ON THE EMBEDDING OF A VECTOR LATTICE IN
A VECTOR LATTICE WITH WEAK UNIT 1)

BY

BARRON BRAINERD

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1. Introduction

A classical problem in the theory of rings is to embed a ring $R$ in a ring $\bar{R}$ with identity so that $R$ has certain properties relative to $\bar{R}$, for example so that $R$ is an ideal of $\bar{R}$. An analogous problem exists in the theory of vector lattices, that is: Can a vector lattice be embedded in a vector lattice with weak unit? The terms vector lattice and weak unit are defined for example in [2]. Just as in the case of rings this problem may be solved by "adjoining" a (weak) unit 2): If $V$ is a vector lattice and $R$ is the vector lattice of real numbers, then let $L = V \oplus R$, the vector space direct sum of $V$ and $R$, where $(v, r) < (v^1, r^1)$ if either $v < v^1$ or $v = v^1$ and $r < r^1$. With respect to this order $L$ is a vector lattice and $i(v) = (v, 0)$ is an injection of $V$ into $L$ which preserves the vector lattice structure of $V$. The reader can verify that $(0, 1)$ is a weak unit for $L$. This embedding is crude in the sense that certain interesting properties which $V$ might have need not be shared by $L$. For example, $L$ is not archimedean and hence not conditionally $\sigma$-complete although $V$ may be. In addition $iV$ is not order-dense in $L$. A vector sublattice $S$ of a vector lattice $Q$ is order-dense in $Q$ if for every non-zero $f \in Q$ there is a non-zero $s \in S$ such that $|s| < |f|$. The purpose of this note is to show that every vector lattice $V$ can be embedded in a vector lattice $L$ with weak unit such that $V$ is order-dense in $L$. The proof of this result depends on a certain extension theorem of AMEMIYA [1] as well as an inverse limit construction. For definitions pertaining to the concept of inverse limit see [4]. As a corollary it is shown that every archimedean vector lattice can be embedded in an order-dense fashion in a conditionally complete $F$-ring.

2. Projectable vector lattices

Let $V$ be a vector lattice. If for a given $x$ in $V$ the supremum $\bigvee_{n=1}^{\infty} n|x|^\wedge y$ exists for every $y \geq 0$ in $V$, then the operator $[x]$ where $[x]y = \bigvee_{n=1}^{\infty} n|x|^\wedge y$ is called the projector of $x$. For arbitrary $y \in V$ the operator $[x]$ is defined as follows: $[x]y = [x]y^+ - [x]y^-$. Projectors on $\sigma$-complete vector lattices

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1) This paper was prepared while the author was a Fellow of the Summer Research Institute of the Canadian Mathematical Congress, 1959.

2) This construction is due to I. AMEMIYA.
(continuous semi-ordered linear spaces) are discussed extensively by Nakano [5]. With Amemiya [1] we say a vector lattice \( V \) is \textit{projectable} if every \( x \in V \) possesses a projector. It can be shown that Nakano's Theorems 7.1 through 7.6 in [5] are valid for projectors in a projectable vector lattice. The main result of this section is that every projectable vector lattice can be embedded (as an order-dense vector sublattice) in a projectable vector lattice with weak unit.

Let \( \mathcal{M} = \{ x_\alpha \in V | \alpha \in \mathfrak{A} \} \) be a set of positive elements of \( V \) with the following properties:

(i) \( x_\alpha \neq x_\beta \Rightarrow x_\alpha \wedge x_\beta = 0 \).

(ii) \( x \in V \) and \( x \neq 0 \Rightarrow |x| \wedge x_\alpha \neq 0 \) for some \( \alpha \in \mathfrak{A} \).

Such a set \( \mathcal{M} \) can always be extracted from \( V \) by Zorn's Lemma. Let \( D = D(\mathcal{M}) \) stand for the set \( \{ z \in V | z \text{ is the supremum (sum) of a finite set of elements of } \mathcal{M} \} \). The set \( D \) is directed by the order relation of \( V \). For each \( z \in D \) let \( P_z \) be the set of all \( x \in V \) such that \( |x| \wedge y = 0 \) if \( z \wedge y = 0 \).

**Lemma 2.1.** The set \( P_z \) is an \( l \)-ideal of \( V \) and the element \( z \) is a weak unit of \( P_z \).

**Proof.** If \( x, y \in P_z \), then \( |x| \wedge w = 0 \) and \( |y| \wedge w = 0 \) for every \( w \in V \) with the property \( z \wedge w = 0 \). Therefore \( (|x| + |y|) \wedge w = 0 \) whenever \( z \wedge w = 0 \) and since \( |x - y| \leq |x| + |y| \), it follows that \( |x - y| \wedge w = 0 \) whenever \( z \wedge w = 0 \). Thus \( P_z \) is a subgroup of \( V \). In addition \( P_z \) is a linear manifold of \( V \) because \( |x| \wedge w = 0 \) if and only if \( |x| \wedge w = 0 \). It is now easy to show \( P_z \) is an \( l \)-ideal. If \( x \in P_z \) and \( x \wedge z = 0 \), then \( x = x \wedge x = 0 \) and hence \( z \) is a weak unit of \( P_z \).

