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# Laguerre–Sobolev orthogonal polynomials: asymptotics for coherent pairs of type II

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**Abstract**

Let $S_n$ be polynomials orthogonal with respect to the inner product

$$(f,g)_S = \int_0^\infty fg\,d\mu_0 + \lambda \int_0^\infty f'g'\,d\mu_1,$$

where $d\mu_0 = x^\alpha e^{-x}\,dx$, $d\mu_1 = \frac{x^{\alpha+1}e^{-x}}{x-\xi}dx + M\delta_\xi$ with $\alpha > -1$, $\xi \leqslant 0$, $M \geqslant 0$, and $\lambda > 0$. A strong asymptotic on $(0, \infty)$, a Mehler–Heine type formula, a Plancherel–Rotach type exterior asymptotic as well as an upper estimate for $S_n$ are obtained. As a consequence, we give asymptotic results for the zeros and critical points of $S_n$ and the distribution of contracted zeros. Some numerical examples are shown.


*Keywords:* Sobolev orthogonal polynomials; Laguerre polynomials; Asymptotics; Zeros

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## 1. Introduction

The asymptotic behaviour of the polynomials and their zeros is one of the central problems of the theory of orthogonal polynomials.

In this paper we are concerned with the asymptotic properties of Sobolev orthogonal polynomials, that is, polynomials orthogonal with respect to an inner product involving derivatives. More precisely, we consider the Sobolev inner product:

$$(f,g)_S = \int_0^\infty fg\, d\mu_0 + \lambda \int_0^\infty f'g'\, d\mu_1, \tag{1.1}$$

where

$$d\mu_0 = x^\alpha e^{-x} dx, \quad d\mu_1 = \frac{x^{\alpha+1}e^{-x}}{x-\xi} dx + M\delta_\xi$$

with $\alpha > -1$, $\xi \leqslant 0$, $M \geqslant 0$, and $\lambda > 0$. The pair of measures $(\mu_0, \mu_1)$ constitutes one of the so-called coherent pairs.

The goal of coherence is the fact we can establish a relation between two consecutive Sobolev orthogonal polynomials and two consecutive orthogonal polynomials associated with the first measure $\mu_0$. This relation plays an important role in the study of Sobolev polynomials and was one of the properties that Iserles et al. looked for in the new polynomials that they introduced in [4] as the solution to an isoperimetric problem. Moreover, the existence of this kind of relation was the reason for the introduction of the concept of coherence. Although this finite relation between Sobolev polynomials and standard orthogonal polynomials is an important feature of coherence, it is *not* exclusive of coherent pairs. This type of relation provides another advantage: if we consider the inner product of the form

$$(f,g)_S = \int fg\, d\mu_0 + \int f'g'\, d\mu_1,$$

both measures having absolutely continuous part non-zero, then if we have an algebraic relation between Sobolev polynomials and standard orthogonal polynomials, we can construct stable numerical algorithms to compute Sobolev orthogonal polynomials of high degrees. Of course, it is possible to study Sobolev orthogonal polynomials without these algebraic relations (see, for example, [7,8]) and very interesting analytic results can be obtained, but it is enough difficult to generate Sobolev polynomials of high degrees in a stable form. An important first step in this direction has been given in [3].

The complete characterization of all coherent pairs of measures was done in [9]. In the case of unbounded support measures, there are two general families of polynomials related with Laguerre polynomials. The first one, usually named as type I, corresponds to the pair $(\mu_0, \mu_1)$ where either $d\mu_0(x) = (x-\xi)x^{\alpha-1}\, e^{-x}\, dx$, $d\mu_1(x) = x^\alpha\, e^{-x}\, dx$ with $\xi \leqslant 0$ and $\alpha > 0$ or $d\mu_0(x) = e^{-x}\, dx + M\,\delta_0(x)$ with $M \geqslant 0$ and $d\mu_1(x) = e^{-x}\, dx$. The second one (type II) is the pair described in (1.1).

The asymptotic behaviour of Sobolev polynomials for coherent pairs of type I has been widely studied (see, for instance, [6,11,12]) while, with respect to type II, only the comparative asymptotics has been treated (see [11]). The aim of this paper is to complete the study of asymptotic properties for polynomials of type II.

The paper is organized as follows. Some properties of classical Laguerre polynomials are exposed in this section. In Section 2, polynomials orthogonal with respect to the measure $\mu_1$ are analyzed. The interest of these polynomials comes from the fact that the absolutely continuous part of $\mu_1$ is a rational perturbation of the Laguerre weight. Section 3 is dedicated to asymptotics of Sobolev polynomials: a strong asymptotic on $(0, +\infty)$, a Mehler–Heine type formula and Plancherel–Rotach type exterior asymptotics are derived. Moreover, as a consequence, asymptotics of zeros and critical points of Sobolev polynomials as well as the distribution of contracted zeros and the $n$th root asymptotic are obtained. Also, some numerical examples are presented. Finally, in the last section an upper estimate for the Sobolev polynomials is given.

Consider the Sobolev inner product (1.1). Denote by $\{S_n\}_n$ and $\{T_n\}_n$ the sequences of polynomials orthogonal with respect to (1.1) and the measure $\mu_1$, respectively, normalized by the condition that $S_n$ and $T_n$ have the same leading coefficient as the classical Laguerre polynomial $L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + \cdots$. Observe that $T_0 = S_0 = L_0^{(\alpha)}$, and $S_1 = L_1^{(\alpha)}$.

Throughout this paper the following notation will be used:

$$||L_n^{(\alpha)}||_{\mu_0}^2 = \int_0^\infty (L_n^{(\alpha)}(x))^2 \, d\mu_0(x), \quad ||T_n||_{\mu_1}^2 = \int_0^\infty (T_n(x))^2 \, d\mu_1(x)$$

and

$$||S_n||_S^2 = (S_n, S_n)_S.$$

Many of the properties of Laguerre polynomials can be seen, for example, in the classical book of Szegő [13]. For the reference, we summarize in the following proposition some of them which play an important role in this paper:

**Proposition 1.1.** *The following properties hold for Laguerre polynomials*:

(a) [13, *formula* (5.1.1)]:

$$||L_n^{(\alpha)}||_{\mu_0}^2 = \int_0^\infty (L_n^{(\alpha)}(x))^2 \, x^\alpha \, e^{-x} \, dx = \frac{\Gamma(n+\alpha+1)}{n!}, \quad \alpha > -1. \tag{1.2}$$

(b) [13, *formula* (5.1.13)]:

$$L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha-1)}(x), \quad \alpha \in \mathbb{R}. \tag{1.3}$$

(c) *Three term recurrence relation* [13, *formula* (5.1.10)]:

$$xL_n^{(\alpha)}(x) = -(n+1)L_{n+1}^{(\alpha)}(x) + (2n+\alpha+1)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x), \tag{1.4}$$

$$L_{-1}^{(\alpha)}(x) = 0 \quad \textit{and} \quad L_0^{(\alpha)}(x) = 1.$$

(d) [13, *formula* (5.1.14)]:

$$\frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x).$$

(e) *The sequence* $\{\frac{L_n^{(\alpha)}(x)}{n^{\alpha/2-1/4}}\}_n$ *is uniformly bounded on compact subsets of* $(0, +\infty)$ ([13, *Theorem* (8.22.1)]).

