A proof of the set-theoretic version of the salmon conjecture

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We show that the irreducible variety of $4 \times 4 \times 4$ complex valued tensors of border rank at most 4 is the zero set of polynomial equations of degree 5 (the Strassen commutative conditions), of degree 6 (the Landsberg–Manivel polynomials), and of degree 9 (the symmetrization conditions).

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1. Introduction

In this paper we identify $C^m \otimes C^n \otimes C^l$ with the space of tensors $T = \{ti,j,k\mid m,n,l \in C^{m\times n\times l}\}$, where we choose the standard bases in $C^m, C^n, C^l$, unless stated otherwise. Let $V_T(m, n, l) \subseteq C^m \otimes C^n \otimes C^l$ be the variety of tensors of border rank at most $r$. The border rank of a tensor $T \neq 0$ is $r$ if $T$ is the limit of a sequence of rank $r$ tensors, and there does not exist a sequence of tensors, $(T_i)$, such that the limit of $(T_i)$ is $T$ and the rank $T_i < r$ for all $i \in \mathbb{N}$. The projectivization of $V_T(m, n, l)$ is the $r$th secant variety of $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{l-1}$.

In 2007, Elizabeth Allman posed the problem of determining the ideal $I_4(4, 4, 4)$ generated by all polynomials vanishing on $V_4(4, 4, 4)$ [2]. Allman offered a prize of a freshly-caught smoked Copper river salmon for the solution, and thus, the problem is colloquially called the salmon problem. Conjecture 3.24 in [8] states that $I_4(4, 4, 4)$ is generated by polynomials of degree 5 and 9. A first nontrivial step in characterizing $V_4(4, 4, 4)$ is to characterize $V_4(3, 3, 4)$. In [6], Landsberg and Manivel show that $V_4(3, 3, 4)$ satisfies a set of polynomial equations of degree 6 which are not in the ideal generated by the equations of degree 5 from the original conjecture. (See also [7, Remark 5.7] and [3].)

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The degree 16 equations in [5] are a result of the condition

\[ \text{Combining this with the results in [5] we deduce the set-theoretic version of the salmon conjecture.} \]

The degree 16 equations are equivalent to the vanishing of the polynomial (1.2) are the polynomials of degree 6 and 9 [3], where the degree 6 polynomials are the LM-polynomials.

In [5, Theorem 4.5], it is shown that \( V_4(4, 4, 4) \) is the zero set of homogeneous polynomials of degree 5, 6 and 9. This in particular implies the set-theoretic salmon conjecture [5, Lemma 4.3]. We call this set of polynomials the symmetrization conditions.

One can choose \( L \) and \( R \) such that their entries are polynomials of degree 8 in the entries of \( X \). The degree 16 equations in [5] are a result of the condition

\[ LR^\top = R^\top L = \frac{\text{tr}(LR^\top)}{3} I_3. \]  

The degree 16 equations are used only in the case A.I.3 of the proof of Theorem 4.5 of [5].

In [6], Landsberg and Manivel give an algorithm to construct polynomials of degree 6, referred here as the LM-polynomials, that vanish on \( V_4(3, 3, 4) \) but are not in the ideal generated by the known polynomials of degree 5. In [3], Bates and Oeding explicitly construct a basis of the these degree 6 polynomials which consist of ten linearly independent polynomials. Using methods from numerical algebraic geometry, Bates and Oeding give numerical confirmation that \( V_4(3, 3, 4) \) is the zero set of a set of polynomials of degree 6 and 9 [3], where the degree 6 polynomials are the LM-polynomials.

The aim of this paper is to show that \( X = [x_{i,j,k}] \in \mathbb{C}^{3 \times 3 \times 4} \) whose four frontal slices are of the form

\[ X_k = \begin{bmatrix} x_{1,1,k} & x_{1,2,k} & 0 \\ x_{2,1,k} & x_{2,2,k} & 0 \\ 0 & 0 & x_{3,3,k} \end{bmatrix}, \quad k = 1, 2, 3, 4, \]  

has border rank at most four if and only if the ten basis LM-polynomials vanish on \( X \).

