ON UNIQUELY 3-COLORABLE PLANAR GRAPHS

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In this paper it is proved that the lower bound for the number of 3-circuits in a uniquely 3-colorable planar graph given by Chartrand and Geller [2, 3] is obtained only if the graph has four points. A complete description is given of uniquely 3-colorable planar graphs containing exactly three 3-circuits.

We shall consider here planar 3-colorable graphs. A graph is called uniquely 3-colorable if there is only one partition of its point set into three independent sets. Planar uniquely 3-colorable graphs were considered by Chartrand and Geller [2] where the following theorem was proved (Theorem 6):

If $G$ is a uniquely 3-colorable planar graph with $p > 4$ points, then $G$ has at least two triangles.

Notation. $m(G)$ is the number of edges in the graph $G$; $n(G)$, the number of points; $f(G)$, the number of faces; $f_i(G)$, the number of $i$-faces, i.e., faces containing $i$ edges in their boundary; $t(G)$, the number of 3-circuits; $s(x)$, the degree of the point $x$.

Let $G$ be a graph $G$ in any planar representation, and let $C$ be a circuit in $G$. We denote by $[C]$ the subgraph of $G$ generated by the points of $C$ and all points in its interior. Similarly, $]\mathcal{C}\mathcal{[}$ is the subgraph of $G$ generated by all the points of $C$ and all points in its exterior. A circuit $C$ is called separating if $n([C]) > n(C)$ and $n(]\mathcal{C}\mathcal{[}) > n(C)$.

$\mathfrak{A}$ is the set of uniquely 3-colorable planar graphs.

Lemma 1. If $G \in \mathfrak{A}$ and $n(G) \geq 4$, then $t(G) \geq 2\Delta(G) + 2$, where $\Delta(G) = m(G) - 2n(G) + 3$.

Proof. If $n(G) = 4$, then the lemma is easily verified. Suppose now that for each graph $G \in \mathfrak{A}$ with $n(G) < k$ Lemma 1 holds. Let $G_0 \in \mathfrak{A}$, where $n(G_0) = k$. We consider two cases:

Case 1. There is a separating 3-circuit $T$ in $G_0$. 
Then \([T], \lfloor T \rfloor \in \mathcal{A}\) and by assumption \(t([T]) \geq 2\Delta([T]) + 2\) and \(t(\lfloor T \rfloor) \geq 2\Delta(\lfloor T \rfloor) + 2\). It is obvious that

\[
m(G_0) = m([T]) + m(\lfloor T \rfloor) - 3,
\]

\[
n(G_0) = n([T]) + n(\lfloor T \rfloor) - 3
\]

and thus

\[
t(G_0) = t([T]) + t(\lfloor T \rfloor) - 1 \geq 2m(G_0) - 4n(G_0) + 9 > 2\Delta(G_0) + 2.
\]

**Case 2.** There are no separating 3-circuits in \(G_0\), i.e., \(t(G_0) = f_3(G_0)\).

Then \(2m(G_0) \geq 3f_3(G_0) + 4(f(G_0) - f_3(G_0))\) or \(f_3(G_0) \geq 4f(G_0) - 2m(G_0)\).

By Euler’s formula we obtain

\[
f_3(G_0) \geq 4m(G_0) - 4n(G_0) + 8 - 2m(G_0) = 2\Delta(G_0) + 2.
\]

Let \(G_q\) be the subgraph of \(G\) induced by the points colored \(i\) or \(j\). If \(G \in \mathcal{A}\) then each \(G_q\) is connected [3, Theorem 12.16], and if \(p_i\) is the number of points having color \(i\), then \(m(G_q) = p_i + p_j - 1\) and \(m(G) = m(G_{12}) + m(G_{13}) + m(G_{23}) \geq 2n(G) - 3\). So \(\Delta(G) = m(G) - 2n(G) + 3 \geq 0\), and if \(\Delta(G) = 0\) then \(G_q\) contains no circuits.