(f) *It holds*

$$\frac{L_n^{(\alpha)}(x)}{n^{\alpha/2}} = e^{x/2}x^{-\alpha/2} J_\alpha(2\sqrt{nx}) + O(n^{-3/4}) \tag{1.5}$$

*uniformly on compact subsets of* $(0, +\infty)$ *where* $J_\alpha$ *is the Bessel function* ([13, Section 8.22 and formula (1.71.7)]).

(g) *Mehler–Heine formula* [13, *Theorem* 8.1.3]:

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(x/n)}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x}) \tag{1.6}$$

*uniformly on compact subsets of* $\mathbb{C}$.

(h) *Ratio asymptotics for scaled Laguerre polynomials*:

$$\lim_{n\to\infty} \frac{L_{n-1}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)} = -\frac{1}{\varphi((x-2)/2)} \tag{1.7}$$

*uniformly on compact subsets of* $\mathbb{C}\backslash[0,4]$, *where* $\varphi$ *is the conformal mapping of* $\mathbb{C}\backslash[-1,1]$ *onto the exterior of the unit circle given by*

$$\varphi(x) = x + \sqrt{x^2-1}, \quad x \in \mathbb{C}\backslash[-1,1], \tag{1.8}$$

*with* $\sqrt{x^2-1} > 0$ *when* $x > 1$.

Formula (1.7) can be deduced from (4.3.5) in [14] taking into account that the $n$th orthonormal Laguerre polynomial with positive leading coefficient is $l_n^\alpha(x) = (-1)^n \frac{L_n^{(\alpha)}(x)}{||L_n^{(\alpha)}||}$.

We want to remark that from (1.6) and (1.7) it can be shown, respectively, that

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(x/(n+j))}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x}), \tag{1.9}$$

holds uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N} \cup \{0\}$ and

$$\lim_{n\to\infty} \frac{L_{n-1}^{(\alpha)}((n+j)x)}{L_n^{(\alpha)}((n+j)x)} = -\frac{1}{\varphi((x-2)/2)}, \tag{1.10}$$

holds uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$ and uniformly on $j \in \mathbb{N} \cup \{0\}$.

## 2. The orthogonal polynomials $T_n$ and the Sobolev orthogonal polynomials $S_n$

Polynomials $T_n$ have an independent interest as orthogonal with respect to a measure whose absolutely continuous component is a rational modification of the Laguerre weight function $x^{\alpha+1}e^{-x}$ on $[0,\infty)$ and possibly with a mass point (a Dirac delta) at $\xi \leqslant 0$. In fact, we use the following results established in [11].

**Lemma 2.1.** (a) [11, *Lemma* 4.1]. *The polynomials $T_n$ satisfy the relation*

$$T_n(x) = L_n^{(\alpha+1)}(x) - c_n L_{n-1}^{(\alpha+1)}(x), \quad n \geqslant 0, \tag{2.1}$$

*where*

$$c_n = \frac{||T_n||^2_{\mu_1}}{||L_n^{(\alpha)}||^2_{\mu_0}}, \quad n \geqslant 0. \tag{2.2}$$

(b) *Relation* (2.1) *can be expressed as*

$$T_n(x) = L_n^{(\alpha)}(x) - d_n L_{n-1}^{(\alpha+1)}(x), \quad n \geqslant 0, \tag{2.3}$$

*where* $d_n = c_n - 1, \ n \geqslant 0$.

(c) [11, *Lemma* 4.4]. *It holds*

$$\lim_n \sqrt{n}\, d_n = d(\xi) = \begin{cases} -\sqrt{-\xi} & \text{if } M = 0, \\ \sqrt{-\xi} & \text{if } M > 0, \end{cases} \tag{2.4}$$

*and therefore* $\lim_n c_n = 1$. *In particular*,

- *If* $\xi = 0$ *and* $M > 0$, *we get*

$$\lim_{n\to\infty} n\, d_n = \alpha + 1. \tag{2.5}$$

- *If* $\xi = M = 0$, *then* $d_n = 0$ *and therefore* $c_n = 1$, *for all* $n$.

We have the following explicit relation between Sobolev orthogonal polynomials and Laguerre polynomials (see [11, Lemma 4.7, 4] in a more general framework):

**Lemma 2.2.** *It holds*

$$L_n^{(\alpha)}(x) - c_{n-1} L_{n-1}^{(\alpha)}(x) = S_n(x) - a_{n-1} S_{n-1}(x), \quad n \geqslant 1, \tag{2.6}$$

*where* $a_n = c_n \frac{||L_n^{(\alpha)}||^2_{\mu_0}}{||S_n||^2_S}$. *Moreover* (*see* [11, *Lemma* 4.10]),

$$\lim_{n\to\infty} a_n = a = \frac{1}{\varphi((\lambda+2)/2)}, \tag{2.7}$$

*where* $\varphi$ *is defined by* (1.8).

It is clear from (2.6) that we can compute $S_n$ in a recursive way, and we can give even an explicit expression for $S_n$ in terms of Laguerre polynomials and the sequences $\{c_n\}$ and $\{a_n\}$. Thus, if we want to compute the polynomials $S_n$, calculate its zeros or realize any numerical experiment with these polynomials, we have to compute effectively the sequence $\{c_n\}$ that appears in relation (2.1) and the sequence $\{a_n\}$.

First, we obtain a nonlinear recurrence relation for $\{c_n\}$.