As we will see later, a tensor \( X \in \mathbb{C}^{3 \times 3 \times 4} \) of the form (1.2) has border rank at most four if and only if either the four matrices \( [x_{1,1,k} x_{1,2,k}] \), \( k = 1, 2, 3, 4 \) are linearly dependent or \( x_{3,3,k} = 0 \) for \( k = 1, 2, 3, 4 \). Note that the condition that the above four \( 2 \times 2 \) matrices are linearly dependent is equivalent to the vanishing of the polynomial

\[ f(X) = \det \begin{bmatrix} x_{1,1,1} & x_{1,1,2} & x_{1,2,1} & x_{2,2,1} \\ x_{1,1,2} & x_{1,2,2} & x_{2,1,1} & x_{2,2,2} \\ x_{1,1,3} & x_{1,2,3} & x_{2,1,3} & x_{2,2,3} \\ x_{1,1,4} & x_{1,2,4} & x_{2,1,4} & x_{2,2,4} \end{bmatrix}. \]  

Computer-aided calculations show that the restrictions of the ten basis LM-polynomials to \( X \) of the form (1.2) are the polynomials

\[ x_{3,3,k} x_{3,3,l} f(X) \quad \text{for} \ 1 \leq k \leq l \leq 4. \]  

Hence \( X \) has a border rank at most four if and only if the ten basis LM-polynomials vanish on \( X \). Combining this with the results in [5] we deduce the set-theoretic version of the salmon conjecture.
We summarize briefly the content of the paper. In Section 2 we restate the characterization of $V_4(3, 3, 4)$ given in [5, Theorem 4.5]. In Section 3 we show that the use of polynomials of degree 16 in the proof of [5, Theorem 4.5] can be replaced by the use of the LM-polynomials. In Section 4 we summarize briefly the characterization of $V_4(4, 4, 4)$ as the zero set of polynomials of degree 5, 6 and 9.

2. A characterization of $V_4(3, 3, 4)$

We now state [5, Theorem 4.5] which characterizes $V_4(3, 3, 4)$. Let $\mathcal{X} = [x_{i,j,k}]_{i,j,k=1}^{3,3,4} \in \mathbb{C}^{3 \times 3 \times 4}$. The four frontal slices of $\mathcal{X}$ are denoted as the matrices $X_k = [x_{i,j,k}]_{i,j=1}^{3,3} \in \mathbb{C}^{3 \times 3}$, $k = 1, 2, 3, 4$. Assume that $\mathcal{X} \in V_4(3, 3, 4)$. A special case of [5, Lemma 4.3] claims that there exist nontrivial matrices $L, R \in \mathbb{C}^{3 \times 3} \setminus \{0\}$ satisfying the conditions

$$LX_k - XX_k^T = 0, \quad k = 1, \ldots, 4, \quad L \in \mathbb{C}^{3 \times 3}. \quad (2.1)$$

$$X_kR - R^TX_k^T = 0, \quad k = 1, \ldots, 4, \quad R \in \mathbb{C}^{3 \times 3}. \quad (2.2)$$

These are the symmetrization conditions.

