Available online at www.sciencedirect.com

ScienceDirect

Journal of Number Theory 128 (2008) 2138–2158

JOURNAL OF Number Theory

www.elsevier.com/locate/jnt

Average values of L-functions in characteristic two ☆

Yen-Mei J. Chen

Department of Mathematics, National Central University, Jhongli City, Taoyuan County, 32001, Taiwan

Received 2 June 2007; revised 19 December 2007

Available online 1 April 2008

Communicated by Dinesh Thakur

Abstract

Gauss made two conjectures about average values of class numbers of orders in quadratic number fields, later on proven by Lipschitz and Siegel. A version for function fields of odd characteristic was established by Hoffstein and Rosen. In this paper, we extend their results to the case of even characteristic. More precisely, we obtain formulas of average values of L-functions associated to orders in quadratic function fields over a constant field of characteristic two, and then derive formulas of average class numbers of these orders.

MSC: 11R11; 11R29; 11R42; 11R58

Keywords: L-functions; Orders; Class numbers; Quadratic function fields

1. Introduction

Let D be an integer, which is a non-square and congruent to 0 or 1 modulo 4. Denote by $\mathcal{O}_D$ the order $\mathbb{Z}+\mathbb{Z}\sqrt{D}$ in the quadratic field $\mathbb{Q}(\sqrt{D})$. And denote by h_D the class number of $\mathcal{O}_D$. Based on extensive numerical evidence, Gauss made two conjectures about the average value of the class numbers h_D, reformulated as follows:

Conjecture 1. *If D is negative and $D=-4k$, then $\sum_{1\leqslant k\leqslant N} h_D \sim \frac{4\pi}{21\zeta(3)}N^{\frac{3}{2}}$.*

☆ Research partially supported by National Science Council, Republic of China.
E-mail address: ymjchen@math.ncu.edu.tw.

doi:10.1016/j.jnt.2007.12.011

Conjecture 2. *If D is positive and $D = 4k$, then $\sum_{1 \leqslant k \leqslant N} h_D R_D \sim \frac{4\pi^2}{21\zeta(3)} N^{\frac{3}{2}}$. Here R_D is the regulator of the order $\mathcal{O}_D$.*

Let $\psi_D(n) = (\frac{D}{n})$ denote the Kronecker symbol and consider the associated L-function

$$L(s, \psi_D) = \sum_{n=1}^{+\infty} \frac{\psi_D(n)}{n^s}.$$

Denote by w_D the number of roots of unity in $\mathcal{O}_D$. Dirichlet proved the following famous class number formula:

$$L(1, \psi_D) = \begin{cases} \frac{2\pi h_D}{w_D \sqrt{|D|}} & \text{if } D < 0, \\ \frac{h_D R_D}{\sqrt{|D|}} & \text{if } D > 0. \end{cases}$$

Then the following result of Siegel implies the two conjectures:

$$\sum L(1, \psi_D) = \frac{1}{2} \frac{\zeta(2)}{\zeta(3)} N + O\big(N^{\frac{1}{2}} \log N\big),$$

where the sum is over all positive discriminants D between 1 and N, or all negative discriminants D such that $1 \leqslant |D| \leqslant N$.

Let $k = \mathbb{F}_q(t)$ be the rational function field over the finite field $\mathbb{F}_q$ with q elements, where q is odd. Let $\mathbb{A} = \mathbb{F}_q[t]$ be the corresponding polynomial ring, and let $\mathbb{A}^+$ be the subset consisting of monic polynomials in $\mathbb{A}$. Consider the quadratic function field $K = k(\sqrt{M})$, where M is a non-square polynomial in $\mathbb{A}$. Let $\mathcal{O}_M$ be the $\mathbb{A}$-order $\mathbb{A} + \mathbb{A}\sqrt{M}$ in the ring of integers $\mathcal{O}_K$; and denote by h_M the class number of $\mathcal{O}_M$. If $P \in \mathbb{A}^+$ is irreducible, define $\chi_M(P)$ according to

$$\chi_M(P) = \begin{cases} 0 & \text{if } P \mid M, \\ 1 & \text{if } P \nmid M \text{ and } M \text{ is a square modulo } P, \\ -1 & \text{if } P \nmid M \text{ and } M \text{ is a non-square modulo } P. \end{cases}$$

One can extend $\chi_M(N)$ to all $N \in \mathbb{A}^+$ multiplicatively. The following analogy of Dirichlet's Theorem is proven in E. Artin's thesis [1]:

Theorem. *(See Artin [1].) Let M be a square-free polynomial in $\mathbb{F}_q[t]$ of degree m, where q is odd. Then*

(a) *if m is odd, $L(1, \chi_M) = \frac{\sqrt{q}}{\sqrt{|M|}} h_M$,*
(b) *if m is even and $\operatorname{sgn}_2(M) = -1$, $L(1, \chi_M) = \frac{q+1}{2\sqrt{|M|}} h_M$,*
(c) *if m is even and $\operatorname{sgn}_2(M) = 1$, $L(1, \chi_M) = \frac{q-1}{\sqrt{|M|}} h_M R_M$.*

Here R_M is the regulator of $\mathcal{O}_M$ and $\operatorname{sgn}_2(M) = 1$ (respectively -1) if the leading coefficient of M is a square (respectively non-square) in $\mathbb{F}_q^\star$.

Recall the Dedekind zeta function of $\mathbb{A}$: for $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$\zeta_{\mathbb{A}}(s) := \sum_{N \in \mathbb{A}^+} \frac{1}{|N|^s} = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{1 - q^{1-s}}.$$

Then the previous theorem of Artin leads to the following average theorem:

Theorem. *(See Hoffstein and Rosen [5], 1992.) Let h_M be the class number of the $\mathbb{A}$-order $\mathcal{O}_M = \mathbb{A} + \mathbb{A}\sqrt{M}$. The following sums are over all non-square monic polynomials in $\mathbb{F}_q[t]$ of degree m, where q is odd.*

(a) *If m is odd, $q^{-m} \sum_M h_M = \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)} q^{\frac{m-1}{2}} - \frac{1}{q}$.*
(b) *If m is even, $q^{-m} \sum_M h_{\gamma M} = \frac{2}{q+1} \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)} q^{\frac{m}{2}} + \frac{2}{q}$, where $\gamma \in \mathbb{F}_q^\star$ is a non-square constant.*
(c) *If m is even, $q^{-m} \sum_M h_M R_M = \frac{1}{q-1} (\frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)} q^{\frac{m}{2}} - \frac{q+1}{q}) - \frac{m}{q}$.*

The previous theorem of Hoffstein and Rosen is for odd characteristic. The main purpose of this paper is to extend it to characteristic 2. In this case, all separable quadratic extensions are Artin–Schreier, and a detailed study of this elementary theory is given in [2]. We will briefly review this theory in next section. However, it is remarkable that separable quadratic extensions of the rational function field $\mathbb{F}_q(t)$ in characteristic 2 are indeed more complicate than the odd characteristic case. First, all ramification is wild. Secondly, the discriminant does not determine the quadratic field alone. There are many quadratic fields with the same discriminant while in the classical case and the odd characteristic case, the discriminant uniquely determines the quadratic field. The crucial point of this paper is that we succeed in finding a "good" parameterization of these quadratic fields. That makes these beautiful averaging formulas come into view. At this moment, we would like to state an average theorem in characteristic 2, that is a combination of Corollaries 4.16, 4.10, and 4.7.

Theorem in characteristic 2. *Let m be a positive integer, and let d be a nonnegative integer.*

(a) *The following sum is over all $\mathbb{A}$-orders with finite discriminants of degree $2m$ in imaginary quadratic fields whose local discriminants at infinity have degree $2d + 2$*

$$\frac{\sum_{[(\widetilde{D},M)] \in \widetilde{\mathcal{D}}_{m,d}} h_{(\widetilde{D},M)}}{\#\widetilde{\mathcal{D}}_{m,d}} = \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)} q^{m+d} + \frac{1}{q}.$$

(b) *The following sum is over all $\mathbb{A}$-orders with discriminants of degree $2m$ in imaginary quadratic fields that are non-constant field extensions and unramified at infinity*

$$\frac{\sum_{[(D',M)] \in \mathcal{D}'_m} h_{(D',M)}}{\#\mathcal{D}'_m} = \frac{2}{q+1} \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)} q^m.$$

(c) *The following sum is over all $\mathbb{A}$-orders with discriminants of degree $2m$ in real quadratic fields*

$$\frac{\sum_{[(D,M)]\in\mathcal{D}_m} R_{(D,M)}h_{(D,M)}}{\#\mathcal{D}_m} = \frac{1}{q-1}\frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)}q^m - \frac{2m}{q} + \frac{2m}{q}\frac{1}{q^m-1}.$$

Here $\mathcal{D}_m$, $\mathcal{D}'_m$, $\widetilde{\mathcal{D}}_{m,d}$ parameterize all the $\mathbb{A}$-orders in the three cases above and will be introduced in Section 4.