**Proposition 2.3.** *It holds, for* $n \geqslant 0$,

$$c_{n+1} = \frac{2n+2+\alpha-\xi}{n+1} - \frac{n+1+\alpha}{(n+1)\, c_n}, \tag{2.8}$$

*with*

$$c_0 = \frac{\int_0^\infty \frac{x^{\alpha+1}e^{-x}}{x-\xi}\,dx + M}{\Gamma(\alpha+1)}.$$

**Proof.** We express the polynomial $-\frac{x-\xi}{n+1} L_n^{(\alpha+1)}(x)$ in terms of the basis $\{T_i\}_{i=0}^{n+1}$ and we obtain

$$-\frac{x-\xi}{n+1} L_n^{(\alpha+1)}(x) = T_{n+1}(x) - \frac{n+1+\alpha}{(n+1)\, c_n} T_n(x), \quad n \geqslant 0. \tag{2.9}$$

Then, multiplying (2.9) by $L_n^{(\alpha+1)}(x)$ and integrating with respect to the measure $x^{\alpha+1}e^{-x}\,dx$ on $[0,\infty)$, we can derive the result using formulas (1.4) and (2.1). $\square$

The sequence $\{c_n\}$ also plays an important role for the polynomials $\{T_n\}$ from computational point of view as well as to obtain asymptotic properties.

It is well known (see [2]) that zeros of polynomials $T_n$ are the eigenvalues of the symmetric tridiagonal Jacobi matrix, whose entries are the coefficients of the three term recurrence relation for the orthonormal polynomials $t_n$ with positive leading coefficient:

$$xt_n(x) = \beta_{n+1} t_{n+1}(x) + \gamma_n t_n(x) + \beta_n t_{n-1}(x), \quad n \geqslant 0,$$

with $t_{-1}(x) = 0,\; t_0(x) = ||T_0||_{\mu_1}^{-1}$.

Expanding the polynomials $xT_n(x)$ in the basis $\{T_n\}$, we get

$$xT_n(x) = -(n+1)T_{n+1}(x) + \left(nc_n + \frac{n+1+\alpha}{c_n} + \xi\right)T_n(x) - (n+\alpha)\frac{c_n}{c_{n-1}}T_{n-1}(x), \quad n \geqslant 0,$$

with $T_{-1}(x) = 0$ and $T_0(x) = 1$. Since $t_n(x) = (-1)^n \frac{T_n(x)}{\|T_n\|_{\mu_1}}$, straightforward computations show that

$$\beta_n = \sqrt{n(n+\alpha)\frac{c_n}{c_{n-1}}} \quad \text{and} \quad \gamma_n = nc_n + \frac{n+\alpha+1}{c_n} + \xi.$$

Now, we present several analytic properties of the polynomials $T_n$.

**Proposition 2.4.** *For $\alpha > -1$, the following properties hold*:

(a) *The sequence $\{\frac{T_n(x)}{n^{\alpha/2-1/4}}\}_n$ is uniformly bounded on compact subsets of $(0, +\infty)$.*

(b) *Asymptotics on $(0, +\infty)$ for $T_n$: if $\xi < 0$,*

$$\frac{T_n(x)}{n^{\alpha/2}} = e^{x/2}x^{-\alpha/2}J_\alpha(2\sqrt{nx}) + O(n^{-1/4}),$$

*and, if $\xi = 0$,*

$$\frac{T_n(x)}{n^{\alpha/2}} = e^{x/2}x^{-\alpha/2}J_\alpha(2\sqrt{nx}) + O(n^{-3/4}).$$

*Both identities hold uniformly on compact subsets of $(0, +\infty)$.*

(c) *Mehler–Heine type formula for $T_n$: if $\xi < 0$,*

$$\lim_{n\to\infty} \frac{T_n(x/(n+j))}{n^{\alpha+1/2}} = -d(\xi)\, x^{-(\alpha+1)/2}J_{\alpha+1}(2\sqrt{x}),$$

*if $\xi = 0$ and $M > 0$,*

$$\lim_{n\to\infty} \frac{T_n(x/(n+j))}{n^{\alpha}} = -x^{-\alpha/2}J_{\alpha+2}(2\sqrt{x}),$$

*and, if $\xi = M = 0$,*

$$\lim_{n\to\infty} \frac{T_n(x/(n+j))}{n^{\alpha}} = x^{-\alpha/2}J_{\alpha}(2\sqrt{x}).$$

*All the limits hold uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N} \cup \{0\}$, where $d(\xi)$ is given by (2.4).*

(d) *Plancherel–Rotach type exterior asymptotics for $T_n$:*

$$\lim_{n\to\infty} \frac{T_n((n+j)x)}{L_n^{(\alpha+1)}((n+j)x)} = 1 + \varphi\left(\frac{x-2}{2}\right)^{-1}$$

*uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$ and uniformly on $j \in \mathbb{N} \cup \{0\}$.*

**Proof.** If $\xi = M = 0$ all the results are true because of $T_n(x) = L_n^{(\alpha)}(x)$, for all $n$.

(a) We divide (2.3) by $n^{\alpha/2-1/4}$. Then, using (2.4) and Proposition 1.1(e) the result follows.

(b) If $\xi < 0$ we divide (2.3) by $n^{\alpha/2}$ and using again Proposition 1.1(e) and (2.4) we get

$$\begin{aligned}\frac{T_n(x)}{n^{\alpha/2}} &= \frac{L_n^{(\alpha)}(x)}{n^{\alpha/2}} - \frac{1}{(n-1)^{1/4}}\sqrt{n}\, d_n \left(\frac{n-1}{n}\right)^{(\alpha+1)/2} \frac{L_{n-1}^{(\alpha+1)}(x)}{(n-1)^{(\alpha+1)/2-1/4}} \\ &= \frac{L_n^{(\alpha)}(x)}{n^{\alpha/2}} + O(n^{-1/4}).\end{aligned}$$

Thus, the result follows from (1.5). On the other hand, if $\xi = 0$ and $M > 0$, we can proceed in the same way using now (2.5).

(c) Whenever $\xi < 0$, scaling the variable as $x \to x/(n+j)$ in relation (2.3) we get

$$\frac{T_n(x/(n+j))}{n^{\alpha+1/2}} = \frac{L_n^{(\alpha)}(x/(n+j))}{n^{\alpha+1/2}} - \sqrt{n} d_n \frac{L_{n-1}^{(\alpha+1)}(x/(n+j))}{n^{\alpha+1}}.$$

It only remains to use (1.9) and (2.4) to reach the result.

If $\xi = 0$ and $M > 0$, proceeding as above and using (2.5) it follows that

$$\lim_{n\to\infty} \frac{T_n(x/(n+j))}{n^{\alpha}} = x^{-\alpha/2} J_\alpha(2\sqrt{x}) - (\alpha+1)\, x^{-(\alpha+1)/2} J_{\alpha+1}(2\sqrt{x}).$$

Now, using

$$2\alpha\, z^{-1} J_\alpha(z) = J_{\alpha-1}(z) + J_{\alpha+1}(z) \tag{2.10}$$

(see, [13, formula (1.71.5)]), we have the result.