If the entries of $R$ and $L$ are viewed as the entries of two vectors with 9 coordinates each, then the systems $(2.1)$ and $(2.2)$ are linear homogeneous equations with coefficient matrices $C_L(\mathcal{X}), C_R(\mathcal{X}) \in \mathbb{C}^{12 \times 9}$ respectively. (Observe that for any $A \in \mathbb{C}^{3 \times 3}$ the matrix $A - A^T$ is skew symmetric, which has, in general, 3 free parameters.) The entries of $C_L(\mathcal{X}), C_R(\mathcal{X})$ are linear functions in the entries of $\mathcal{X}$. For a generic $\mathcal{X} \in V_4(3, 3, 4)$, $\text{rank} C_L(\mathcal{X}) = \text{rank} C_R(\mathcal{X}) = 8$ [5]. Hence we can express the entries of $L$ and $R$ in terms of corresponding 8 $\times$ 8 minors of $C_L(\mathcal{X}), C_R(\mathcal{X})$ respectively. There are a finite number of ways to express $L$ and $R$ in this way, and some of these expressions may be zero matrices. Nonetheless, the entries of $L$ and $R$ are polynomials of degree 8 in the entries of $\mathcal{X}$. If $\text{rank} C_L(\mathcal{X}) = \text{rank} C_R(\mathcal{X}) = 8$ then it is necessary that the condition $(1.1)$ holds for every expression of $L$ and $R$ [5]. Furthermore, if $\text{rank} C_L(\mathcal{X}) < 8$ then each possible expression of $L$ in terms of 8 $\times$ 8 minors of $C_L(\mathcal{X})$ is a zero matrix, and a similar statement holds for $R$, so $(1.1)$ holds trivially.

Thus, the characterization of $V_4(3, 3, 4)$ is given by [5, Theorem 4.5].

**Theorem 2.1.** $\mathcal{X} = [x_{i,j,k}]_{i,j,k=1}^{3,3,4} \in \mathbb{C}^{3 \times 3 \times 4}$ has border rank at most 4 if and only if the following conditions hold.

1. Let $X_k := [x_{i,j,k}]_{i,j=1}^{3,3} \in \mathbb{C}^{3 \times 3}$, $k = 1, \ldots, 4$ be the four frontal slices of $\mathcal{X}$. Then the ranks of $C_L(\mathcal{X}), C_R(\mathcal{X})$ are less than 9. (These are degree 9 equations.)
2. Let $R, L$ be solutions of $(2.1)$ and $(2.2)$ respectively given by $8 \times 8$ minors of $C_L(\mathcal{X}), C_R(\mathcal{X})$. Then $(1.1)$ holds. (These are degree 16 equations.)

The proof of Theorem 2.1 in [5] consists of discussing a number of cases. The degree 16 polynomial conditions $(1.1)$ are used only in the case A.I.3. In the next section we show how to prove the theorem in the case A.I.3 using only the ten basis LM-polynomials of degree 6.

3. The case A.I.3 of [5, Theorem 4.5]

Suppose $\mathcal{X} \in \mathbb{C}^{3 \times 3 \times 4}$ and there exist two nonzero matrices $L, R \in \mathbb{C}^{3 \times 3}$ such that $(2.1)$–$(2.2)$ hold. The case A.I.3 assumes that $L$ and $R$ are rank one matrices. The degree 16 equations yield that $LR^T = R^TL = 0$, thus, the remainder of the proof of [5, Theorem 4.5] in the case A.I.3 resolves the case where $LR^T = R^TL = 0$. Therefore, to eliminate the use of polynomial conditions of degree 16 we need to show the following.

**Claim 3.1.** Let $\mathcal{X} \in \mathbb{C}^{3 \times 3 \times 4}$. Let $R, L \in \mathbb{C}^{3 \times 3}$ be rank one matrices satisfying the conditions $(2.1)$–$(2.2)$ respectively. Suppose furthermore that either $LR^T \neq 0$ or $R^TL \neq 0$. If the ten LM-polynomials vanish on $\mathcal{X}$ then $\mathcal{X} \in V_4(3, 3, 4)$. 

In the rest of this section we prove Claim 3.1. Assume that \( L = uv^\top, R = xy^\top \). The following claim is straightforward:

\[
\begin{align*}
uv^\top A & \text{ is symmetric if and only if } v^\top A = bu^\top \quad \text{for some } b \in \mathbb{C}, \\
Axy^\top & \text{ is symmetric if and only if } Ax = cy \quad \text{for some } c \in \mathbb{C}.
\end{align*}
\] (3.1) (3.2)

By changing bases in two copies of \( \mathbb{C}^3 \) we can assume that \( u = v = e_3 = (0, 0, 1)^\top \). (Changes of bases do not affect the vanishing condition of either \( LR^\top \) or \( R^\top L \) [5].) Let \( P, Q \in \text{GL}(3, \mathbb{C}) \) such that

\[
P^\top e_3, Q^\top e_3 \in \text{span}(e_3). \] (3.3)