In Section 2 we describe some elementary facts about quadratic function fields of characteristic 2, then recall the Hasse symbols [4], and finally prove the analogue of Dirichlet's class number formula in characteristic 2 (cf. Theorem 2.5). In Section 3 we define several exponential sums and prove some lemmas that will be used later. Finally in Section 4, we derive some formulas (Theorems 4.5, 4.9, and 4.14 below) about sums of values of L-functions associated to $\mathbb{A}$-orders in quadratic function fields over a constant field of characteristic 2 and hence obtain formulas of average class numbers of these $\mathbb{A}$-orders.

2. Quadratic function fields of characteristic two

Let $k = \mathbb{F}_q(t)$ be the rational function field over the finite field $\mathbb{F}_q$ with q elements, where q is even. Let $\mathbb{A} = \mathbb{F}_q[t]$ be the corresponding polynomial ring, and let $\mathbb{A}^+$ be the subset consisting of monic polynomials in $\mathbb{A}$. Let $\wp : k \to k$ be the additive homomorphism defined by $\wp(x) = x^2 + x$. For $f \in \mathbb{A}$, denote by $\operatorname{sgn}(f)$ the leading coefficient of f. From now on, we fix a constant $\xi \in \mathbb{F}_q \setminus \wp(\mathbb{F}_q)$.

Let K/k be a separable quadratic extension. Then $K = k(\alpha)$, where α is a root of $x^2 + x = \frac{D_1}{D_2}$. Here $D_1, D_2 \in \mathbb{A}$ can be normalized to satisfy the following conditions (cf. [4]):

$$\text{(1)}\quad \operatorname{sgn}(D_2) = 1, \quad \gcd(D_1, D_2) = 1, \quad \text{and} \quad D_2 = \prod_{i=1}^{s} P_i^{2e_i - 1},$$

$$\text{(2)}\quad \text{if } \deg(D_1) = \deg(D_2), \quad \text{then } \operatorname{sgn}(D_1) = \xi,$$

$$\text{(3)}\quad \text{if } \deg(D_1) > \deg(D_2), \quad \text{then } 2 \nmid \deg(D_1) - \deg(D_2),$$

where P_i's are distinct monic irreducible polynomials in $\mathbb{A}$ and e_i's are positive integers. Note that D_2 is uniquely determined, but D_1 is not.

There are three possibilities about the ramification of the place at infinity (cf. [4]):

$$\begin{cases} \infty \text{ ramifies} & \text{if } \deg(D_1) > \deg(D_2) \text{ and } 2 \nmid \deg(D_1) - \deg(D_2), & \text{(I)} \\ \infty \text{ is inert} & \text{if } \deg(D_1) = \deg(D_2) \text{ and } \operatorname{sgn}(D_1) = \xi, & \text{(II)} \\ \infty \text{ splits} & \text{if } \deg(D_1) < \deg(D_2). & \text{(III)} \end{cases}$$

In cases (I) and (II), K is said to be *imaginary*; in case (III), K is said to be *real*. Let $G(K) = \prod_{i=1}^{s} P_i^{e_i}$. Then $\mathcal{O}_K$ is a rank 2 $\mathbb{A}$-module, which can be explicitely described as follows:

Fact. $\mathcal{O}_K = \mathbb{A} + G(K)\alpha\mathbb{A}$.

Proof. Denote $G(K)$ by G for simplicity. Given any $a, b \in \mathbb{A}$, one can see that $a + bG\alpha$ is a root of the monic quadratic polynomial

$$x^2 + bGx + a^2 + abG + b^2G^2D_1/D_2,$$

which is a polynomial in $\mathbb{A}[x]$ by the definition of G. Hence $a + bG\alpha \in \mathcal{O}_K$.

Conversely, assume $a + bG\alpha \in \mathcal{O}_K$, where $a, b \in k$. Then bG and $a^2 + abG + b^2G^2D_1/D_2$ are both elements in $\mathbb{A}$. For monic irreducible P with $P \nmid G$, it is clear that $\mathrm{ord}_P(b) \geqslant 0$ and thus $\mathrm{ord}_P(a) \geqslant 0$. For monic irreducible P with $P \mid G$, observe that $\mathrm{ord}_P(a^2)$ is even and $\mathrm{ord}_P(b^2G^2D_1/D_2)$ is odd. One can conclude that both $\mathrm{ord}_P(a)$ and $\mathrm{ord}_P(b)$ are always non-negative. Therefore, a and b are both elements in $\mathbb{A}$. $\square$

Denote by $g(K)$ the genus of K. Recall that the discriminant of K is of degree $2g(K) + 2$ in all cases (cf. [4]). By the previous fact, the discriminant of $\mathcal{O}_K$ is equal to $G(K)^2$. In case (I), one can check that the local discriminant of K at infinity has degree $\deg(D_1) - \deg(D_2) + 1$ and thus

$$g(K) = \deg\bigl(G(K)\bigr) + \frac{\deg(D_1) - \deg(D_2) - 1}{2}.$$

In cases (II) and (III), it is clear that the local discriminant of K at infinity has degree 0 and thus $g(K) = \deg(G(K)) - 1$.

Now we recall the definition of Hasse symbols in characteristic 2. Let $P \in \mathbb{A}^+$ be irreducible. For $a \in k$ which is P-integral, define the Hasse symbol $[a, P)$ with value in $\mathbb{F}_2$ by

$$[a, P) = \begin{cases} 0 & \text{if } \mathbf{x}^2 + \mathbf{x} \equiv a \pmod{P} \text{ is solvable in } \mathbb{A}, \\ 1 & \text{otherwise.} \end{cases}$$

For $N \in \mathbb{A}$ prime to the denominator of a, write $N = \epsilon \prod_{i=1}^{s} P_i^{e_i}$, where $\epsilon = \mathrm{sgn}(N)$, P_i's are distinct monic irreducible in $\mathbb{A}^+$ and e_i's are positive integers, and define $[a, N)$ to be $\sum_{i=1}^{s} e_i[a, P_i)$.

For $a \in k$, $N \in \mathbb{A}$, $N \neq 0$ also define the quadratic symbol:

$$\left\{\frac{a}{N}\right\} = \begin{cases} (-1)^{[a,N)} & \text{if } N \text{ is prime to the denominator of } a, \\ 0 & \text{otherwise.} \end{cases}$$

This symbol is clearly additive in its first variable, and multiplicative in its second variable.

For the quadratic field K, one associates a character χ_K on $\mathbb{A}^+$ that is defined by $\chi_K(N) = \{\frac{D_1/D_2}{N}\}$. Consider the L-function of K: for $s \in \mathbb{C}$ with $\Re(s) \geqslant 1$,

$$L_K(s) = \sum_{N \in \mathbb{A}^+} \frac{\chi_K(N)}{|N|^s} = \prod_{P \in \mathbb{A}^+} \left(1 - \frac{\chi_K(P)}{|P|^s}\right)^{-1}.$$

Recall the Dedekind zeta function of $\mathcal{O}_K$: for $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$\zeta_{\mathcal{O}_K}(s) = \sum_{\mathfrak{A}} N(\mathfrak{A})^{-s},$$

where the sum ranges over nonzero integral ideals in K. Then one has $\zeta_{\mathcal{O}_K}(s) = \zeta_{\mathbb{A}}(s)L_K(s)$ (cf. [4]). Note that $L_K(s)$ is not the L-function associated to the field extension K/k, but is obtained from the latter by suppressing the Euler factors at infinity.

The following theorem relates the class number h_K of the maximal order $\mathcal{O}_K$ to the special value of the L-function of K at $s = 1$.

Theorem 2.1. *(See Chen and Yu [2], Theorem 5.2.) Let K/k be quadratic of genus g. Then*

$$L_K(1) = \begin{cases} h_K/q^g & \text{if } \infty \text{ ramifies,} \\ (q+1)h_K/2q^{g+1} & \text{if } \infty \text{ is inert,} \\ (q-1)R_K h_K/q^{g+1} & \text{if } \infty \text{ splits.} \end{cases}$$

Note that one has $L_K(s) = \sum_{n=0}^{\infty} \frac{\sigma_n}{q^{ns}}$ where $\sigma_n = \sum_{\substack{N \in \mathbb{A}^+ \\ \deg(N)=n}} \chi_K(N)$.

Remark 2.2. (See [2, Proposition 5.1] and [4, Eq. (31)].) If K/k is quadratic of genus g, $L_K(s)$ is a polynomial in q^{-s} of degree less than $2g+2$ and hence $\sigma_n = 0$ for all $n \geqslant 2g+2$.

All the orders considered in this paper are $\mathbb{A}$-orders. The following proposition can be proven almost in the same way as in classical case (cf. [3] or [6]).