(d) In the same way as in (c), scaling the variable as $x \to (n+j)x$ in relation (2.1), dividing by $L_n^{(\alpha+1)}((n+j)x)$ and using (1.10) and $\lim_n c_n = 1$, the result arises. □

## 3. Asymptotics of Sobolev orthogonal polynomials $S_n$

In this section, first of all, we will obtain the strong asymptotics of $S_n$ on the positive semiaxis and analogues of the Mehler–Heine and Plancherel–Rotach type asymptotic formulas for the Sobolev polynomials.

If we look for analytic properties of the Sobolev orthogonal polynomials $S_n$, we have to pay attention to the polynomials on the left-hand side of (2.6), that is

$$\begin{aligned} V_n(x) &:= L_n^{(\alpha)}(x) - c_{n-1} L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha-1)}(x) - d_{n-1} L_{n-1}^{(\alpha)}(x), \quad n \geqslant 0, \\ &\text{with } c_{-1} = 0 = d_{-1}, \end{aligned} \tag{3.1}$$

where the last equality is a consequence of (1.3) and the relation between the coefficients $c_n$ and $d_n$. We can observe that the polynomials $V_n$ are, in some sense, close to the polynomials $T_n$, namely $V_n$ is a primitive of $-T_{n-1}$, i.e., $V_n'(x) = -T_{n-1}(x)$.

First, we give the strong asymptotics of $S_n$ on $(0,+\infty)$. In order to do this, we will use several analytic properties of polynomials $V_n$. Notice that, to establish Proposition 2.4 it was only necessary to know the asymptotic behaviour of the sequence $\{d_n\}$ and of the corresponding Laguerre polynomials involved in the algebraic relation: in the case of $T_n$ they are the Laguerre polynomials with parameter $\alpha+1$ and in the case of $V_n$ the Laguerre polynomials with parameter $\alpha$.

**Theorem 3.1.** *For* $\alpha>-1$, *we have*

$$\frac{S_n(x)}{n^{(\alpha-1)/2}}=e^{x/2}x^{-(\alpha-1)/2}J_{\alpha-1}(2\sqrt{nx})+O(n^{-1/4})$$

*uniformly on compact subsets of* $(0,+\infty)$.

**Proof.** From (2.6) and (3.1)

$$S_n(x)=V_n(x)+a_{n-1}S_{n-1}(x) \tag{3.2}$$

so,

$$\frac{S_n(x)}{n^{\alpha/2-3/4}}=\frac{V_n(x)}{n^{\alpha/2-3/4}}+a_{n-1}\left(\frac{n-1}{n}\right)^{\alpha/2-3/4}\frac{S_{n-1}(x)}{(n-1)^{\alpha/2-3/4}}.$$

Dividing in (3.1) by $n^{\alpha/2-3/4}$ and taking into account Proposition 1.1(e) and (2.4), we have that $\{V_n(x)/n^{\alpha/2-3/4}\}_n$ is uniformly bounded on compact sets of $(0,+\infty)$. Since $a_{n-1}(\frac{n-1}{n})^{\alpha/2-3/4}\to a\in(0,1)$, standard arguments yield that $\{S_n(x)/n^{\alpha/2-3/4}\}_n$ is also uniformly bounded.

On the other hand, using Proposition 1.1(e) and Lemma 2.1(c), it can be deduced that if $\xi<0$,

$$\frac{V_n(x)}{n^{(\alpha-1)/2}}=\frac{L_n^{(\alpha-1)}(x)}{n^{(\alpha-1)/2}}+O(n^{-1/4})$$

and if $\xi=0$

$$\frac{V_n(x)}{n^{(\alpha-1)/2}}=\frac{L_n^{(\alpha-1)}(x)}{n^{(\alpha-1)/2}}+O(n^{-3/4}),$$

where the bound for the remainder holds uniformly on compact subsets of $(0,+\infty)$, for all $\xi\leqslant 0$.

Finally, observe that

$$\begin{aligned}\frac{S_n(x)}{n^{(\alpha-1)/2}}&=\frac{V_n(x)}{n^{(\alpha-1)/2}}+\frac{a_{n-1}}{(n-1)^{1/4}}\left(\frac{n-1}{n}\right)^{(\alpha-1)/2}\frac{S_{n-1}(x)}{(n-1)^{\alpha/2-3/4}}\\&=\frac{V_n(x)}{n^{(\alpha-1)/2}}+O(n^{-1/4})=\frac{L_n^{(\alpha-1)}(x)}{n^{(\alpha-1)/2}}+O(n^{-1/4})\end{aligned}$$

uniformly on compact subsets of $(0,+\infty)$.

Using (1.5), the theorem follows. $\square$

As we mention in Section 2, we can express the polynomials $S_n$ in terms of the Laguerre polynomials with parameter $\alpha$, that is, using (3.2) in a recursive way and taking into account (3.1) we obtain

$$S_n(x) = \sum_{i=0}^{n} b_i^{(n)} V_{n-i}(x) = \sum_{i=0}^{n} b_i^{(n)} (L_{n-i}^{(\alpha)}(x) - c_{n-i-1} L_{n-i-1}^{(\alpha)}(x)), \quad n \geqslant 0, \tag{3.3}$$

where $b_i^{(n)} = \prod_{j=1}^{i} a_{n-j}$ and $b_0^{(n)} = 1$.

Moreover, from (2.7) we have

$$\lim_{n\to\infty} b_i^{(n)} = \varphi\left(\frac{\lambda+2}{2}\right)^{-i} = a^i \quad \text{for all } i. \tag{3.4}$$

Next, we obtain further asymptotic results for the Sobolev orthogonal polynomials $S_n$. Before, we want to remark that for the case corresponding to $\xi = M = 0$, that is, $d\mu_0 = d\mu_1 = x^\alpha e^{-x} dx$, $\alpha > -1$, Mehler–Heine type formula and Plancherel–Rotach type exterior asymptotics were obtained in Theorem 5 of [6], in other framework. Here, we include this case for completeness.