Then if \( A \in \mathbb{C}^3 \times \mathbb{C}^3 \) such that (3.1) and (3.2) hold, \( e_3 e_3^\top (PAQ) \) is symmetric. Observe next that \( PAQ(\mathbb{Q}^{-1}x)(Py)^\top \) is also symmetric. Thus we need to analyze what kind of vectors can be obtained from two nonzero vectors \( x, y \) by applying \( \mathbb{Q}^{-1}x, Py \), where \( P, Q \) satisfy (3.3). By letting \( Q_1 := \mathbb{Q}^{-1} \) we see that \( Q_1 \) satisfies the same conditions \( Q \) in (3.3). Hence \( Q_1, P \) have the zero pattern

\[
\begin{bmatrix}
* & * & * \\
* & * & * \\
0 & 0 & *
\end{bmatrix}.
\] (3.4)

**Lemma 3.2.** Let \( y \in \mathbb{C}^3 \setminus \{0\} \). If \( e_3^\top y \neq 0 \) then there exists \( P \in \text{GL}(3, \mathbb{C}) \) of the form (3.4) such that \( Py = e_3 \). If \( e_3^\top y = 0 \) then there exists \( P \in \text{GL}(3, \mathbb{C}) \) of the form (3.4) such that \( Py = e_2 \).

**Proof.** Assume first that \( e_3^\top y \neq 0 \). Let \( f = (f_1, 0, f_3)^\top, g = (0, g_2, g_3)^\top \in \mathbb{C}^3 \setminus \{0\} \) such that \( f^\top y = g^\top y = 0 \). Then \( f_1, f_3 \neq 0 \). Hence there exists \( P \in \text{GL}(3, \mathbb{C}) \) of the form (3.4), whose first and second rows are \( f^\top, g^\top \) respectively, such that \( Py = e_3 \).

Suppose now that \( e_3^\top y = 0 \). Hence there exists \( P = P_1 \oplus [1], P_1 \in \text{GL}(2, \mathbb{C}) \) such that \( Py = e_2 \). \( \square \)

**Corollary 3.3.** Let \( A \in \mathbb{C}^{3 \times 3} \) and assume that \( LA \) and \( AR \) are symmetric matrices for some rank one matrices \( L, R \in \mathbb{C}^{3 \times 3} \). Then there exist \( P, Q \in \text{GL}(3, \mathbb{C}) \) such that by replacing \( A, L, R \) by \( A_1 := PAQ, L_1 := Q^{-1}LP^{-1}, R_1 = Q^{-1}RP^{-1} \) we can assume \( L_1 = e_3 e_3^\top \) and \( R_1 \) has one of the following 4 forms

\[
e_3 e_3^\top, \quad e_2 e_2^\top, \quad e_2 e_2^\top, \quad e_2 e_2^\top.
\] (3.5)

To prove Claim 3.1 we need to consider the first three choices of \( R_1 \) in (3.5) since the last choice implies \( LR^\top = R^\top L = 0 \). Note that by changing the first two indices in \( \mathcal{X} \in \mathbb{C}^{3 \times 3 \times 4} \) we need to consider only the first two choices of \( R_1 \) in (3.5).

### 3.1. The case \( L = R = e_3 e_3^\top \)

In the remainder of this section we say that a tensor \( T \in \mathbb{C}^{m \times n \times k} \) is represented as a tensor \( T' = [t_{i,j,k}]_{1 \leq i \leq m', 1 \leq j \leq n', 1 \leq k \leq l} \) if the following condition holds. There exist bases in \( \mathbb{C}^m, \mathbb{C}^n, \mathbb{C}^l \) such that the tensor \( T \) is represented by the tensor \( T' = [t_{i,j,k}] \in \mathbb{C}^{m \times n \times k} \), where the following conditions hold. First \( t_{i,j,k} = t_{i,j,k} \) for \( i = 1, \ldots, m', j = 1, \ldots, n', k = 1, \ldots, l \). Second \( t_{i,j,k} = 0 \) if \( t_{i,j,k} \) is not a coordinate of \( T' \). Clearly, \( \text{rank} T' = \text{rank} T, \text{brank} T' = \text{brank} T \).