An \( l \)-ideal \( I \) of a vector lattice \( V \) is said to be closed if from

\[ \{ x_\beta \in V | x_\beta > 0, \beta \in \mathfrak{B} \} \subseteq I \text{ and } x = \vee_{\beta \in \mathfrak{B}} x_\beta \]

belongs to \( V \) it follows that \( x \in I \).

**Corollary 2.1.** For each \( z \in D \), \( P_z \) is closed.

**Proof.** Suppose \( \{ x_\beta \in V | x_\beta > 0, \beta \in \mathfrak{B} \} \subseteq P_z \) and \( x = \vee_{\beta \in \mathfrak{B}} x_\beta \) belongs to \( V \). Then for each \( \beta \in \mathfrak{B} \), \( x_\beta \wedge w = 0 \) whenever \( z \wedge w = 0 \). However,

\[ (\vee_{\beta \in \mathfrak{B}} x_\beta) \wedge w = \vee_{\beta \in \mathfrak{B}} (x_\beta \wedge w) = 0, \]

and therefore \( x \wedge w = 0 \) whenever \( z \wedge w = 0 \).

If \( u < v \) for \( u, v \in D \), then define the mapping \( \Pi_{uv}x = [u]x \) for \( x \in P_u \).

**Lemma 2.2.** The mapping \( \Pi_{uv} \) is a homomorphism of the vector lattice \( P_u \) onto \( P_u \).

**Proof.** From [5, Theorem 3.2] it follows that \([u]x \in P_u \) for each \( x \in V \) and \( u \in D \). Since Nakano's proof [5, Theorem 7.1] of the linearity of \([u]\) does not depend on \( \sigma \)-completeness, \([u]\) is linear, and since \( u \geq 0 \), \([u]\) is order preserving. From [5, Theorem 7.6] it follows that \( \Pi_{uv} = [u] \) is a lattice homomorphism.
Lemma 2.3. If $w<u<v$, then $\Pi_{uw}\Pi_{vq}=\Pi_{wq}$ for $w, v, q \in D$.

Proof. This lemma is valid if it can be shown that $[w][v]x=[w]x$ for each $x>0$ in $P_q$. By [5, Theorem 7.2] $[w][v]=|[w]|$; hence $[w][v]<[w]x$. Therefore the lemma is valid.

Corollary 2.2. For $v \in D$, $\Pi_{vu}x=x$.

From the preceding lemmas and corollaries it follows that $\mathcal{B} = \{(P_z, \Pi_{zw}) | z, w \in D\}$ is an inverse system. Let $L=L(\mathcal{R})$ stand for the inverse limit of $\mathcal{B}$. $L$ is a partially ordered vector space with respect to the relation: $f<g$ if and only if $f(u)<g(u)$ for all $u \in D$. Here $f(u)$ and $g(u)$ stand respectively for the components of $f$ and $g$ corresponding to $u \in D$.

Proposition 2.1. The partially ordered vector space $L$ is a vector lattice which possesses a weak unit.

Proof. Let $f^+$ be the function on $D$ to $V$ for which $f^+(u) = (f(u))^+$. From [5, Theorem 7.6] it follows that for $u < v$,

$$\Pi_{uv}f^+(v) = [u](f(v) \lor 0) = f(u) \lor 0 = f^+(u).$$

Therefore $f^+ \in L$. To show $f^+$ is indeed the supremum of $f$ and $0$ in $L$, suppose there is $h \in L$ such that $h > 0$ and $h > f$. Then $h(u) > f(u)$ and $h(u) > 0$ for each $u \in D$. Hence $h(u) > f^+(u)$ for each $u \in D$ and $h \geq f^+$. Therefore $f^+ = \sup \{f, 0\}$ in $L$ and $L$ is a vector lattice.

Consider the function $q$ defined as follows: $q(u) = u$ for each $u \in D$. If $u < v$, then $\Pi_{uv}q(u) = [u]v$, but $v = u + w$ where $w \in D$ and $u \land w = 0$. Therefore $[u]v = [u]u + [u]w = u$. Thus $\Pi_{uv}q(v) = q(u)$ and $q \in L$. The element $q$ is a weak unit of $L$. In fact if $q \land f = 0$, then $u \land f(u) = 0$ for all $u \in D$, and by Lemma 2.1, $f(u) = 0$ for all $u \in D$.