First, we give the following technical result:

**Lemma 3.2.** *There exist constants $C$ and $r$ with $C > 1$ and $0 < r < 1$ such that the coefficients $b_i^{(n)}$ in* (3.3) *verify $0 < b_i^{(n)} < C\, r^i$ for all $n \geqslant 0$ and $0 \leqslant i \leqslant n$.*

**Proof.** From Lemma 2.2 we know that $a_n > 0$ and $\lim_n a_n = a < 1$, then there exists $r \in (a, 1)$ such that $0 < a_n < r < 1$ for all $n \geqslant n_0$. Therefore, whenever $1 \leqslant i \leqslant n - n_0$, $b_i^{(n)} < r^i$ and for the remaining values of $i$, taking $M = \max\{1, a_0, a_1, \ldots, a_{n_0-1}\}$, we have

$$b_i^{(n)} = \prod_{j=1}^{n-n_0} a_{n-j} \prod_{j=n-n_0+1}^{i} a_{n-j} < r^{n-n_0} M^{i-n+n_0} \leqslant r^{n-n_0} M^{n_0} \leqslant r^i \left(\frac{M}{r}\right)^{n_0}.$$

The result follows with $C = (\frac{M}{r})^{n_0}$. $\square$

**Theorem 3.3.** *Let $\alpha > -1$, the polynomials $S_n$ orthogonal with respect to the inner product* (1.1) *satisfy*

(a) *A Mehler–Heine type formula. It holds: if $\xi < 0$,*

$$\lim_{n\to\infty} \frac{S_n(x/n)}{n^{\alpha-1/2}} = -\frac{d(\xi)}{1-a} x^{-\alpha/2} J_\alpha(2\sqrt{x}),$$

*if $\xi = 0$ and $M > 0$,*

$$\lim_{n\to\infty} \frac{S_n(x/n)}{n^{\alpha-1}} = \frac{1}{1-a} s(x),$$

*and, if $\xi = M = 0$,*

$$\lim_{n\to\infty} \frac{S_n(x/n)}{n^{\alpha-1}} = \frac{1}{1-a} x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}),$$

*where a and* $d(\xi)$ *are given by* (2.7) *and* (2.4), *respectively, and*

$$s(x) = x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}) - (\alpha+1) x^{-\alpha/2} J_\alpha(2\sqrt{x}). \tag{3.5}$$

*All the limits hold uniformly on compact subsets of* $\mathbb{C}$.

(b) *Plancherel–Rotach type exterior asymptotics. It holds*

$$\lim_{n\to\infty} \frac{S_n(nx)}{L_n^{(\alpha)}(nx)} = \frac{\varphi(\frac{x-2}{2}) + 1}{\varphi(\frac{x-2}{2}) + a}$$

*uniformly on compact subsets of* $\mathbb{C}\backslash[0,4]$ *where* $\varphi$ *and a are given by* (1.8) *and* (2.7), *respectively*.

**Proof.** (a) From (3.3), we have

$$\frac{S_n(x/n)}{n^{\alpha-1/2}} = \sum_{i=0}^{n} b_i^{(n)} \frac{V_{n-i}(x/n)}{n^{\alpha-1/2}} =: \sum_{i=0}^{n} v_{n,i}(x/n). \tag{3.6}$$

Whenever $\xi < 0$, dividing by $n^{\alpha-1/2}$ in formula (3.1) evaluated at $x/(n+j)$, and using (1.9) and (2.4), we deduce that

$$\lim_{n\to\infty} \frac{V_n(x/(n+j))}{n^{\alpha-1/2}} = -d(\xi) x^{-\alpha/2} J_\alpha(2\sqrt{x}), \tag{3.7}$$

holds uniformly on compact sets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N} \cup \{0\}$ and therefore

$$\lim_{n\to\infty} \frac{V_{n-i}(x/n)}{n^{\alpha-1/2}} = -d(\xi) x^{-\alpha/2} J_\alpha(2\sqrt{x}),$$

holds uniformly on compact sets of $\mathbb{C}$ and uniformly on $i \in \{0, 1, \dots, n\}$.

Given a compact set $K \subset \mathbb{C}$, because of this last result and Lemma 3.2, there exists a constant $D$, depending only on $K$, such that $|v_{n,i}(x/n)| < Dr^i$ for $i = 0, \dots, n$ and $x \in K$. Therefore, by Lebesgue's dominated convergence theorem, (3.7) and (3.4), we have

$$\lim_{n\to\infty} \sum_{i=0}^{n} v_{n,i}(x/n) = \sum_{i=0}^{\infty} \lim_{n\to\infty} v_{n,i}(x/n) = -d(\xi) x^{-\alpha/2} J_\alpha(2\sqrt{x}) \sum_{i=0}^{\infty} a^i$$

uniformly on compact subsets of $\mathbb{C}$ and the result follows.

Whenever $\xi = 0$, formula (3.7) takes the form

$$\lim_{n\to\infty} \frac{V_n(x/(n+j))}{n^{\alpha-1}} = x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}) - (\alpha+1) x^{-\alpha/2} J_\alpha(2\sqrt{x}), \quad M > 0,$$

$$\lim_{n\to\infty} \frac{V_n(x/(n+j))}{n^{\alpha-1}} = x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}), \quad M = 0.$$

Now we can conclude the proof in the same way as we did in the case $\xi < 0$.

(b) From (3.3) we can write

$$\frac{S_n(nx)}{L_n^{(\alpha)}(nx)} = \sum_{i=0}^{n} b_i^{(n)} \frac{V_{n-i}(nx)}{L_n^{(\alpha)}(nx)}, \quad x \in \mathbb{C}\backslash[0,4].$$

The polynomials $V_n$ satisfy the following Plancherel–Rotach type exterior asymptotics

$$\lim_{n\to\infty} \frac{V_n((n+j)x)}{L_n^{(\alpha)}((n+j)x)} = 1 + \varphi\left(\frac{x-2}{2}\right)^{-1} \tag{3.8}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$ and uniformly on $j \in \mathbb{N} \cup \{0\}$. This is a simple consequence of (1.10) and (3.1).

Now, handling in the same way as in (a) and using (1.7), we can deduce

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=0}^{n} b_i^{(n)} \frac{V_{n-i}(nx)}{L_n^{(\alpha)}(nx)} &= \lim_{n\to\infty} \sum_{i=0}^{n} b_i^{(n)} \frac{V_{n-i}(nx)}{L_{n-i}^{(\alpha)}(nx)} \frac{L_{n-i}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)} \\ &= \sum_{i=0}^{\infty} \lim_{n\to\infty} \left( b_i^{(n)} \frac{V_{n-i}(nx)}{L_{n-i}^{(\alpha)}(nx)} \frac{L_{n-i}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)} \right) \\ &= \left(1 + \varphi\left(\frac{x-2}{2}\right)^{-1}\right) \sum_{i=0}^{\infty} \left(\frac{-a}{\varphi(\frac{x-2}{2})}\right)^i,\end{aligned}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$, and thus, the result follows. □

The above theorem allows us to obtain additional results about asymptotic properties of zeros and critical points of Sobolev polynomials $S_n$. First, recall that $S_n$ has $n$ different, real zeros, and at most one of them is outside $(0,+\infty)$, they interlace with those of $L_n^{(\alpha)}$ and the zeros of $S_n'$ with those of $T_{n-1}$ (for more information about location of these zeros, see [10]). Moreover, from Theorem 4.11 in [11], it follows that they accumulate on $\{\xi\} \cup [0,+\infty)$ when $M>0$ and in $[0,+\infty)$ when $M=0$.