From the last inequality in the proof of Lemma 1, Case 1 we have:

**Corollary 1.** If \(G \in \mathcal{A}\) and there is a separating 3-circuit in \(G\), then \(t(G) > 2\Delta(G) + 2\).

From Case 2, we have:

**Corollary 2.** If \(G \in \mathcal{A}\) and \(t(G) = 2\Delta(G) + 2\), then \(f_i(G) = 0\) for \(i \geq 5\).

**Corollary 3.** If \(G \in \mathcal{A}\), \(t(G) = f_3(G)\), and \(t(G) = 2\Delta(G) + 3\), then \(f_3(G) = 1\) and \(f_i(G) = 0\) for \(i \geq 6\).

**Proof.** If \(f_3(G) \geq 2\) then

\[
2m(G) \geq 3f_3(G) + 4(f(G) - f_3(G) - 2) + 10
\]

If \(f_i(G) \geq 1\), \(i \geq 6\), then

\[
2m(G) \geq 3f_3(G) + 4(f(G) - f_3(G) - 1) + 6.
\]

In both cases we have \(f_i(G) \geq 4f(G) - 2m(G) + 2\), and by Euler’s formula:

\[
2\Delta(G) + 3 = f_3(G) \geq 4m(G) - 4n(G) + 8 - 2m(G) + 2
\]

\[
= 2\Delta(G) + 4.
\]

Contradiction.
The extension of coloring used in this paper is based on Theorems 1 and 2 in [1]. First of all, we need two definitions. In any 3-coloring of a 5-circuit, two nonadjacent vertices are colored one color, two other nonadjacent vertices are colored a second color, and the remaining vertex, the special vertex is colored the third color. The edge of a 5-circuit opposite to the special vertex is the special edge.

**Theorem (See Theorem 1 in [1]).** Let $G$ be a plane graph having only 3-, 4-, and 5-faces. Then:

(a) if $t(G) \leq 3$ then $G$ is 3-colorable.

(b) if $t(G) \leq 1$ then (i) every 3-coloring of an arbitrary 4-face can be extended to the whole graph $G$ and (ii) every 3-coloring of an arbitrary 5-face can be extended to the whole graph $G$ if the special edge of the 5-face does not belong to a 3-circuit.

**Theorem (See Theorem 2 in [1]).** Let $G$ be a plane graph having only 3-, 4-, and 5-faces, but which has no separating 3- or 4-circuits, and where $t(G) = 1$. If the special edge of a 3-colored 5-face is contained in the unique 3-circuit, then the 3-coloring of the 5-face can be extended to the whole graph $G$ if and only if $G$ has more than one 5-face.

**Theorem 1.** If $G \in \mathfrak{A}$ and $n(G) \geq 5$, then $t(G) \geq 3$.

**Proof.** Let $G \in \mathfrak{A}$, and $n(G) \geq 5$, but $t(G) = 2$. By Corollary 1 of Lemma 1, $t(G) = f_s(G)$. Consider any 3-face $T = (x_0, x_1, x_2)$. Clearly $s(x_i) > 2$, $i = 0, 1, 2$, since otherwise after removing the point having degree 2 we obtain a graph $G'$ where $t(G') = 1$; by Lemma 1, $G' \notin \mathfrak{A}$, thus $G \notin \mathfrak{A}$, Contradiction.

Since $t(G) = 2$, there exists an edge having only one point in common with $T$ not belonging to the second 3-circuit. Suppose this edge is $(x_0, x_3)$. We split the point $x_3$ and edge $(x_0, x_3)$ as shown in Fig. 1 obtaining graph $G''$. Edges $(x'_0, x_3)$ and $(x''_0, x_1)$ do not belong to the 3-circuit in $G''$.

Consider two 3-colorings of 5-face $(x_3, x'_0, x_1, x_2, x''_0)$ such that points $x'_0$ and $x''_0$ have the same color in both, and such that points $x_1$ and $x_3$ have the same color in one coloring, and $x_3$ and $x_2$ have the same color in the second coloring. By Theorem
Or Theorem 2 of [1], these two colorings can each be extended to \( G' \). However, by then identifying \( x_a \) and \( x_b \) we obtain two different colorings of \( G \), so \( G \not\in \mathfrak{A} \). Contradiction.