Let \( X_1, X_2, X_3, X_4 \in \mathbb{C}^{3 \times 3} \) be the four frontal sections of \( \mathcal{X} = [x_{i,j,k}] \in \mathbb{C}^{3 \times 3 \times 4} \). Assume that (2.1)–(2.2) hold. Then each \( X_k \) has the form of (1.2). (This is the case discussed in [5, (4.7)].)

Using Mathematica, we took the ten basis LM-polynomials available in the ancillary material of [3, deg_6_salmon.txt] and let \( x_{1,3,k} = 0, x_{2,3,k} = 0, x_{3,1,k} = 0, x_{3,2,k} = 0 \) for \( k = 1, 2, 3, 4 \). The resulting polynomials had 24 terms. We then factored \( f(X) \) from these restricted polynomials. This symbolic
computation shows that the restriction of the ten basis LM-polynomials to $X$ satisfying (2.1)--(2.2) are the polynomials given in (1.4). Therefore, by the result of Landsberg–Manivel [6], if $X \in V_4(3,3,4)$ then all polynomials in (1.4) vanish on $X$.

Vice versa, suppose that all polynomials in (1.4) vanish on $X$. Let

$$Y = \begin{bmatrix} x_{1,1,k} & x_{1,2,k} \\ x_{2,1,k} & x_{2,2,k} \end{bmatrix}, \quad k = 1, 2, 3, 4.$$

(3.6)

be the projection of the four frontal sections of $X$ by (1.3) on $\mathbb{C}^{2 \times 2}$. Then $f(X) = 0$ if and only if $Y_1, Y_2, Y_3, Y_4$ are linearly dependent. Decompose the tensor $X$ to a sum $Y + Z$. The four frontal sections of $Y$ are block diagonal matrices $\text{diag}(Y_k, 0), k = 1, 2, 3, 4$ and the four frontal sections of $Z$ are $\text{diag}(0, 0, x_{3,3,k}), k = 1, 2, 3, 4$.

Assume first that the polynomial $f(X)$ given by (1.3) vanishes in $X$. Since $Y_1, Y_2, Y_3, Y_4$ are linearly dependent, it follows the tensor $Y$ is represented as a $2 \times 2 \times 3$ tensor.

A particular case of [4, Theorem 3.1] tells us

$$\text{rank} T \leq \min(n, 2m) \quad \text{for any } T \in \mathbb{C}^{2 \times m \times n} \text{ where } 2 \leq m \leq n.$$

(3.7)

Hence the border rank of $Y$ is at most 3. (It is straightforward to show that any three dimensional subspace of $\mathbb{C}^{2 \times 2}$ is spanned by 3 rank one matrices. Hence [5, Theorem 2.1] implies that rank $Y \leq 3$.) Clearly rank $Z \leq 1$. Therefore rank $X \leq 4$. (More precisely rank $X = 4$.)

Assume now that $f(X) \neq 0$. Since the ten polynomials in (1.4) vanish on $X$, it follows that $x_{3,3,k} = 0$ for $k = 1, 2, 3, 4$. So $Z = 0$. In this case $X$ is represented a $2 \times 2 \times 4$ tensor. Hence, by (3.7), its border rank is at most 4. (More precisely, [5, Theorem 2.1] implies that rank $Y \leq 4$.)