**Corollary 3.4.** *For $\alpha > -1$, denote with $j_{\alpha,i}$ the ith positive zero of the Bessel function $J_\alpha(x)$. Let $\{x_{n,i}\}_{i=1}^n$ be the zeros in increasing order of the polynomial $S_n$ orthogonal with respect to the inner product* (1.1) *and $\{\tilde{x}_{n,i}\}_{i=1}^{n-1}$ be the critical points of $S_n$. Then,*

(a) *If $\xi<0$, we have*

$$\lim_{n\to\infty} nx_{n,i} = \frac{j_{\alpha,i}^2}{4} \quad \text{and} \quad \lim_{n\to\infty} n\tilde{x}_{n,i} = \frac{j_{\alpha+1,i}^2}{4}.$$

(b) *If $\xi=0$ and $M>0$, we have*

$$\lim_{n\to\infty} nx_{n,i} = s_{\alpha,i},$$

$$\lim_{n\to\infty} n\tilde{x}_{n,1} = 0 \quad \text{and} \quad \lim_{n\to\infty} n\tilde{x}_{n,i} = \frac{j_{\alpha+2,i-1}^2}{4}, \quad i \geqslant 2,$$

*where $s_{\alpha,i}$ denotes the ith real zero of function $s(x)$ defined in* (3.5).

(c) *If $\xi = M = 0$, we have*

$$\lim_{n\to\infty} nx_{n,i} = \frac{j_{\alpha-1,i}^2}{4} \quad \text{and} \quad \lim_{n\to\infty} n\tilde{x}_{n,i} = \frac{j_{\alpha,i}^2}{4},$$

*where three cases are possible*:

- *If* $-1<\alpha<0$, (*that is* $-2<\alpha-1<-1$) $j_{\alpha-1,1}$ *is any of the two purely imaginary zeros of* $J_{\alpha-1}(x)$ *and, for* $i\geqslant 2$, $j_{\alpha-1,i}$ *is the* $(i-1)$*th positive real zero of* $J_{\alpha-1}(x)$.
- *If* $\alpha=0$, $j_{\alpha-1,1}=j_{-1,1}=0$ *and, for* $i\geqslant 2$, $j_{-1,i}$ *is the* $(i-1)$*th positive real zero of* $J_{-1}(x)$.
- *If* $\alpha>0$, $j_{\alpha-1,i}$ *is the ith positive real zero of* $J_{\alpha-1}(x)$.

**Proof.** (a) The result for the zeros is a consequence of Theorem 3.3(a) and Hurwitz's theorem. Concerning the critical points, since we have uniform convergence in the Mehler–Heine type formula (Theorem 3.3(a)), taking derivatives and using properties of Bessel functions ([13, Section 1.7]) we get

$$\lim_{n\to\infty}\frac{S_n'(x/n)}{n^{\alpha+1/2}}=\frac{d(\xi)}{1-a}x^{-(\alpha+1)/2}J_{\alpha+1}(2\sqrt{x}),$$

uniformly on compact subsets of $\mathbb{C}$, which yields the result.

(b) Denote $g_\alpha(x)=x^{-\alpha/2}J_\alpha(2\sqrt{x})=\sum_{i=0}^{\infty}\frac{(-x)^i}{i!\Gamma(i+\alpha+1)}$, $x\in\mathbb{C}$. From the definition of $s(x)$ (see (3.5) and (2.10)), we can write

$$s(x)=-g_\alpha(x)-x\,g_{\alpha+1}(x)=\sum_{i=0}^{\infty}\frac{(i-1)}{i!}\frac{(-x)^i}{\Gamma(i+\alpha+1)},$$

for $\alpha>-1$ and $x\in\mathbb{C}$.

Observe that, if $x\in(-\infty,0)$, then $g_\alpha(x)>0$, $\lim_{x\to-\infty}g_\alpha(x)=+\infty$ and $\lim_{x\to-\infty}s(x)=+\infty$.

Using formula (1.71.5) in [13] we have $s'(x)=xg_{\alpha+2}(x)$, $x\in\mathbb{C}$ and therefore $s(x)$ is a decreasing function on $(-\infty,0)$. Since $s(0)<0$, we have that $s(x)$ has only one negative zero. Moreover, because the positive zeros of $J_\alpha(x)$ interlace with those of $J_{\alpha+1}(x)$, we can deduce that there is precisely one zero of $s(x)$ between two consecutive positive zeros of $J_{\alpha+1}(2\sqrt{x})$.

Now, again by Hurwitz's theorem the result for the zeros follows. Finally, we have

$$\lim_{n\to\infty}\frac{S_n'(x/n)}{n^{\alpha}}=\frac{1}{1-a}s'(x)=\frac{1}{1-a}x^{-\alpha/2}\,J_{\alpha+2}(2\sqrt{x})$$

uniformly on compact subsets of $\mathbb{C}$, which implies the result.

(c) It can be obtained in a similar way as we did in (a) (see also Proposition 4 and Remark 2 in [6]). □

**Remark.** The existence of a negative zero of $S_n$ is an interesting problem (see, for example, [10, Section 5]). Here, we have found the range of values of the parameters $\alpha$, $\xi$, and $M$ for which the polynomials $S_n$ have a negative zero for $n$ sufficiently large, i.e.:

- The polynomials $S_n$ have one negative zero for $n$ sufficiently large if and only if either $\alpha>-1$, $\xi=0$, and $M>0$ or $-1<\alpha<0$ and $\xi=M=0$.
- Moreover, the critical points of $S_n$ for $n$ sufficiently large lie on $[0,+\infty)$.

Finally, observe that, $i$ a fixed positive integer, the zeros of $S_n$ satisfy $\lim_n x_{ni} = 0$; more precisely $x_{ni} = O(1/n)$. Even, whenever $S_n$ has a negative zero $x_{n1}$, $\lim_n x_{n1} = 0$.