**Lemma 2.** If \( G \in \mathfrak{A}, n(G) > 5 \), and \( G \) has a separating 3-circuit, then \( t(G) \geq 4 \).

**Proof.** Let \( G \) have separating 3-circuit \( C \). Then \( [C], j \subset \mathfrak{A} \). Since \( n(G) > 5 \), clearly \( n([C]) > 5 \) or \( n(\{C\}) > 5 \). Suppose \( n([C]) > 5 \). By Theorem 1 \( t([C]) \geq 3 \), and by Lemma 1 \( t(\{C\}) \geq 2 \). Thus \( t(G) = t([C]) + t(\{C\}) - 1 \geq 4 \).

**Remark.** Every plane graph has an even number of odd faces.

Let \( G^* \) denote the graph of Fig. 2.

![Fig. 2](image-url)

**Theorem 2.** If \( G \in \mathfrak{A}, n(G) > 5 \), and \( t(G) = 3 \), then \( G^* \subseteq G \).

**Proof.** We will prove the theorem by induction with respect to number of points.

If \( n(G) = 6 \), then the theorem is easily verified. Suppose the theorem is true for every graph \( G \) with \( n(G) < k \). Let \( G_0 \in \mathfrak{A}, n(G_0) = k \), and \( t(G_0) = 3 \).

Two cases are possible:

**Case 1.** \( G_0 \) contains a separating 4-circuit. Denote this 4-circuit by \( Q \). Then either \( t([Q]) \leq 1 \) or \( t(\{Q\}) \leq 1 \). Suppose \( t([Q]) \leq 1 \). Then \( [Q] \in \mathfrak{A} \); since otherwise \( \{Q\} \) has two different 3-colorings, and by Theorem 1 [1] each of them can be extended to \([Q]\), implying \( G_0 \not\in \mathfrak{A} \). If \( t(\{Q\}) = 3 \), then by the induction hypothesis \( G^* \subseteq \{Q\} \), thus \( G^* \subseteq G_0 \). If \( t(\{Q\}) = 2 \), then by Theorem 1 \( \{Q\} \not\in \mathfrak{A} \). Contradiction.

**Case 2.** \( G \) contains no separating 4-circuit. By Corollary 3 to Lemma 1, \( G_0 \) contains a unique 5-face. Let \( G_0 \) be represented in the plane such that the 5-face is the infinite face. Consider any 4-face \((x_1, x_2, x_3, x_4)\). Let \( G_1 \) be a graph obtained from \( G_0 \) by the identification of points \( x_1 \) and \( x_3 \); let \( G_2 \) be obtained from \( G_0 \) by identifying \( x_2 \) and \( x_4 \).

If \( t(G_1) > 3 \) and \( t(G_2) > 3 \), then \( G_0 \) has 5-circuit \( F = (x_1, y_1, y_2, y_3, x_4) \) (see Fig. 3). By Lemma 2, \( t(G_0) = f_3(G_0) \). Thus, since \( G_0 \) has no separating 4-circuits, \( t(\{F\}) = 2 \) and \( n(\{F\}) > 5 \). Thus, by Theorem 1 \( F \not\in \mathfrak{A} \). Any 3-coloring of \( F \) can be extended to \( x_2 \) and \( x_3 \). Thus \( G_0 \not\in \mathfrak{A} \). Contradiction.

Clearly if \( G^* \subseteq G_1 \) and \( G^* \subseteq G_2 \), then \( G^* \subseteq G_0 \).
Now let $t(G_1) = 3$. Then by assumption $G^* \subseteq G_1$. Suppose $G^* \not\subseteq G_0$, then $G^* \not\subseteq G_2$ and $t(G_2) > 3$. So in $G_0$ there is the configuration of Fig. 4.