Proposition 2.3. *Let O be an order in a quadratic field K of conductor F. Then*

$$h_{\mathcal{O}}\big(\mathcal{O}_K^\star : \mathcal{O}^\star\big) = h_K|F| \prod_{P|F} \left(1 - \frac{\chi_K(P)}{|P|}\right).$$

Proof. For any order $\mathcal{O}$ in K, denote by $I(\mathcal{O})$ and $P(\mathcal{O})$ the groups of proper ideals and proper principal ideals. And for $M \in \mathbb{A}^+$, denote by $I(\mathcal{O}, M)$ and $P(\mathcal{O}, M)$ the subgroups of ideals prime to M. Then

$$I(\mathcal{O})/P(\mathcal{O}) \cong (\mathcal{O}, F)/P(\mathcal{O}, F) \cong I(\mathcal{O}_K, F)/P_{\mathbb{A}}(\mathcal{O}_K, F). \tag{1}$$

Here $P_{\mathbb{A}}(\mathcal{O}_K, F)$ is the subgroup of principal ideals of the form $\beta\mathcal{O}_K$, where $\beta \in \mathcal{O}_K$ satisfies $\beta \equiv N \bmod F\mathcal{O}_K$ for some $N \in \mathbb{A}$ (with $\gcd(N, F) = 1$). One can see immediately that the following sequence is exact:

$$1 \to \frac{I(\mathcal{O}_K, F) \cap P(\mathcal{O}_K)}{P_{\mathbb{A}}(\mathcal{O}_K, F)} \to \frac{I(\mathcal{O}_K, F)}{P_{\mathbb{A}}(\mathcal{O}_K, F)} \to \frac{I(\mathcal{O}_K)}{P(\mathcal{O}_K)} \to 1. \tag{2}$$

To complete the proof, for any order $\mathcal{O}$ in K, we denote by $\mathcal{O}^{(1)}$ the subgroup of $\mathcal{O}^\star$ consisting of norm 1 elements, namely,

$$\mathcal{O}^{(1)} = \big\{\beta \in \mathcal{O}^\star\colon\ N_k^K(\beta) = 1\big\}.$$

Then $\mathcal{O}^\star \cong \mathbb{F}_q^\star \times \mathcal{O}^{(1)}$ and $\mathcal{O}_K^\star/\mathcal{O}^\star \cong \mathcal{O}_K^{(1)}/\mathcal{O}^{(1)}$. Define two functions f and g as follows:

$$f:\left(\frac{\mathbb{A}}{F\mathbb{A}}\right)^{\star}\times\frac{\mathcal{O}_K^{(1)}}{O^{(1)}}\to\left(\frac{\mathcal{O}_K}{F\mathcal{O}_K}\right)^{\star}$$
$$\left(a+F\mathbb{A},u\mathcal{O}^{(1)}\right)\mapsto au+F\mathcal{O}_K,$$
$$g:\left(\frac{\mathcal{O}_K}{F\mathcal{O}_K}\right)^{\star}\to\frac{I(\mathcal{O}_K,F)\cap P(\mathcal{O}_K)}{P_{\mathbb{A}}(\mathcal{O}_K,F)}$$
$$\beta+F\mathcal{O}_K\mapsto(\beta)P_{\mathbb{A}}(\mathcal{O}_K,F).$$

Then one can check directly that f and g are both well-defined homomorphisms, f is injective, g is surjective and $\mathrm{im}(f)\subset\ker(g)$ (since the composition of f and g is the trivial homomorphism). Note that, for any $(\beta)\in P_{\mathbb{A}}(\mathcal{O}_K,F)$, can write $\beta=au$ for some $a\in\mathbb{A}$ and $u\in\mathcal{O}_K^{\star}$. Since $N_k^K(u)\in\mathbb{F}_q^{\star}=(\mathbb{F}_q^{\star})^2$ (because q is even), say $N_k^K(u)=c^2$ where $c\in\mathbb{F}_q^{\star}$. Then $ca\in\mathbb{A}$, $c^{-1}u\in\mathcal{O}_K^{(1)}$ and $\beta=(ca)(c^{-1}u)$. This implies that $\mathrm{im}(f)=\ker(g)$. So we can conclude that the following sequence is exact:

$$1\to\left(\frac{\mathbb{A}}{F\mathbb{A}}\right)^{\star}\times\frac{\mathcal{O}_K^{(1)}}{O^{(1)}}\to\left(\frac{\mathcal{O}_K}{F\mathcal{O}_K}\right)^{\star}\to\frac{I(\mathcal{O}_K,F)\cap P(\mathcal{O}_K)}{P_{\mathbb{A}}(\mathcal{O}_K,F)}\to 1. \tag{3}$$

Combining (1), (2) and (3), one has

$$\begin{aligned}h(\mathcal{O})\big(\mathcal{O}_K^{\star}:\mathcal{O}^{\star}\big)&=h(\mathcal{O}_K)\times\left|(\mathcal{O}_K/F\mathcal{O}_K)^{\star}\right|\times\left|(\mathbb{A}/F\mathbb{A})^{\star}\right|^{-1}\\&=h(\mathcal{O}_K)|F|\prod_{P|F}\left(1-\frac{\chi_K(P)}{|P|}\right).\qquad\square\end{aligned}$$

Let

$$\begin{aligned}\mathcal{D}=\big\{&(D,M)\colon\ D\in\mathbb{A},\ M\in\mathbb{A}^{+},\ \gcd(D,M)=1,\ D/M^2\notin\wp(k),\\&\mathrm{sgn}(D)=\xi\ \text{if}\ \deg(D)=2\deg(M),\ 2\nmid\deg(D)\ \text{if}\ \deg(D)>2\deg(M)\big\}.\end{aligned}$$

For $(D,M)\in\mathcal{D}$, denote by $K_{(D,M)}$ the field $k(\alpha)$, where α is a root of $x^2+x=\frac{D}{M^2}$ (cf. [8]). Then, for any $(D,M)\in\mathcal{D}$, $K_{(D,M)}$ is a quadratic field with $G(K_{(D,M)})$ dividing M. Let $\mathcal{O}_{(D,M)}=\mathbb{A}+M\alpha\mathbb{A}$, which is an order in the quadratic field $K_{(D,M)}$ of conductor equal to $M/G(K_{(D,M)})$. Thus the discriminant of $\mathcal{O}_{(D,M)}$ is M^2. Conversely, we have

Proposition 2.4. *Given any quadratic function field K and any order $\mathcal{O}$ of conductor F in K, there exists $(D,M)\in\mathcal{D}$ such that $K=K_{(D,M)}$ and $\mathcal{O}=\mathcal{O}_{(D,M)}$.*

Proof. Suppose that the discriminant of the maximal order $\mathcal{O}_K$ is G^2. Let Q be the product of distinct monic irreducible factors of G and let $D_2=G^2/Q$. Then $K=k(\alpha)$, where α is a root of the equation $x^2+x=D_1/D_2$ for some $D_1\in\mathbb{A}$ satisfying $\gcd(D_1,D_2)=1$, $\mathrm{sgn}(D_1)=\xi$ if $\deg(D_1)=\deg(D_2)$, and $2\nmid\deg(D_1)-\deg(D_2)$ if $\deg(D_1)>\deg(D_2)$. Recall that $\mathcal{O}_K=\mathbb{A}+G\alpha\mathbb{A}$ and thus $\mathcal{O}=\mathbb{A}+FG\alpha\mathbb{A}$, since $\mathcal{O}$ is of conductor F. Let $M=FG$. Then take any $B\in\mathbb{A}$ with $\deg(B)<\deg(M)$ and $\gcd(B,M)=1$. Let

$$D=D_1QF^2+B^2+BM.$$

Note that $\gcd(D, M) = 1$ by the definition of Q and $\gcd(B, M) = 1$. In the definition of D, we divide by M^2 and use $M = FG$ to derive

$$D/M^2 = D_1/D_2 + (B/M)^2 + B/M,$$

which shows that $(D, M) \in \mathcal{D}$ and the roots of the equations $x^2 + x = D_1/D_2$ and $x^2 + x = D/M^2$ give the same field extension of k. So $K = K_{(D,M)}$. Finally, note $\mathcal{O}_{(D,M)} = \mathbb{A} + M(\alpha + B/M)\mathbb{A} = \mathbb{A} + M\alpha\mathbb{A} = \mathcal{O}$. $\square$

For $(D, M) \in \mathcal{D}$, let $h_{(D,M)}$ denote the class number of the order $\mathcal{O}_{(D,M)}$. For simplicity, denote $K(D, M)$ by K. Then, by applying Proposition 2.3, one can obtain the equality

$$\frac{h_{(D,M)}(\mathcal{O}_K^\star : \mathcal{O}_{(D,M)}^\star)}{|M|} = \frac{h_K}{|G(K)|} \prod_{P|M} \left(1 - \frac{\chi_K(P)}{|P|}\right).$$

In particular, if K is real, then

$$\frac{h_{(D,M)} R_{(D,M)}}{|M|} = \frac{h_K R_K}{|G(K)|} \prod_{P|M} \left(1 - \frac{\chi_K(P)}{|P|}\right),$$

where R_K and $R_{(D,M)}$ denote the regulators of $\mathcal{O}_K$ and $\mathcal{O}_{(D,M)}$ respectively.