Proposition 2.2. The mapping $x \rightarrow f_x$ where $f_x(u) = [u]x$ for each $u \in D$ is an injection of $V$ into $L$.

Proof. The reader can verify that $f_x$ belongs to $L$ and $x \rightarrow f_x$ is linear and preserves lattice operations.

To show $x \rightarrow f_x$ is biunique, suppose $f_x = 0$. Then $[u]x^+ = [u]x^- = 0$ for all $u \in D$. Hence $[x_a]x^+ = [x_a]x^- = 0$ for all $x_a \in \mathcal{R}$. Therefore $x_a \land |x| = 0$ for all $x_a \in \mathcal{R}$ and $|x| = x = 0$ by the definition of $\mathcal{R}$.

Proposition 2.3. The vector lattice $L$ is projectable.

Proof. For $f, g \in L$ and $g \geq 0$ consider the function $h(u) = [f(u)]g(u)$. To show $h \in L$ suppose $u < v$. Then

$$\Pi_{uv}h(v) = [u][f(v)]g(v)$$

$$= \land_{n}[u]\land \{\land_{m}f(v) \land g(v)\}$$

$$= \land_{n}\land_{m}[u]f(v) \land g(v).$$
However, $\bigvee_m mf(u) \wedge g(u)$ exists in $V$; hence by [5, Theorem 2.5]

$$II_{uv}h(v) = \bigvee_m [u](mf(v) \wedge g(v))$$

$$= \bigvee_m mf(u) \wedge g(u)$$

$$= [f(u)]g(u)$$

$$= h(u).$$

Therefore $h \in L$.

To show that $h = \bigvee_m \nu f \wedge g$, that is to show $([\nu]g)(u) = [f(u)]g(u)$, suppose $k > m \nu f \wedge g$ for $m > 1$. Then $k(u) > [f(u)]g(u)$ for each $u \in D$ and hence $k > h$. Therefore since $h > m \nu f \wedge g$ for each $m$, $h = [f]g$. Thus every element of $L$ possesses a projector and $L$ is projectable.

Let $i$ designate the mapping $x \to f_x$ described in Proposition 2.2. Then $iV$ stands for the image of $V$ in $L$ under $i$.

**Proposition 2.4.** The vector lattice $iV$ is order-dense in $L$.

**Proof.** If $0 < f \in L$, then $[\bar{w}]f \in L$ for $\bar{w} \in iD$ and $([\bar{w}]f)(u) = [\bar{w}(u)]f(u)$ by Proposition 2.3. Since $\bar{w}(u) = [u]w$ where $\bar{w} = i(w)$, we have

$$([\bar{w}]f)(u) = \bigvee_n n([u]w) \wedge f(u)$$

$$= \bigvee_n n(\bigvee_m nmu \wedge w) \wedge f(u)$$

$$= \bigvee_n \bigvee_m nmu \wedge nw \wedge f(u).$$

If $u \triangleright w$, then

$$([\bar{w}]f)(u) = \bigvee_n n w \wedge f(u)$$

$$= \bigvee_n f(u) = [w][u]f(u) = [u]f(w).$$

Otherwise $w \lor u \triangleright w$ and

$$([\bar{w}]f)(w \lor u) = [w \lor u]f(w);$$

hence

$$([\bar{w}]f)(u) = [u][w \lor u]f(w)$$

$$= [u]f(w).$$

Therefore $[\bar{w}]f \in iV$ and since $[\bar{w}]f \triangleleft f$, it follows that $iV$ is order-dense in $L$.

**Proposition 2.5.** For every $f \triangleright 0$ in $L$,

$$f = \bigvee_{u \in ID}[\bar{u}]f,$$

that is, every $f \triangleright 0$ in $L$ is the supremum of elements of $iV$.

**Proof.** From the proof of Proposition 2.4 we deduce that $[\bar{u}]f \triangleleft f$ for every $\bar{u} \in iD$. Suppose $h \in L$ and $[\bar{u}]f \triangleleft h$ for each $\bar{u} \in iD$. Then since $([\nu]g)(u) = [f(u)]g(u)$, it follows that

$$([\bar{u}]f)(v) = f(u) \triangleleft h(v)$$

for $u \lhd v$ in $D$ where $\bar{u} = i(u)$. Hence $f \succeq h$ and equation (*) is valid.
Corollary 2.3. The weak unit \( q \in L \) where \( q(u) = u \) for all \( u \in D \) is the supremum of the elements of \( iD \).

The preceding results may be collected in the following theorem.

Theorem 2.1. Every projectable vector lattice \( V \) can be embedded as an order-dense vector sublattice of a projectable vector lattice \( L \) which possesses a weak unit. In addition every non-negative element of \( L \) is a supremum of elements of \( V \).

Proposition 2.6. If \( V \) is archimedean or \( \sigma \)-complete, then \( L \) is as well.