Let $C = (x_1, x_2, y_1, y_2, x_3)$. Since the 5-face in $G_0$ is the infinite face, we have that $t([C]) = 1$ by our Remark. If $z \neq y_2$, then any 3-coloring of $C$ can be extended to $[C]$ by Theorem 1 [1]. By our Theorem 1 above, $C \not\subseteq \mathcal{A}$, hence $G_0 \not\subseteq \mathcal{A}$. Contradiction.

If $z = y_2$, then $(x_2, y_1, y_2, x_3)$ is a 4-face (see Fig. 5). Let $G_2$ be the graph obtained from $G_0$ by the identification of points $x_2$ and $y_2$; let $G_4$ be obtained from $G_0$ by identifying $x_3$ and $y_1$. Since $t(G_3) > 3$, then using similar reasoning to the above we conclude that $t(G_4) = 3$ and $G^* \subseteq G_4$. Thus, we have that $G^* \subseteq G_1$ and $G^* \subseteq G_4$, but $G^* \not\subseteq G_0$, so 3-faces $T_1$ and $T_2$ must have an edge in common. This is impossible since $t(G_0) = 3$.

**Theorem 3.** If $G \in \mathcal{A}$, $n(G) > 5$, and $t(G) = 3$, then there is a point having degree 2 on the boundary of a 5-face of $G$. 
Proof. If \( n(G) = 6 \), then the theorem is easily verified. Suppose that \( G_0 \in \mathcal{A} \), \( t(G_0) = 3 \), but that \( s(x) > 2 \) for each point \( x \) belonging to a 5-face, and for any other graph \( G' \) with the same properties the inequality \( n(G') \geq n(G_0) \) holds. By Lemma 2 we have \( t(G_0) = f_0(G_0) \). By Corollary 3 to Lemma 1, \( G_0 \) contains a unique 5-face \( F \). Let \( G_0 \) be represented in the plane such that the 5-face is the infinite face.

Two cases are possible:

Case 1. \( G_0 \) contains a separating 4-circuit. Let \( Q \) be a separating 4-circuit such that for any other separating 4-circuit \( Q' \) the inequality \( n([Q]) \leq n([Q']) \) holds. By our remark above, either \( t([Q]) = 2 \) or \( t([Q]) = 0 \). If \( t([Q]) = 2 \), then \([Q] \notin \mathcal{A}\) by Theorem 1. Since \( t([Q]) = 1 \) any 3-coloring of \( Q \) can be extended to \([Q] \) by Theorem 1 [1]. Hence \( G_0 \notin \mathcal{A} \).

Now let \( t([Q]) = 0 \) and \( t([Q]) = 3 \). Then \([Q] \in \mathcal{A}\). By Lemma 1 \( \Delta(G_0) = 0 \), so \( G_0 \) has no circuit whose points are colored in two colors. Each 3-coloring of the 4-circuit \( Q \) can be fixed by the addition of an edge \( e \) to \( Q \).

If \( t([Q] \cup \{e\}) = 2 \), then by Theorem 1 \([Q] \cup \{e\} \notin \mathcal{A}\), i.e., there exists two different colorings of \([Q]\) in which the coloring of the 4-circuit \( Q \) coincides with the coloring of \( Q \) in \([Q]\). Hence \( G_0 \notin \mathcal{A} \). Contradiction.

Suppose \( t([Q] \cup \{e\}) > 2 \), \( Q = (x_1, x_2, x_3, x_4) \), and \( e = (x_2, x_4) \). Then according to the choice of \( Q \), \( n([Q]) = 5 \) and the whole subgraph \([Q]\) is as in Fig. 6. Points \( x_2 \) and \( x_4 \) have different colors, and \( s(x_4) = 2 \). Since \( n([Q]) < n(G_0) \) and \( t([Q]) = 3 \), then \([Q]\) contains a point \( y \) of degree 2 which belongs to the 5-face \( F \). Clearly \( y = x_1 \) or \( y = x_2 \). Suppose \( y = x_2 \) and \( s(y) = 2 \) in \([Q]\). Then \( x_2 \) is adjacent to three points of the same color: \( x_1, x_3, x_5 \). So \( G_0 \notin \mathcal{A} \). Contradiction.