Now, for $(D, M) \in \mathcal{D}$, one associates a character $\chi_{(D,M)}$ on $\mathbb{A}^+$ that is defined by

$$\chi_{(D,M)}(N) = \left\{\frac{D/M^2}{N}\right\}.$$

One may then consider its L-function

$$L(s, \chi_{(D,M)}) = \sum_{N \in \mathbb{A}^+} \frac{\chi_{(D,M)}(N)}{|N|^s} = \prod_{P \in \mathbb{A}^+} \left(1 - \frac{\chi_{(D,M)}(P)}{|P|^s}\right)^{-1}.$$

Observe that $\prod_{P|M}(1 - \frac{\chi_K(P)}{|P|}) = \frac{L(1,\chi_{(D,M)})}{L_K(1)}$. So Theorem 2.1 and Proposition 2.3 imply the following theorem that is an analogue of Dirichlet's class number formula.

Theorem 2.5. *Let $(D, M) \in \mathcal{D}$ with $\deg(D) = d$ and $\deg(M) = m$. Let $h_{(D,M)}$ and $R_{(D,M)}$ denote the class number and the regulator of the order $\mathcal{O}_{(D,M)}$ respectively. Then*

$$L(1, \chi_{(D,M)}) = \begin{cases} h_{(D,M)}/q^{\frac{d-1}{2}} & \text{if } d > 2m, \\ (q+1)h_{(D,M)}/2q^m & \text{if } d = 2m, \\ (q-1)R_{(D,M)}h_{(D,M)}/q^m & \text{if } d < 2m. \end{cases}$$

3. Some lemmas

For $M, N \in \mathbb{A}^+$, we define two sums $T_{N,M}$ and $\Gamma_{N,M}$:

$$T_{N,M} = \sum_{\substack{D\in\mathbb{A} \\ \deg(D)<\deg(M),\, \gcd(D,M)=1}} \left\{\frac{D/M}{N}\right\},$$

$$\Gamma_{N,M} = \sum_{\substack{D\in\mathbb{A} \\ \deg(D)<\deg(M)}} \left\{\frac{D/M}{N}\right\}.$$

For $M = \prod_{i=1}^{s} P_i^{e_i} \in \mathbb{A}^+$, let $D_2(M) = \prod_{i=1}^{s} P_i^{2e_i-1}$. Then one has

Lemma 3.1. *Let $M, N \in \mathbb{A}^+$ with $\deg(N) \leqslant 2\deg(M) - 1$. Suppose that $\gcd(M, N) = 1$ and N is not a perfect square. Then $\Gamma_{N,D_2(M)} = 0$.*

Proof. Since $\gcd(N, M) = 1$ and $\deg(N) \leqslant 2\deg(M) - 1$, the set

$$\left\{D/M^2\colon\ D \in \mathbb{A},\ \deg(D) < 2\deg(M)\right\}$$

contains a complete residue system modulo N. So the map $D \mapsto \{\frac{D/M^2}{N}\}$ is a surjective additive character from the set $\{D \in \mathbb{A}\colon\ \deg(D) < 2\deg(M)\}$ onto $\{\pm 1\}$. Hence there exists some $D \in \mathbb{A}$ of degree less than $2\deg(M)$ satisfying $\{\frac{D/M^2}{N}\} = -1$. Then one can normalize D/M^2 to $D_1/D_2(M)$, where $D_1 \in \mathbb{A}$ satisfies $\deg(D_1) < \deg(D_2(M))$, namely

$$D_1/D_2(M) - D/M^2 \in \wp(k).$$

So $\{\frac{D_1/D_2(M)}{N}\} = \{\frac{D/M^2}{N}\} = -1$. Therefore, the map $D \mapsto \{\frac{D/D_2(M)}{N}\}$ is also a surjective additive character from $\{D \in \mathbb{A}\colon\ \deg(D) < \deg(D_2(M))\}$ onto $\{\pm 1\}$. Hence one can conclude that $\Gamma_{N,D_2(M)} = 0$. $\square$

Observe that the following equality holds:

$$\Gamma_{N,M} = \sum_{\bar{M}\in\mathbb{A}^+,\, \bar{M}\mid M} T_{N,\bar{M}}.$$

By the Möbius Inversion formula, one has

$$T_{N,M} = \sum_{\bar{M}\in\mathbb{A}^+,\, \bar{M}\mid M} \mu(\bar{M})\Gamma_{N,M/\bar{M}}.$$

Fact 3.2. *Let $M, \bar{M} \in \mathbb{A}^+$ with $\bar{M} \mid M$. If $\Gamma_{N,\bar{M}} = 0$, then $\Gamma_{N,M} = 0$.*

Proof. Since $\Gamma_{N,\bar{M}} = 0$, there exists some $\bar{D} \in \mathbb{A}$ with $\deg(\bar{D}) < \deg(\bar{M})$ such that $\{\frac{\bar{D}/\bar{M}}{N}\} = -1$. Consider $D = \bar{D} \cdot M/\bar{M} \in \mathbb{A}$. Then one can see that $\deg(D) < \deg(M)$ and $\{\frac{D/M}{N}\} = -1$. Therefore, $\Gamma_{N,M} = 0$. $\square$

Observe that if M and N are not relatively prime, then T_{N,M^2} is always zero by the definition of the quadratic symbol. If $\gcd(N, M) = 1$ and N is a perfect square, then $T_{N,M^2} = |M|\Phi(M)$. The following lemma follows from Lemma 3.1 and Fact 3.2:

Lemma 3.3. *Let $M, N \in \mathbb{A}^+$ with $\deg(N) \leqslant 2\deg(M) - 1$. Then*

(a) *if M and N are not relatively prime, then $T_{N,M^2} = 0$,*
(b) *if $\gcd(N, M) = 1$ and N is not a perfect square, then $T_{N,M^2} = 0$,*
(c) *if $\gcd(N, M) = 1$ and N is a perfect square, then $T_{N,M^2} = |M|\Phi(M)$.*

For $M, N \in \mathbb{A}^+$ and positive integer d, we define another two sums $\widetilde{T}_{N,M,d}$ and $\widetilde{\Gamma}_{N,M,d}$:

$$\widetilde{T}_{N,M,d} = \sum_{\substack{D \in \mathbb{A} \\ \deg(D)-\deg(M)=d,\ \gcd(D,M)=1}} \left\{\frac{D/M}{N}\right\},$$

$$\widetilde{\Gamma}_{N,M,d} = \sum_{\substack{D \in \mathbb{A} \\ \deg(D)-\deg(M)=d}} \left\{\frac{D/M}{N}\right\}.$$

Lemma 3.4. *Let $M, N \in \mathbb{A}^+$ and let d be a nonnegative integer with $\deg(N) \leqslant 2\deg(M) + 2d + 1$. Suppose that $\gcd(M, N) = 1$ and N is not a perfect square. Then $\widetilde{\Gamma}_{N,D_2(M),2d+1} = 0$.*

Proof. By Lemma 3.1, one has $\Gamma_{N,D_2(M)} = 0$, so there exists some $D_1 \in \mathbb{A}$ with $\deg(D_1) < \deg(D_2(M))$ satisfying $\{\frac{D_1/D_2(M)}{N}\} = -1$. Define two sets $\mathcal{D}^+$ and $\mathcal{D}^-$ as follows:

$$\mathcal{D}^+ = \left\{D\colon \deg(D) = \deg\bigl(D_2(M)\bigr) + 2d + 1,\ \left\{\frac{D/D_2(M)}{N}\right\} = 1\right\},$$

$$\mathcal{D}^- = \left\{D\colon \deg(D) = \deg\bigl(D_2(M)\bigr) + 2d + 1,\ \left\{\frac{D/D_2(M)}{N}\right\} = -1\right\}.$$

Then the mapping $D \mapsto D + D_1$ establishes a bijection between $\mathcal{D}^+$ and $\mathcal{D}^-$. Hence one can conclude that $\widetilde{\Gamma}_{N,D_2(M),2d+1} = 0$. $\square$

Observe that the following equality holds:

$$\widetilde{\Gamma}_{N,M,d} = \sum_{\bar{M} \in \mathbb{A}^+,\ \bar{M} | M} \widetilde{T}_{N,\bar{M},d}.$$

By the Möbius Inversion formula, one has

$$\widetilde{T}_{N,M,d} = \sum_{\bar{M} \in \mathbb{A}^+,\ \bar{M} | M} \mu(\bar{M}) \widetilde{\Gamma}_{N,M/\bar{M},d}.$$

