k \left( \frac{c_{44} - c_{33} s_{2}^2}{s_k} \right) \left( \frac{1}{R_1} \left( \frac{x^2}{R_1 R_0^*} \right) \right) \right),
\]

\[
\hat{u}^c(x,y,z) = \frac{-c_{13} + c_{44}}{4\pi c_{11} c_{44} (s_{2}^2 - s_{1}^2)} \sum_{k=1}^{2} (1)^k \left( \frac{y}{R_1 R_0^*} \right) + \frac{1}{4\pi c_{44}} \left( \frac{1}{S_0} \left( \frac{x}{R_0 R_0^*} + \frac{1}{c_{13} (s_{2}^2 - s_{1}^2)} \right) \sum_{k=1}^{2} (1)^k \left( \frac{c_{44} - c_{33} s_{2}^2}{s_k} \right) \left( \frac{1}{R_1} \left( \frac{y^2}{R_1 R_0^*} \right) \right) \right),
\]

\[
\hat{u}^c(x,y,z) = \frac{-c_{13} + c_{44}}{4\pi c_{11} c_{44} (s_{2}^2 - s_{1}^2)} \sum_{k=1}^{2} (1)^k \left( \frac{c_{44} s_{2}^2 - c_{11}}{s_k R_1^2} \right) - \frac{1}{4\pi c_{33} c_{44}} \left( \frac{1}{S_1} \left( \frac{x}{R_0 R_0^*} \right) \sum_{k=1}^{2} (1)^k \left( \frac{c_{44} - c_{33} s_{2}^2}{s_k} \right) \left( \frac{xz}{R_1 R_0^*} \right) \right),
\]

(A.1)

for \( s_1 \neq s_2 \), and

\[
\hat{u}^c(x,y,z) = \frac{c_{13} + c_{44}}{8\pi c_{44}} \left( \frac{1}{S_0} \left( \frac{y^2}{R_0 R_0^*} \right) + \frac{c_{44} s_{1}^2 - c_{11}}{c_{11} s_1 R_1^2} \left( \frac{1}{R_1} \left( \frac{x^2}{R_1 R_1^*} \right) \right) + \frac{1}{s_1} \left( \frac{1}{R_1} - \frac{x^2}{R_1 R_1^*} \right) \right) + \frac{c_{13} + c_{44}}{8\pi c_{11} c_{44}} \left( \frac{1}{S_1} \left( \frac{y^2}{R_1 R_1^*} \right) + \frac{c_{44} s_{1}^2 - c_{11}}{c_{11} s_1 R_1^2} \left( \frac{1}{R_1} \left( \frac{x^2}{R_1 R_1^*} \right) \right) \right),
\]

(A.2)

\[
\hat{u}^c(x,y,z) = \frac{c_{13} + c_{44}}{8\pi c_{44}} \left( \frac{c_{44} s_{1}^2 - c_{11}}{c_{11} s_1 R_1^2} \right) + \frac{1}{8\pi c_{11} c_{44}} \left( \frac{xz}{R_1^2} + \frac{c_{44} s_{1}^2 - c_{11}}{s_1} \right).
\]
References