Case 2. There are no separating 4-circuits in \( G_0 \). Consider any 4-face \((x_1, x_2, x_3, x_4)\) which has only one edge in common with 5-face \( F \) (see Fig. 7). Such a 4-face can always be found since \( n(G_0) > 5 \), \( G^* \subset G_0 \), and \( G_0 \) has no separating 4-circuits. In this 4-face there are two nonadjacent points having different colors.

Fig. 6.

Suppose these points are \( x_1 \) and \( x_5 \). Let \( G' \) be obtained from \( G_0 \) by the identification of \( x_1 \) and \( x_5 \). The graph \( G' \) has no 3-coloring, otherwise \( G_0 \notin \mathcal{A} \). By Theorem 1 [1] \( t(G') > 3 \), i.e., there must be a path \((x_3, y_1, y_2, x_1)\) in \( G_0 \) (see Fig. 7). Let \( C_1 = (x_3, x_2, x_3, y_1, y_2) \) and \( C_2 = (x_1, x_4, x_3, y_1, y_2) \). According to our remark, we have either \( t([C_1]) = 1 \) or \( t([C_1]) = 3 \).
(1) Suppose \( t([C_1]) = 1 \). Since \( G_0 \) has no separating 4-circuits, \((x_3, y_2), (x_1, y_1) \not\in G_0 \) and \( y_1 \neq y_3 \).

(1.1) Let \( C_1 \) be a separating 5-circuit. Then \( t([C_2]) = 2 \), but \( f_5([C_2]) > 0 \), so by Corollary 1 to Lemma 1 \( [C_2] \not\in \mathcal{A} \). If every 3-coloring of \( C_2 \) can be extended to \([C_2]\), then \( G_0 \not\in \mathcal{A} \). Now let there exist a 3-coloring of \( C_2 \) which cannot be extended to \([C_2]\). Then by Theorem 2 [1], \( C_2 \) has an edge in common with a 3-circuit \( T \) belonging to \([C_2]\). As \( G^* \subset G_0 \), we have that one of the following holds: \((x_1, y_2) \in T, (x_3, y_1) \in T, (y_1, y_2) \in T \). If \( T \) contains \((x_1, y_2)\), then \( t([C_1] \cup (x_4, y_2)) = 3 \) and \([C_1] \cup (x_4, y_2) \not\in \mathcal{A} \). So \( G_0 \not\in \mathcal{A} \), since by Theorem 2 [1] each coloring of \( C_2 \) obtained in \([C_1] \cup (x_4, y_2) \) can be extended to \([C_2]\). When \((x_3, y_1) \in T \), the same result can be obtained by considering the graph \([C_1] \cup (x_4, y_1)\).

If \((y_1, y_2) \in T\), we consider graphs \( G_1 = [C_1] \cup (x_1, y_1) \) and \( G_2 = [C_1] \cup (x_3, y_2) \). Since \( t(G_0) = 3 \), we have that \( t(G_1) = 3 \) or \( t(G_2) = 3 \). Let \( t(G_1) = 3 \); then \( G_1 \not\in \mathcal{A} \). As in the preceding case, each coloring of \( C_2 \) in \( G_1 \) can be extended to \([C_2]\), so \( G_0 \not\in \mathcal{A} \).

(1.2) Let \( C_2 \) not be separating. Then either \((x_4, y_2) \in G_0 \) or \((x_4, y_1) \in G_0 \).

If \((x_4, y_2) \in G_0 \), then by Theorem 2 \( y_2 \neq y_4 \). Thus \([C_1] \cup (x_2, y_2) \not\in \mathcal{A} \) since \( s(x_4) > 2 \) and \( s(x_2) > 2 \) in \([C_1]\). Hence \( G_0 \not\in \mathcal{A} \).