Fact 3.5. *Let $M, \bar{M} \in \mathbb{A}^+$ with $\bar{M} \mid M$. If $\widetilde{\Gamma}_{N,\bar{M},d} = 0$, then $\widetilde{\Gamma}_{N,M,d} = 0$.*

Proof. Since $\widetilde{\Gamma}_{N,\bar{M},d} = 0$, there exists $\bar{D} \in \mathbb{A}$ with $\deg(\bar{D}) - \deg(\bar{M}) = d$ such that $\{\frac{\bar{D}/\bar{M}}{N}\} = -1$. Consider $D = \bar{D} \cdot M/\bar{M} \in \mathbb{A}$. Then one can see that $\deg(D) - \deg(M) = d$ and $\{\frac{D/M}{N}\} = -1$. Therefore, $\widetilde{\Gamma}_{N,M,d} = 0$. □

Observe that if M and N are not relatively prime, then $\widetilde{T}_{N,M^2,d}$ is zero by the definition of the quadratic symbol. If $\gcd(N, M) = 1$ and N is a perfect square, then $\widetilde{T}_{N,M^2,2d+1} = (q-1)q^{2d+1}|M|\Phi(M)$. The following lemma follows from Lemma 3.4 and Fact 3.5:

Lemma 3.6. *Let $M, N \in \mathbb{A}^+$ and let d be a nonnegative integer with $\deg(N) \leqslant 2\deg(M) + 2d + 1$. Then*

(a) *if M and N are not relatively prime, then $\widetilde{T}_{N,M^2,2d+1} = 0$,*
(b) *if $\gcd(N, M) = 1$ and N is not a perfect square, then $\widetilde{T}_{N,M^2,2d+1} = 0$,*
(c) *if $\gcd(N, M) = 1$ and N is a perfect square, then*

$$\widetilde{T}_{N,M^2,2d+1} = (q-1)q^{2d+1}|M|\Phi(M).$$

4. Average value theorems in characteristic two

Given $(D, M), (D', M') \in \mathcal{D}$, we say that they are equivalent if

$$M = M' \quad \text{and} \quad D/M^2 + D'/M^2 \in \wp(k).$$

Fix $(D, M) \in \mathcal{D}$, let

$$\bigl[(D, M)\bigr] = \bigl\{(D', M) \in \mathcal{D}\colon\ (D', M) \text{ is equivalent to } (D, M)\bigr\}.$$

Denote by Φ the Euler-phi function on $\mathbb{A}$. Then one has

Fact 4.1.

(a) *Let $(D, M), (D', M') \in \mathcal{D}$. Then $K_{(D,M)} = K_{(D',M')}$ and $O_{(D,M)} = O_{(D',M')}$ if and only if (D, M) and (D', M') are equivalent.*
(b) *For any $(D, M) \in \mathcal{D}$,*

$$\#\bigl[(D, M)\bigr] = \begin{cases} \Phi(M) & \text{if } \deg(D) \leqslant 2\deg(M), \\ \frac{1}{2}q^{\frac{\deg(D) - 2\deg(M) + 1}{2}}\Phi(M) & \text{if } \deg(D) > 2\deg(M). \end{cases}$$

Proof. (a) It is clear that $K_{(D,M)} = K_{(D',M')}$ and $O_{(D,M)} = O_{(D',M')}$ if (D, M) and (D', M') are equivalent. Conversely, recall that the discriminant of $O_{(D,M)}$ is equal to M^2. So $O_{(D,M)} = O_{(D',M')}$ implies that $M = M'$. Then one can check directly that $D/M^2 + D'/M^2 \in \wp(k)$ if the roots of $x^2 + x = D/M^2$ and $x^2 + x = D'/M^2$ give the same field extension of k.

(b) Write $M = \prod_{i=1}^{s} P_i^{e_i}$, where P_i are distinct monic irreducible factors of M. Given any $D' \in \mathbb{A}$, there is a unique solution of the congruence equation $x^2 \equiv D' \bmod P_i$, say $D_i^2 \equiv D'$

mod P_i. Observe that, for any $B \in \mathbb{A}$, $\gcd(D' + B^2 + BM, M) = 1$ if and only if $B \not\equiv D_i \bmod P_i$ for all $1 \leqslant i \leqslant s$.

If $(D', M) \in \mathcal{D}$ is equivalent to (D, M) where $\deg(D) \leqslant 2\deg(M)$, then there exists a unique $B \in \mathbb{A}$ with $\deg(B) < \deg(M)$ and $B \not\equiv D_i \bmod P_i$ for all $1 \leqslant i \leqslant s$ such that $D' = D + B^2 + BM$. There are in total $\Phi(M)$ such B's. Hence $[(D, M)] = \Phi(M)$.

Now assume $\deg(D) > 2\deg(M)$. For any $D' \in \mathbb{A}$, one can uniquely write $D' = Q_{D'}M^2 + R_{D'}$, where $Q_{D'}, R_{D'} \in \mathbb{A}$ satisfy $\deg(R_{D'}) < 2\deg(M)$. Note that the homomorphism $\wp : \mathbb{A} \to \mathbb{A}$ is two-to-one since $\ker(\wp) = \mathbb{F}_2$. If $(D', M) \in \mathcal{D}$ is equivalent to (D, M), then there are exactly two $A \in \mathbb{A}$ with $\deg(A) < (\deg(D) - 2\deg(M) + 1)/2$ such that $Q_{D'} = Q_D + A^2 + A$ and there exists a unique $B \in \mathbb{A}$ with $\deg(B) < \deg(M)$ and $B \not\equiv D_i \bmod P_i$ for all $1 \leqslant i \leqslant s$ such that $R_{D'} = R_D + B^2 + BM$. There are $q^{\frac{\deg(D)-2\deg(M)+1}{2}}$ possible A's and $\Phi(M)$ possible B's. So $[(D, M)] = \frac{1}{2}q^{\frac{\deg(D)-2\deg(M)+1}{2}}\Phi(M)$. $\square$

In the sequel, we proceed to averaging values over orders in real quadratic function fields. For positive integer m, denote by $\mathbb{A}_m^+$ the subset of $\mathbb{A}$ consisting of monic polynomials of degree m and let

$$\mathcal{D}_m = \big\{\big[(D, M)\big]\colon\ (D, M) \in \mathcal{D},\ M \in \mathbb{A}_m^+,\ \deg(D) < 2m\big\}.$$

By Proposition 2.4 and Fact 4.1, one can conclude that there is a one-to-one correspondence between the set $\mathcal{D}_m$ and the set of orders with discriminants of degree $2m$ in real quadratic function fields. Observe that, for any $M \in \mathbb{A}^+$,

$$\#\big\{D \in \mathbb{A}\colon\ \gcd(D, M) = 1,\ \deg(D) < 2\deg(M),\ D/M^2 \in \wp(k)\big\} = \Phi(M).$$

Therefore, one has

$$\#\mathcal{D}_m = \sum_{M \in \mathbb{A}_m^+} \big(|M|\Phi(M) - \Phi(M)\big)/\Phi(M) = q^{2m} - q^m.$$

For $[(D, M)] \in \mathcal{D}_m$ and nonnegative integer n, let

$$\sigma_n(D, M) = \sum_{N \in \mathbb{A}_n^+} \left\{\frac{D/M^2}{N}\right\}.$$

Then $L(s, \chi_{(D,M)}) = \sum_{n=0}^{+\infty} \frac{\sigma_n(D,M)}{q^{ns}}$. To sum up the L-functions, set

$$S_{m,n} = \sum_{[(D,M)] \in \mathcal{D}_m} \sigma_n(D, M).$$

By Remark 2.2, one has

Fact 4.2. *If m is a positive integer and n is a nonnegative integer with $n \geqslant 2m$, then, for any $[(D, M)] \in \mathcal{D}_m$, one has $\sigma_n(D, M) = 0$ and $S_{m,n} = 0$.*

In order to compute $S_{m,n}$ with $n \leqslant 2m-1$, one just need to recall the definition of the following arithmetic function on $\mathbb{A}$:

$$\Phi_n(M) = \#\{N \in \mathbb{A}_n^+ \colon \gcd(M, N) = 1\},$$

where n is a nonnegative integer. Then, for positive integer m, if n is a nonnegative integer, define $\Phi_n^{(m)} = \sum_{M \in \mathbb{A}_m^+} \Phi_n(M)$; otherwise set $\Phi_n^{(m)} = 0$.