In order to illustrate these analytic results, we show numerically the behaviour of the first zero, $x_{n1}$, of $S_n$ in the cases of Corollary 3.4 where the nonlinear recurrence relation satisfied by $c_n$ (formula (2.8)) and $a_n$ (formula (4.7) in [11]) have been used.
For better reading we have rounded the numerical results in the case (c) to six digits and we also eliminated the column $x_{n,1}$ for $\alpha = 0$, see Table 1.

Using the zero distribution of the orthonormal Laguerre polynomials $l_n^{(\alpha)}$ and the $n$th root asymptotics for the scaled $l_n^{(\alpha)}(nx)$ polynomials (see [14,15]), and Theorem 3.3, the asymptotic distribution of the contracted zeros and the $n$th root asymptotics for the scaled Sobolev polynomials can be derived:

**Corollary 3.5.** (a) *The contracted zeros of* $S_n$, $\frac{x_{ni}}{n}$, *accumulate on* $[0,4]$ *and they have the same asymptotic distribution as the contracted zeros of the orthonormal Laguerre polynomials* $l_n^{(\alpha)}$, *that is, it has density* $dv(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,dx$.

(b) *The formula*

$$\lim_n |S_n(nx)|^{1/n} = \exp\left\{1 + \int_0^4 \log|x-y|\,dv(y)\right\}$$

*is true uniformly on compact subsets of* $\mathbb{C}\backslash[0,4]$.

**Remark.** For monic Sobolev polynomials $\widehat{S_n}$ we have

$$\lim_n \frac{1}{n}|\widehat{S_n}(nx)|^{1/n} = \exp\left\{\frac{1}{2\pi}\int_0^4 \log|x-y|\sqrt{\frac{4-y}{y}}\,dy\right\}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$ or, equivalently,

$$\lim_n \frac{1}{2n}|\widehat{S_n}(2nx)|^{1/n} = \exp\left\{\frac{1}{\pi}\int_0^2 \log|x-y|\sqrt{\frac{2-y}{y}}\,dy\right\}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,2]$.

Observe that this is exactly the result for monic Laguerre–Sobolev polynomials of type I obtained in [12, Theorem 2.2], using potential theory. (In all the results in [12] concerned with $n$th root asymptotic, the locally uniformly convergence holds in $\mathbb{C}\backslash[0,2]$ instead of in $[0,2]$).

## 4. Upper bound for Sobolev orthogonal polynomials $S_n$

To obtain an upper bound for Sobolev orthogonal polynomials our starting point will be formula (3.3). A global estimate for classical Laguerre polynomials with

Table 1

(a) $\alpha = -0.5,\ \xi = -10,\ \lambda = 1.$

| | $M = 0$ | | $M = 2$ | |
|---|---|---|---|---|
| $n$ | $nx_{n,1}$ | $x_{n,1}$ | $nx_{n,1}$ | $x_{n,1}$ |
| 50 | 0.5985025263 | 0.0119700505 | 0.6560458759 | 0.0131209175 |
| 100 | 0.6022670343 | 0.0060226703 | 0.6421427906 | 0.0064214279 |
| 150 | 0.6042891938 | 0.0040285946 | 0.6366175985 | 0.0042441173 |
| 200 | 0.6056173146 | 0.0030280866 | 0.6335134224 | 0.0031675671 |
| 250 | 0.6065803700 | 0.0024263219 | 0.6314766976 | 0.0025259068 |
| | 0.6168502751 | | 0.6168502751 | |
| | $\frac{j_{\alpha,1}^2}{4}$ | | $\frac{j_{\alpha,1}^2}{4}$ | |

(b) $\xi = 0,\ M = 2,\ \lambda = 1.$

| | $\alpha = -0.5$ | | $\alpha = 2.5$ | |
|---|---|---|---|---|
| $n$ | $nx_{n,1}$ | $x_{n,1}$ | $nx_{n,1}$ | $x_{n,1}$ |
| 50 | −0.9995290524 | −0.0199905810 | −4.4617547547 | −0.0892350951 |
| 100 | −1.0118720710 | −0.0101187207 | −4.3961517653 | −0.0439615177 |
| 150 | −1.0191664985 | −0.0067944433 | −4.3745120007 | −0.0291634133 |
| 200 | −1.0240655338 | −0.0051203277 | −4.3637453016 | −0.0218187265 |
| 250 | −1.0276502314 | −0.0041106009 | −4.3573033494 | −0.0174292134 |
| | −1.066582516 | | −4.3325842295 | |
| | $s_{\alpha,1}$ | | $s_{\alpha,1}$ | |

(c) $\xi = M = 0,\ \lambda = 1.$

| | $\alpha = -0.5$ | | $\alpha = 0$ | $\alpha = 2.5$ | |
|---|---|---|---|---|---|
| $n$ | $nx_{n,1}$ | $x_{n,1}$ | $nx_{n,1}$ | $nx_{n,1}$ | $x_{n,1}$ |
| 50 | −0.366308 | −0.007326 | $1.41153\ 10^{-19}$ | 4.98876 | 0.09978 |
| 100 | −0.362992 | −0.003630 | $3.56416\ 10^{-40}$ | 5.01701 | 0.05017 |
| 150 | −0.361917 | −0.002413 | $6.74969\ 10^{-61}$ | 5.02696 | 0.03351 |
| 200 | −0.361384 | −0.001807 | $1.13621\ 10^{-81}$ | 5.03204 | 0.02516 |
| 250 | −0.361066 | −0.001444 | $1.79310\ 10^{-102}$ | 5.03512 | 0.02014 |
| | −0.359807 | | 0 | 5.04768 | |
| | $\frac{j_{\alpha-1,1}^2}{4}$ | | $\frac{j_{\alpha-1,1}^2}{4}$ | $\frac{j_{\alpha-1,1}^2}{4}$ | |

respect to $n,\ x$, and $\alpha$ is known (see formulas (22.14.13) and (22.14.14) in [1]): For $x \geqslant 0$, $n \geqslant 0$ and $\alpha > -1$, the inequality

$$|L_n^{(\alpha)}(x)| \leqslant A(n,\alpha)\, e^{x/2}, \tag{4.1}$$

where

$$A(n,\alpha)=\begin{cases}\dfrac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} & \text{if } \alpha\geqslant 0,\\[2mm] 2-\dfrac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} & \text{if } -1<\alpha\leqslant 0,\end{cases}\tag{4.2}$$

holds.

Therefore, we need upper estimates for the coefficients $b_i^{(n)}$ (that is, for $a_n$) and $c_n$. This is done in the next lemma.