If \((x_4, y_1) \in G_0 \) and \( s(x_1) > 3 \), we consider the graphs \( G_3 = [C_1] \cup (x_1, y_1) \) and \( G_4 = [C_1] \cup (x_3, y_2) \). Since \( t(G_0) = 3 \), we have that \( t(G_3) = 3 \) or \( t(G_4) = 3 \). Suppose \( t(G_3) = 3 \), then \( G_3 \not\in \mathcal{A} \) and \( G_0 \not\in \mathcal{A} \).

(2) Suppose \( t([C_1]) = 3 \).

(2.1) \( C_2 \) is a separating circuit. Consider a separating 5-circuit \( C_1^* \) containing points \( x_1, x_4, x_5 \) and \( x \), such that for any other separating 5-circuit \( C' \) containing points \( x_1, x_4, x_5, x \), the inequality \( n([C_1]) > n([C']) \) holds. We add two edges to \([C_1] \cup (x_1, x_2) \cup (x_1, z) \). Then \( s(x_1) > 2 \), \( s(x_2) > 2 \), and \( s(y_2) > 2 \), so \( G_1 \not\in \mathcal{A} \) and \( G_0 \not\in \mathcal{A} \).

(2) Suppose \( t([C_1]) = 3 \).

(2.1) \( C_2 \) is a separating circuit. Consider a separating 5-circuit \( C_1^* \) containing points \( x_1, x_4, x_5, x \), such that for any other separating 5-circuit \( C' \) containing points \( x_1, x_4, x_5, x \), the inequality \( n([C_1]) > n([C']) \) holds. We add two edges to \([C_1] \cup (x_1, x_2) \cup (x_1, z) \). Then \( s(x_1) > 2 \), \( s(x_2) > 2 \), and \( s(y_2) > 2 \), so \( G_1 \not\in \mathcal{A} \) and \( G_0 \not\in \mathcal{A} \).
this graph by \( G_0 \). By the choice of \( C'_1 \), \( t(G_0) = 3 \) and \( y_1 \notin C'_1 \), so by the induction hypothesis \( G_0 \notin \mathcal{A} \) and hence \( G_0 \notin \mathcal{A} \).

(2.2) \( C_2 \) is not a separating circuit. By Lemma 2, \( t(G_0) = f_1(G_0) \), so \((x_1, x_3) \notin G_0 \). Since \( G_0 \) has no separating 4-circuits, \((x_1, y_2), (x_3, y_2) \notin G_0 \). Thus \((x_4, y_1), (x_4, y_2) \in G_0 \). Consider a separating 5-circuit \( C'_2 \) containing points \( x_1, x_2, y_1, y_2 \) such that for any other separating 5-circuit \( C' \) containing points \( x_1, x_2, y_1, y_2 \), the inequality \( n(|C'_2|) \geq n(|C'|) \) holds. Let \( C'_2 = (x_1, x_2, y_1, y_2) \) (see Fig. 9), and \( G_7 = [C'_2 \cup (x_2, y_1) \cup (x_2, y_2)] \). By the choice of \( C'_2 \), \( t(G_7) = 3 \). If \( y_2 \neq y_4 \), then \( G_7 \notin \mathcal{A} \), hence \( G_0 \notin \mathcal{A} \). If \( y_2 = y_4 \), then by Theorem 1, \( G_7 \backslash \{x_1\} \notin \mathcal{A} \), so \( G_0 \notin \mathcal{A} \). Contradiction.

There are only two uniquely 3-colorable planar graphs with three 3-circuits and five points, namely \( G^* \) and the graph of Fig. 10.

Fig. 9. Fig. 10.

Theorem 3 gives a complete description of uniquely 3-colorable planar graphs containing exactly three 3-circuits, i.e., every such graph contains a point of degree 2.

Problem 1. In a uniquely 3-colorable planar graph there are two 3-circuits having an edge in common.

Problem 2. If \( G \in \mathcal{A} \) and for every edge \( e \), \( G \backslash e \notin \mathcal{A} \), then \( m(G) = 2n(G) - 3 \).

References