Proposition 4.3. *If m is a positive integer and n is a nonnegative integer with $n \leqslant 2m-1$, then $S_{m,n} = q^m \Phi_{\frac{n}{2}}^{(m)} - \Phi_n^{(m)}$.*

Proof. By Fact 4.1 and the definition of T_{N,M^2}, one has

$$\begin{aligned} S_{m,n} &= \sum_{[(D,M)] \in \mathcal{D}_m} \sum_{N \in \mathbb{A}_n^+} \left\{ \frac{D/M^2}{N} \right\} \\ &= \sum_{M \in \mathbb{A}_m^+} \frac{1}{\Phi(M)} \sum_{N \in \mathbb{A}_n^+} \bigl(T_{N,M^2} - \Phi(M)\bigr) \\ &= \sum_{M \in \mathbb{A}_m^+} \frac{1}{\Phi(M)} \sum_{\substack{N \in \mathbb{A}_n^+ \\ \gcd(M,N)=1}} \bigl(T_{N,M^2} - \Phi(M)\bigr). \end{aligned}$$

Note that an odd-degree polynomial will never be a perfect square. For odd integer n, by applying Lemma 3.3, one has

$$S_{m,n} = -\sum_{M \in \mathbb{A}_m^+} \Phi_n(M) = -\Phi_n^{(m)} = q^m \Phi_{\frac{n}{2}}^{(m)} - \Phi_n^{(m)}.$$

For even integer n, by applying Lemma 3.3 again, one has

$$S_{m,n} = \sum_{M \in \mathbb{A}_m^+} |M| \Phi_{\frac{n}{2}}(M) - \sum_{M \in \mathbb{A}_m^+} \Phi_n(M) = q^m \Phi_{\frac{n}{2}}^{(m)} - \Phi_n^{(m)}. \qquad \square$$

In the paper of Hoffstein and Rosen [5, Proposition 1.2] (or cf. [7, Proposition 17.11]), the authors proved the following:

Proposition 4.4. *Let m be a positive integer and let n be a nonnegative integer. Then*

$$\Phi_n^{(m)} = \begin{cases} q^m & \textit{if } n = 0, \\ (1 - \frac{1}{q}) q^{m+n} & \textit{otherwise.} \end{cases}$$

Consider the following sum of L-functions:

$$\sum_{[(D,M)] \in \mathcal{D}_m} L(s, \chi_{(D,M)}) = \sum_{[(D,M)] \in \mathcal{D}_m} \sum_{n=0}^{+\infty} \frac{\sigma_n(D, M)}{q^{ns}} = \sum_{[(D,M)] \in \mathcal{D}_m} \sum_{n=0}^{2m-1} \frac{\sigma_n(D, M)}{q^{ns}}$$

$$
\begin{aligned}
&= \sum_{n=0}^{2m-1} \frac{S_{m,n}}{q^{ns}} = q^m \sum_{n=0}^{2m-1} \frac{\Phi_{\frac{n}{2}}^{(m)}}{q^{ns}} - \sum_{n=0}^{2m-1} \frac{\Phi_n^{(m)}}{q^{ns}} \\
&= q^m \sum_{n=0}^{m-1} \frac{\Phi_n^{(m)}}{q^{2ns}} - \sum_{n=0}^{2m-1} \frac{\Phi_n^{(m)}}{q^{ns}}. \qquad (\star)
\end{aligned}
$$

Now we arrive at our first main theorem:

Theorem 4.5. *Let m be a positive integer.*

(a) *If $s \in \mathbb{C}$ with $s \neq 1, 1/2$, then*

$$
\begin{aligned}
\sum_{[(D,M)]\in\mathcal{D}_m} L(s, \chi_{(D,M)}) = \frac{1}{q}\left(q^{2m} - q^m\right) + q^{2m}\left(1 - \frac{1}{q}\right)\left(1 - q^{m(1-2s)}\right)\zeta_{\mathbb{A}}(2s) \\
- q^m\left(1 - \frac{1}{q}\right)\left(1 - q^{2m(1-s)}\right)\zeta_{\mathbb{A}}(s).
\end{aligned}
$$

(b) *If $s = 1$, then*

$$
\sum_{[(D,M)]\in\mathcal{D}_m} L(1, \chi_{(D,M)}) = \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)}\left(q^{2m} - q^m\right) - 2mq^m\left(1 - \frac{1}{q}\right).
$$

Proof. (a) By Proposition 4.4 and $(\star)$, one has

$$
\begin{aligned}
&\sum_{[(D,M)]\in\mathcal{D}_m} L(s, \chi_{(D,M)}) \\
&= q^m\left(q^m + \left(1 - \frac{1}{q}\right)\sum_{n=1}^{m-1} \frac{q^{n+m}}{q^{2ns}}\right) - \left(q^m + \left(1 - \frac{1}{q}\right)\sum_{n=1}^{2m-1} \frac{q^{n+m}}{q^{ns}}\right) \\
&= q^{2m}\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{m-1} q^{n(1-2s)}\right) - q^m\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{2m-1} q^{n(1-s)}\right) \\
&= \frac{1}{q}\left(q^{2m} - q^m\right) + q^{2m}\left(1 - \frac{1}{q}\right)\left(1 - q^{m(1-2s)}\right)\zeta_{\mathbb{A}}(2s) \\
&\quad - q^m\left(1 - \frac{1}{q}\right)\left(1 - q^{2m(1-s)}\right)\zeta_{\mathbb{A}}(s).
\end{aligned}
$$

(b) For the special value at $s = 1$, by using the same argument, we have

$$
\begin{aligned}
&\sum_{[(D,M)]\in\mathcal{D}_m} L(1, \chi_{(D,M)}) \\
&= q^m\left(q^m + \left(1 - \frac{1}{q}\right)\sum_{n=1}^{m-1} \frac{q^{n+m}}{q^{2n}}\right) - \left(q^m + \left(1 - \frac{1}{q}\right)\sum_{n=1}^{2m-1} \frac{q^{n+m}}{q^{n}}\right)
\end{aligned}
$$

$$= q^{2m}\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{m-1} q^{-n}\right) - q^m\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{2m-1} 1\right)$$
$$= \left(1 + \frac{1}{q}\right)\left(q^{2m} - q^m\right) - 2mq^m\left(1 - \frac{1}{q}\right).$$

Observe that $\zeta_{\mathbb{A}}(2)/\zeta_{\mathbb{A}}(3) = 1 + 1/q$, hence the desired equality holds. □

Recall that $\#\mathcal{D}_m = q^{2m} - q^m$. Thus we can obtain the following limits:

Corollary 4.6.

(a) *If* $s \in \mathbb{C}$ *with* $\Re(s) \geqslant 1$, *then*

$$\lim_{m \to +\infty} \frac{\sum_{[(D,M)] \in \mathcal{D}_m} L(s, \chi_{(D,M)})}{\#\mathcal{D}_m} = \frac{1}{q} + \left(1 - \frac{1}{q}\right)\zeta_{\mathbb{A}}(2s).$$

(b) *If* $s = 1$, *then*

$$\lim_{m \to +\infty} \frac{\sum_{[(D,M)] \in \mathcal{D}_m} L(1, \chi_{(D,M)})}{\#\mathcal{D}_m} = \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)}.$$

Recall that, for $[(D, M)] \in \mathcal{D}_m$, $L(1, \chi_{(D,M)}) = (q-1)R_{(D,M)}h_{(D,M)}/q^m$, where $R_{(D,M)}$ and $h_{(D,M)}$ denote the regulator and the ideal class number of the order $O_{(D,M)}$ in the real quadratic field $K_{(D,M)}$. The following sum is over all $\mathbb{A}$-orders with discriminants of degree $2m$ in real quadratic function fields.

Corollary 4.7. *Let m be a positive integer. Then*

$$\frac{\sum_{[(D,M)] \in \mathcal{D}_m} R_{(D,M)}h_{(D,M)}}{\#\mathcal{D}_m} = \frac{1}{q-1}\frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)}q^m - \frac{2m}{q}\frac{q^m}{q^m - 1}.$$

Now we turn to average values over imaginary quadratic function fields that are unramified at infinity. For positive integer m, denote by $\mathcal{D}'_m$ the following set:

$$\left\{\left[(D', M)\right]\colon (D', M) \in \mathcal{D},\ M \in \mathbb{A}_m^+,\ \deg(D') = 2m,\ K_{(D',M)} \ncong \mathbb{F}_{q^2}(t)\right\}.$$

Again, by Proposition 2.4 and Fact 4.1, one can conclude that there is a one-to-one correspondence between the set $\mathcal{D}'_m$ and the set of orders with discriminants of degree $2m$ in imaginary quadratic function fields that are non-constant field extensions and unramified at infinity. Observe that

$$\#\mathcal{D}'_m = \#\mathcal{D}_m = q^{2m} - q^m.$$

For $[(D', M)] \in \mathcal{D}'_m$ and nonnegative integer n, let

$$\sigma'_n(D', M) = \sum_{N \in \mathbb{A}_n^+} \left\{\frac{D'/M^2}{N}\right\}.$$

Then $L(s,\chi_{(D',M)})=\sum_{n=0}^{+\infty}\frac{\sigma_n'(D',M)}{q^{ns}}$. To sum up the L-functions, set

$$S'_{m,n}=\sum_{[(D',M)]\in\mathcal{D}'_m}\sigma_n'(D',M).$$

Note that for any $N\in\mathbb{A}_n^+$, $\{\frac{\xi}{N}\}=(-1)^n$. So $\sigma_n'(D',M)=(-1)^n\sigma_n(D,M)$, where $D=D'+\xi$ and thus one has $S'_{m,n}=(-1)^nS_{m,n}$ for all positive integer m and nonnegative integer n. Therefore, by Fact 4.2 and Proposition 4.3, one has