**Lemma 4.1.** *For $n\geqslant 1$, the coefficients $c_n$ and $a_n$ in Lemma 2.2 satisfy*

$$\frac{n+1+\alpha}{2(n+1)+\alpha-\xi}<c_n<2+\frac{\alpha-\xi}{n},\quad n\geqslant 1,\tag{4.3}$$

*and*

$$a_n<\left(2+\frac{\alpha-\xi}{n}\right)\frac{2n+\alpha-\xi}{(2+\lambda)n+\alpha-\xi},\quad n\geqslant 1.\tag{4.4}$$

**Proof.** From recurrence relation (2.8) for the parameters $c_n$, since $c_n>0$ for every $n$, we get inequalities (4.3).

On the other hand, recall that the coefficients $a_n$ in formula (2.6) are defined by $a_n=c_n\frac{||L_n^{(\alpha)}||^2_{\mu_0}}{||S_n||^2_S}$. As a consequence of the extremal property of the norms of the monic orthogonal polynomials, we have

$$||S_n||^2_S\geqslant ||L_n^{(\alpha)}||^2_{\mu_0}+\lambda||T_{n-1}||^2_{\mu_1},\quad n\geqslant 1,$$

which, by the definition of $c_n$, (see (2.2)), and (1.2) leads to

$$\frac{||S_n||^2_S}{||L_n^{(\alpha)}||^2_{\mu_0}}\geqslant 1+\lambda\frac{||T_{n-1}||^2_{\mu_1}}{||L_n^{(\alpha)}||^2_{\mu_0}}=1+\lambda\frac{n}{n+\alpha}c_{n-1}.\tag{4.5}$$

Thus, from (4.3) and (4.5), we obtain $\frac{||L_n^{(\alpha)}||^2_{\mu_0}}{||S_n||^2_S}\leqslant\left(1+\frac{\lambda n}{2n+\alpha-\xi}\right)^{-1}$ and so (4.4) holds. $\square$

A global estimate for Sobolev orthogonal polynomials is now deduced:

**Theorem 4.2.** *For $x\geqslant 0$, $\alpha>-1$ and $n\geqslant 1$ we have*

$$|S_n(x)|\leqslant Cf_n(r)A(n,\alpha)\,e^{x/2},$$

*where*

$$C = \begin{cases} 3+\alpha-\xi & \textit{if } \alpha \geqslant \xi, \\ 3 & \textit{if } \alpha \leqslant \xi, \end{cases} \qquad r = \begin{cases} \frac{(2+\alpha-\xi)^2}{2+\lambda+\alpha-\xi} & \textit{if } \alpha \geqslant \xi, \\ \frac{4}{2+\lambda} & \textit{if } \alpha \leqslant \xi. \end{cases}$$

$$f_n(r) = \begin{cases} n & \textit{if } r = 1, \\ \frac{1-r^n}{1-r} & \textit{if } r \neq 1 \end{cases} \quad \textit{and } A(n,\alpha) \textit{ is given by } (4.2).$$

**Proof.** Observe that, using $b_n^{(n)} = a_0 b_{n-1}^{(n)}$, formula (3.3) can be written in the form

$$S_n(x) = \sum_{i=0}^{n-2} b_i^{(n)} (L_{n-i}^{(\alpha)}(x) - c_{n-i-1}\, L_{n-i-1}^{(\alpha)}(x)) + b_{n-1}^{(n)} (L_1^{(\alpha)}(x) - c_0 + a_0).$$

Then, as $a_0 = c_0$,

$$|S_n(x)| \leqslant \sum_{i=0}^{n-2} b_i^{(n)} (|L_{n-i}^{(\alpha)}(x)| + c_{n-i-1} |L_{n-i-1}^{(\alpha)}(x)|) + b_{n-1}^{(n)} |L_1^{(\alpha)}(x)|. \tag{4.6}$$

It is easy to prove that, for $\alpha > -1$ and $i = 0, 1, \dots, n$, $A(n-i,\alpha) \leqslant A(n,\alpha)$ and therefore, by (4.1), $|L_{n-i}^{(\alpha)}(x)| \leqslant A(n,\alpha) e^{x/2}$ which leads to

$$|S_n(x)| \leqslant \left[ \sum_{i=0}^{n-2} b_i^{(n)} (1 + c_{n-i-1}) + b_{n-1}^{(n)} \right] A(n,\alpha) e^{x/2}.$$

From (4.3), analysing separately the cases $\alpha - \xi < 0$ (that is, $-1 < \alpha < \xi \leqslant 0$) and $\alpha - \xi \geqslant 0$, we get

$$c_n < c = \begin{cases} 2+\alpha-\xi & \text{if } \alpha \geqslant \xi, \\ 2 & \text{if } \alpha \leqslant \xi \end{cases}.$$

In a similar way, from (4.4) we deduce that

$$a_n < r = \begin{cases} \frac{(2+\alpha-\xi)^2}{2+\lambda+\alpha-\xi} & \text{if } \alpha \geqslant \xi, \\ \frac{4}{2+\lambda} & \text{if } \alpha \leqslant \xi \end{cases}.$$

It suffices to write $C = 1 + c$ and the result follows. □

In some particular cases the upper estimate for the Sobolev polynomials $S_n$ can be improved. One of them occurs when $M = 0$ in the inner product (1.1), that is $d\mu_1 = \frac{x^{\alpha+1}\, e^{-x}}{x-\xi}\, dx$. In this situation, integrating in formula (2.1) with respect to the measure $\mu_1$, we have

$$c_n \int_0^\infty L_{n-1}^{(\alpha+1)}(x)\, d\mu_1(x) = \int_0^\infty L_n^{(\alpha+1)}(x)\, d\mu_1(x), \quad n \geqslant 1.$$

Using Rodrigues' formula for Laguerre polynomials and after integration by parts $n-1$ times, it can be derived, see ([11]),

$$\int_0^\infty L_{n-1}^{(\alpha+1)}(x)\, d\mu_1(x) = \int_0^\infty \frac{x^{n+\alpha}\, e^{-x}}{(x-\xi)^n}\, dx.$$

This implies that, for every $n \geqslant 1$, $c_n \leqslant 1$. (Observe that $c_n = 1$ only if $\xi = 0$).

As a consequence, we have $a_n \leqslant 1$ and $b_i^n \leqslant 1$ for every $n \geqslant 1$ and $i = 0, \dots, n-1$. Thus, the upper estimate for $S_n$ in Theorem 4.2 becomes

$$|S_n(x)| \leqslant (2\,n-1) A(n,\alpha) e^{x/2}.$$

Improvements of the estimates for $|L_n^\alpha(x)|$ lead to improvements of the ones for $|S_n(x)|$, according to formula (4.6) (see for instance [5]).

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