3.2. The case $L = e_3 e_2^T$, $R = e_3 e_2^T$

Let $X_1, X_2, X_3, X_4 \in \mathbb{C}^{3 \times 3}$ be the four frontal sections of $X = [x_{i,j,k}] \in \mathbb{C}^{3 \times 3 \times 4}$. Assume that (2.1)--(2.2) hold. This means that our tensor $X = [x_{i,j,k}] \in \mathbb{C}^{3 \times 3 \times 4}$ has the following zero entries $x_{1,1,k} = x_{3,1,k} = x_{3,2,k} = x_{3,3,k} = 0$ for $k = 1, 2, 3, 4$. So our tensor is represented a $2 \times 3 \times 4$ tensor and hence, by (3.7), its border rank is at most 4.

4. The defining polynomials of $V_4(4,4,4)$

In this section we state for the reader’s convenience the defining equations of $V_4(4,4,4)$. We briefly repeat the arguments in [5] by replacing the degree 16 polynomial equations with the degree 6 polynomial equations. Let $X = [x_{i,j,k}] \in \mathbb{C}^{4 \times 4 \times 4}$. For each $l \in \{1,2,3\}$ we fix $l$ while we let $l_p, l_q = 1, 2, 3, 4$ where $\{p, q\} = \{1,2,3\} \setminus \{l\}$. In this way we obtain four $l$-sections $X_{1,l}, \ldots, X_{4,l} \in \mathbb{C}^{4 \times 4}$. (Note that $X_{k,3} = [x_{i,j,k}]_{i,j=1}^{4}$, $k = 1, 2, 3, 4$ are the four frontal sections of $X$.) Denote by $X_{l} = \text{span}(X_{1,l}, \ldots, X_{4,l}) \subset \mathbb{C}^{4 \times 4}$ the $l$-section subspace corresponding to $X$. For each $l \in \{1,2,3\}$ we define the following linear subspaces of polynomials of degree 5, 6, 9 respectively in the entries of $X$.

The defining polynomials could be any basis in each of these linear subspaces.

We first describe the Strassen commutative conditions [9]. (These conditions were rediscovered independently in [1].) Take $U_1, U_2, U_3 \in X_l$. View $U_i = \sum_{j=1}^{3} u_{j,i} X_{j,l}$ for $i = 1, 2, 3$. So the entries of each $X_{j,l}$ are fixed scalars and $u_{j,i}, i = 1, 2, 3, j = 1, 2, 3, 4$ are viewed as variables. Let $\text{adj } U_2$ be the adjoint matrix of $U_2$. Then the Strassen commutative conditions are

$$U_1(\text{adj } U_2)U_3 - U_3(\text{adj } U_2)U_1 = 0.$$

Since the values of $u_{j,i}, i = 1, 2, 3, j = 1, 2, 3, 4$ are arbitrary, we regroup the above condition for each entry as a polynomial in $u_{j,i}$. The coefficient of each monomial in the $u_{j,i}$ variables is a polynomial of degree 5 in the entries of $X$ and must be equal to zero. The set of all such polynomials of degree 5 span a linear subspace, and we can choose any basis in this subspace.
The degree 6 and 9 polynomial conditions are obtained in a slightly different way. Let $P = [p_{ij}]$, $Q = [q_{ij}] \in \mathbb{C}^{4 \times 4}$ be matrices with entries viewed as variables. View $PX_k, l Q, k = 1, 2, 3, 4$ as the four frontal slices of the $4 \times 4 \times 4$ tensor $X(P, Q, l) = [x_{i, j, k}(P, Q, l)]_{i, j, k=1}^4$.

Let $Y = [x_{i, j, k}(P, Q, l)]_{i, j, k=1}^{3, 3, 4}$. Now $Y$ must satisfy the degree 6 polynomial conditions of Landsberg–Manivel and the degree 9 symmetrization conditions. Since the entries of $P, Q$ are variables, this means that the coefficients of the monomials in the variables $p_{ij}, q_{ij}, i, j = 1, 2, 3, 4$ must vanish identically. This procedure gives rise to 10 polynomial conditions of degree 6 [6], which are linearly independent, and 440 polynomial conditions of degree 9 [5], which may be linearly dependent. Using appropriate software one may reduce the number of linearly independent conditions of degree 9.

The zero set of the above polynomials of degree 5, 6 and 9 defines $V_4(4, 4, 4)$.

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