Fact 4.8. *If m is a positive integer and n is a nonnegative integer, then*

$$S'_{m,n}=\begin{cases}q^m\Phi^{(m)}_{\frac{n}{2}}-(-1)^n\Phi^{(m)}_n & \text{if } n\leqslant 2m-1,\\ 0 & \text{if } n\geqslant 2m.\end{cases}$$

Similar to the real case, one has:

$$\sum_{[(D',M)]\in\mathcal{D}'_m}L(s,\chi_{(D',M)})=q^m\sum_{n=0}^{m-1}\frac{\Phi_n^{(m)}}{q^{2ns}}-\sum_{n=0}^{2m-1}(-1)^n\frac{\Phi_n^{(m)}}{q^{ns}}. \qquad (\star\star)$$

Now we arrive at our second main theorem:

Theorem 4.9. *Let m be a positive integer.*

(a) *If $s\in\mathbb{C}$ with $s\neq 1,1/2$, then*

$$\begin{aligned}&\sum_{[(D',M)]\in\mathcal{D}'_m}L(s,\chi_{(D',M)})\\&\quad=\frac{1}{q}\left(q^{2m}-q^m\right)+q^{2m}\left(1-\frac{1}{q}\right)\left(1-q^{m(1-2s)}\right)\zeta_{\mathbb{A}}(2s)\\&\qquad+q^m\left(1-\frac{1}{q}\right)\left(1-q^{2m(1-s)}\right)\zeta_{\mathbb{A}}(s)-2q^m\left(1-\frac{1}{q}\right)\left(1-q^{2m(1-s)}\right)\zeta_{\mathbb{A}}(2s-1).\end{aligned}$$

(b) *If $s=1$, then*

$$\sum_{[(D',M)]\in\mathcal{D}'_m}L(1,\chi_{(D',M)})=\frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)}\left(q^{2m}-q^m\right).$$

Proof. (a) By Proposition 4.4 and $(\star\star)$, one has

$$\begin{aligned}&\sum_{[(D',M)]\in\mathcal{D}'_m}L(s,\chi_{(D',M)})\\&\quad=q^m\left(q^m+\left(1-\frac{1}{q}\right)\sum_{n=1}^{m-1}\frac{q^{n+m}}{q^{2ns}}\right)-\left(q^m+\left(1-\frac{1}{q}\right)\sum_{n=1}^{2m-1}\frac{(-1)^nq^{n+m}}{q^{ns}}\right)\end{aligned}$$

$$
\begin{aligned}
&= q^{2m}\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{m-1} q^{n(1-2s)}\right) - q^m\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{2m-1} (-1)^n q^{n(1-s)}\right) \\
&= q^{2m}\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{m-1} q^{n(1-2s)}\right) + q^m\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{2m-1} q^{n(1-s)}\right) \\
&\quad - 2q^m\left(1 - \frac{1}{q}\right)\sum_{n=0}^{m-1} q^{2n(1-s)} \\
&= \frac{1}{q}\left(q^{2m} - q^m\right) + q^{2m}\left(1 - \frac{1}{q}\right)\left(1 - q^{m(1-2s)}\right)\zeta_{\mathbb{A}}(2s) \\
&\quad + q^m\left(1 - \frac{1}{q}\right)\left(1 - q^{2m(1-s)}\right)\zeta_{\mathbb{A}}(s) - 2q^m\left(1 - \frac{1}{q}\right)\left(1 - q^{2m(1-s)}\right)\zeta_{\mathbb{A}}(2s-1).
\end{aligned}
$$

(b) For the special value at $s = 1$, by using the same argument, we have

$$
\begin{aligned}
&\sum_{[(D',M)]\in\mathcal{D}'_m} L(1, \chi_{(D',M)}) \\
&\quad = q^m\left(q^m + \left(1 - \frac{1}{q}\right)\sum_{n=1}^{m-1} \frac{q^{n+m}}{q^{2n}}\right) - \left(q^m + \left(1 - \frac{1}{q}\right)\sum_{n=1}^{2m-1} \frac{(-1)^n q^{n+m}}{q^n}\right) \\
&\quad = q^{2m}\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{m-1} q^{-n}\right) - q^m\left(\frac{1}{q} + \left(1 - \frac{1}{q}\right)\sum_{n=0}^{2m-1} (-1)^n\right) \\
&\quad = \left(1 + \frac{1}{q}\right)\left(q^{2m} - q^m\right). \qquad \square
\end{aligned}
$$

Recall that $\#\mathcal{D}'_m = q^{2m} - q^m$. Then we can obtain the following limits:

Corollary 4.10. *Let m be a positive integer. Then*

(a) *If $s \in \mathbb{C}$ with $\Re(s) \geqslant 1$, then*

$$
\lim_{m\to+\infty} \frac{\sum_{[(D',M)]\in\mathcal{D}'_m} L(s, \chi_{(D',M)})}{\#\mathcal{D}'_m} = \frac{1}{q} + \left(1 - \frac{1}{q}\right)\zeta_{\mathbb{A}}(2s).
$$

(b) *If $s = 1$, then*

$$
\lim_{m\to+\infty} \frac{\sum_{[(D',M)]\in\mathcal{D}'_m} L(1, \chi_{(D',M)})}{\#\mathcal{D}'_m} = \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)}.
$$

Recall that, for $[(D', M)] \in \mathcal{D}'_m$, $L(1, \chi_{(D',M)}) = (q+1)h_{(D',M)}/2q^m$, where $h_{(D',M)}$ denotes the ideal class number of the order $O_{(D',M)}$ in the imaginary quadratic function field $K_{(D',M)}$. The following sum is over all $\mathbb{A}$-orders with discriminants of degree $2m$ in imaginary quadratic function fields that are non-constant field extensions and unramified at infinity.

Corollary 4.11. *Let m be a positive integer. Then*

$$\frac{\sum_{[(D',M)]\in\mathcal{D}'_m} h_{[(D',M)]}}{\#\mathcal{D}'_m} = \frac{2}{q+1}\frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)}q^m.$$

Finally, we continue to average values over imaginary quadratic function fields that are ramified at infinity. For positive integers m and nonnegative integer d, let

$$\widetilde{\mathcal{D}}_{m,d} = \big\{\big[(\widetilde{D}, M)\big]:\ (\widetilde{D}, M) \in \mathcal{D},\ M \in \mathbb{A}_m^+,\ \deg(\widetilde{D}) = 2m + 2d + 1\big\}.$$

Again, by Proposition 2.4 and Fact 4.1, one can conclude that there is a one-to-one correspondence between the set $\widetilde{\mathcal{D}}_{m,d}$ and the set of orders with finite discriminants of degree $2m$ in imaginary quadratic function fields that are ramified at infinity with local discriminants at infinity of degree $2d+2$. Observe that, for any $M \in \mathbb{A}_m^+$ and $\widetilde{D} \in \mathbb{A}$ with $\deg(\widetilde{D}) - 2\deg(M) = 2d+1$, $\widetilde{D}/M^2 \notin \wp(k)$. So one has

$$\#\widetilde{\mathcal{D}}_{m,d} = \sum_{M\in\mathbb{A}_m^+} \frac{(q-1)q^{2d+1}|M|\Phi(M)}{\frac{1}{2}q^{d+1}\Phi(M)} = 2(q-1)q^{2m+d}.$$

For $[(\widetilde{D}, M)] \in \widetilde{\mathcal{D}}_{m,d}$ and nonnegative integer n, let

$$\widetilde{\sigma}_n(\widetilde{D}, M) = \sum_{N\in\mathbb{A}_n^+} \left\{\frac{\widetilde{D}/M^2}{N}\right\}.$$

Then $L(s, \chi_{(\widetilde{D},M)}) = \sum_{n=0}^{+\infty} \frac{\widetilde{\sigma}_n(\widetilde{D},M)}{q^{ns}}$. To sum up the L-functions, set

$$\widetilde{S}_{m,d,n} = \sum_{[(\widetilde{D},M)]\in\widetilde{\mathcal{D}}_m} \widetilde{\sigma}_n(\widetilde{D}, M).$$

By Remark 2.2, one has

Fact 4.12. *If m is a positive integer and d, n are nonnegative integers with $n \geqslant 2m + 2d + 2$, then, for any $(\widetilde{D}, M) \in \widetilde{\mathcal{D}}_{m,d}$, $\widetilde{\sigma}_n(\widetilde{D}, M) = 0$ and $\widetilde{S}_{m,d,n} = 0$.*

Proposition 4.13. *If m is a positive integer and d, n are nonnegative integers with $n \leqslant 2m + 2d + 1$, then $\widetilde{S}_{m,d,n} = 2(q-1)q^{m+d}\Phi_{\frac{n}{2}}^{(m)}$.*

Proof. By Fact 4.1 and the definition of $\widetilde{T}_{N,M^2,2d+1}$, one has

$$\widetilde{S}_{m,d,n} = \sum_{[(\widetilde{D},M)]\in\widetilde{\mathcal{D}}_{m,d}} \sum_{N\in\mathbb{A}_n^+} \left\{\frac{\widetilde{D}/M^2}{N}\right\}$$

$$= \sum_{M \in \mathbb{A}_m^+} \frac{1}{\frac{1}{2} q^{d+1} \Phi(M)} \sum_{N \in \mathbb{A}_n^+} \widetilde{T}_{N, M^2, 2d+1}$$

$$= \sum_{M \in \mathbb{A}_m^+} \frac{2}{q^{d+1} \Phi(M)} \sum_{\substack{N \in \mathbb{A}_n^+ \\ \gcd(M,N)=1}} \widetilde{T}_{N, M^2, 2d+1}.$$

Note that an odd-degree polynomial will never be a perfect square. For odd integer n, by applying Lemma 3.6, one has $\widetilde{S}_{m,d,n} = 0$.

For even integer n, by applying Lemma 3.6 again, one has

$$\widetilde{S}_{m,d,n} = \sum_{M \in \mathbb{A}_m^+} \sum_{\substack{N_1 \in \mathbb{A}_{\frac{n}{2}}^+ \\ \gcd(M,N_1)=1}} 2(q-1) q^{m+d} = 2(q-1) q^{m+d} \Phi_{\frac{n}{2}}^{(m)}. \qquad \square$$

Consider the following sum of L-functions:

$$\begin{aligned} \sum_{[(\widetilde{D},M)] \in \widetilde{\mathcal{D}}_{m,d}} L(s, \chi_{(\widetilde{D},M)}) &= \sum_{[(\widetilde{D},M)] \in \widetilde{\mathcal{D}}_{m,d}} \sum_{n=0}^{+\infty} \frac{\widetilde{\sigma}_n(\widetilde{D}, M)}{q^{ns}} \\ &= \sum_{[(\widetilde{D},M)] \in \widetilde{\mathcal{D}}_{m,d}} \sum_{n=0}^{2m+2d+1} \frac{\widetilde{\sigma}_n(\widetilde{D}, M)}{q^{ns}} \\ &= \sum_{n=0}^{2m+2d+1} \frac{\widetilde{S}_{m,d,n}}{q^{ns}} \\ &= 2(q-1) q^{m+d} \sum_{n=0}^{m+d} \frac{\Phi_n^{(m)}}{q^{2ns}}. \end{aligned} \qquad (\star\star\star)$$

Now we arrive at our third main theorem:

Theorem 4.14. *Let m be a positive integer and let d be a nonnegative integer.*

(a) *If $s \in \mathbb{C}$ with $s \neq 1, 1/2$, then*

$$\sum_{[(\widetilde{D},M)] \in \widetilde{\mathcal{D}}_{m,d}} L(s, \chi_{(\widetilde{D},M)}) = 2(q-1) q^{2m+d-1} + 2(q-1) q^{2m+d} \left(1 - \frac{1}{q}\right) \left(1 - q^{(m+d+1)(1-2s)}\right) \zeta_{\mathbb{A}}(2s).$$

(b) *If $s = 1$, then*

$$\sum_{[(\widetilde{D},M)] \in \widetilde{\mathcal{D}}_{m,d}} L(1, \chi_{(\widetilde{D},M)}) = 2(q-1) \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)} q^{2m+d} + 2(q-1) q^{m-1}.$$

Proof. (a) By Proposition 4.4 and ($\star\star\star$), one has

$$\begin{aligned}
&\sum_{[(\widetilde{D},M)]\in\widetilde{\mathcal{D}}_{m,d}} L(s,\chi_{(\widetilde{D},M)}) \\
&\quad = 2(q-1)q^{m+d}\left(q^m + \left(1-\frac{1}{q}\right)\sum_{n=1}^{m+d}\frac{q^{n+m}}{q^{2ns}}\right) \\
&\quad = 2(q-1)q^{2m+d}\left(\frac{1}{q} + \left(1-\frac{1}{q}\right)\sum_{n=0}^{m+d} q^{n(1-2s)}\right) \\
&\quad = 2(q-1)q^{2m+d-1} + 2(q-1)q^{2m+d}\left(1-\frac{1}{q}\right)\left(1-q^{(m+d+1)(1-2s)}\right)\zeta_{\mathbb{A}}(2s).
\end{aligned}$$

(b) To obtain the special value at $s=1$, one can simply substitute $s=1$ in (a):

$$\begin{aligned}
&\sum_{[(\widetilde{D},M)]\in\widetilde{\mathcal{D}}_{m,d}} L(1,\chi_{(\widetilde{D},M)}) \\
&\quad = 2(q-1)q^{2m+d-1} + 2(q-1)q^{2m+d}\left(1-\frac{1}{q}\right)\left(1-q^{-(m+d+1)}\right)\zeta_{\mathbb{A}}(2) \\
&\quad = 2(q-1)\left(1+\frac{1}{q}\right)q^{2m+d} + 2(q-1)q^{m-1}. \qquad \square
\end{aligned}$$

Recall that $\#\widetilde{\mathcal{D}}_{m,d} = 2(q-1)q^{2m+d}$. Then we can obtain the following limits:

Corollary 4.15. *Let m be a positive integer and let d be a nonnegative integer.*

(a) *If $s\in\mathbb{C}$ with $\Re(s)\geqslant 1$, then*

$$\lim_{m\to+\infty}\frac{\sum_{[(\widetilde{D},M)]\in\widetilde{\mathcal{D}}_{m,d}} L(s,\chi_{(\widetilde{D},M)})}{\#\widetilde{\mathcal{D}}_{m,d}} = \frac{1}{q} + \left(1-\frac{1}{q}\right)\zeta_{\mathbb{A}}(2s).$$

(b) *If $s=1$, then*

$$\lim_{m\to+\infty}\frac{\sum_{[(\widetilde{D},M)]\in\widetilde{\mathcal{D}}_{m,d}} L(1,\chi_{(\widetilde{D},M)})}{\#\widetilde{\mathcal{D}}_{m,d}} = \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)}.$$

Recall that, for $[(\widetilde{D},M)]\in\widetilde{\mathcal{D}}_{m,d}$, $L(1,\chi_{(\widetilde{D},M)}) = h_{(\widetilde{D},M)}/q^{m+d}$, where $h_{(\widetilde{D},M)}$ denotes the class number of the order $O_{(\widetilde{D},M)}$ in the imaginary quadratic function field $K_{(\widetilde{D},M)}$. The following sum is over all $\mathbb{A}$-orders with finite discriminants of degree $2m$ in imaginary quadratic function fields whose local discriminants at infinity have degree $2d+2$.

Corollary 4.16. *Let m be a positive integer and let d be a nonnegative integer. Then*

$$\frac{\sum_{[(\widetilde{D},M)]\in\widetilde{\mathcal{D}}_{m,d}} h_{(\widetilde{D},M)}}{\#\widetilde{\mathcal{D}}_{m,d}} = \frac{\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}(3)}q^{m+d} + \frac{1}{q}.$$

To end this paper, we would like to make the final remark that, for the problem of averaging over all fundamental discriminants, it is extremely difficult even in odd characteristic. However, Hoffstein and Rosen succeeded in obtaining beautiful results on this problem in their 1992 paper (cf. [5]). We believe that this is also a very hard problem in characteristic 2.

Acknowledgments

The author would like to thank Mike Rosen, Joseph Silverman, and Jing Yu for helpful conversations and e-mails.

References

[1] E. Artin, Quadratische Körper in Gebiete der höheren Kongruenzen I, II, Math. Z. 19 (1924) 153–246.
[2] Y.-M.J. Chen, J. Yu, On class number relations in characteristic two, Math. Z. 259 (1) (2008) 197–216.
[3] David A. Cox, Primes of the Form $x^2 + ny^2$, John Wiley & Sons, 1989.
[4] H. Hasse, Theorie der relativ-zyklischen algebraischen Funktionenkörper, insbesondere bei endlichem Konstantenkörper, J. Reine Angew. Math. 172 (1934) 37–54.
[5] J. Hoffstein, M. Rosen, Average values of L-series in function fields, J. Reine Angew. Math. 426 (1992) 117–150.
[6] Serge Lang, Elliptic Functions, Addison–Wesley, Reading, MA, 1973.
[7] M. Rosen, Number Theory in Function Fields, Grad. Texts in Math., vol. 210, Springer-Verlag, New York, 2001.
[8] J.T.-Y. Wang, J. Yu, On class number relation over function fields, J. Number Theory 69 (1998) 181–196.