Proof. If \( 0 < f \in L \), then \( 0 < \frac{1}{n}f(u) \) for each \( n > 1 \) and each \( u \in D \). Suppose \( h \in L \) and

\[
0 < h(u) < \frac{1}{n}f(u)
\]

for each \( u \in D \) and each \( n > 1 \). Then \( h(u) = 0 \) for each \( u \in D \) because \( V \) is archimedean. Therefore \( h = 0 \) and \( L \) is archimedean.

A similar argument using [5, Theorem 7.6] proves that \( L \) is \( \sigma \)-complete if \( V \) is.

The following example shows that for a given projectable vector lattice \( V \) the extension \( L(\mathbb{R}) \) is dependent on the \( \mathbb{R} \) used in its construction:

Let \( V \) stand for the vector lattice of lebesgue measurable real functions on the line with compact support. Let \( \mathbb{R}_1 \) be the set of characteristic functions of points and \( \mathbb{R}_2 \) the set of characteristic functions of intervals \( \{ x \mid N < x < N + 1 \} \) where \( N \) is an integer. \( L(\mathbb{R}_1) \) is the vector lattice of all real functions on the line while \( L(\mathbb{R}_2) \) is the vector lattice of all real lebesgue measurable functions on the line.

3. Main result

With AMEMIYA [1] we call an extension \( \tilde{R} \) of a vector lattice \( R \) a \( p \)-extension if

1) \( \tilde{R} \) is projectable,
2) \( R \) is order-dense in \( \tilde{R} \),
3) the totality of projectors in \( \tilde{R} \) coincides with the least (Boolean) ring containing all projectors of elements of \( R \),
4) for every \( \tilde{a} \in \tilde{R} \), there exists \( a_1, ..., a_k \in R \) and \( \bar{a}_1, ..., \bar{a}_k \in \tilde{R} \) such that \( \tilde{a} = \sum_{n=1}^{k-1}[\bar{a}_n]a_n \).

Theorem 3.1. (Amemiya). Every vector lattice \( R \) has a \( p \)-extension \( \tilde{R} \) which is unique up to isomorphism.

AMEMIYA's method of proof [1] is first to define a spectral function \( \varphi \) of \( R \) as a lattice homomorphism of \( R \) into the three-point-lattice \( \{ -\infty, 0, +\infty \} \) such that \( \varphi(a) \neq 0 \) for some \( a \in R \) and \( \varphi(\alpha a) = \alpha \varphi(a) \) for all real \( \alpha \neq 0 \). These spectral functions are partially ordered by the relation: \( \psi < \varphi \) if \( |\psi(a)| < |\varphi(a)| \) for all \( a \in R \). The proper space \( \mathfrak{P} \) of \( R \) is defined to be the set of all maximal spectral functions and is given the topology
generated by the sets \( U_a = \{ \varphi \in \mathcal{R} | \varphi(a) \neq 0 \} \). The least algebra of subsets of \( \mathcal{R} \) containing the \( U_a \)'s is designated by \( \mathfrak{A} \), and \( \mathfrak{M} \) stands for the vector lattice of all functions from \( \mathcal{R} \) to \( R \) such that \( f(\mathcal{R}) \) is a finite subset of \( R \) and \( f^{-1}(a) \in \mathfrak{A} \) for \( a \in R \). If \( \mathfrak{R} = \{ f \in \mathfrak{M} | C_f = 0 \} \) where \( C_f = \{ \varphi | \varphi \in U_f, \mathfrak{A} \} \), then \( \mathfrak{M}/\mathfrak{R} \) is the unique \( p \)-extension of \( R \).

The main result of this note then follows from Amemiya’s theorem.

**Theorem 3.2.** Every vector lattice \( R \) can be embedded in a projectable vector lattice \( L \) with weak unit such that \( R \) is order-dense in \( L \).

**Proof.** From Theorem 3.1, it follows that \( R \) can be embedded in its \( p \)-extension \( \tilde{R} \) and by Theorem 2.1, \( \tilde{R} \) can be embedded in \( L \), a projectable vector lattice with weak unit. Therefore \( R \) can be embedded in \( L \). Since both the embedding of \( R \) in \( \tilde{R} \) and the embedding of \( \tilde{R} \) in \( L \) are order-dense, so is the composition of the two and the theorem follows.

**Corollary 3.1.** If \( R \) is archimedean, then so is \( L \).

**Proof.** This follows from a remark of Amemiya [1, p. 135].

If \( A \) is archimedean, then \( A \) has a cut extension \( \tilde{A} \) [5, Chapter V] which by definition is a complete vector lattice and hence is projectable. Theorem 2.1 can be applied to \( \tilde{A} \) to prove the following theorem.

**Theorem 3.3.** For each archimedean vector lattice \( R \) there exists a projectable vector lattice \( L \) with a weak unit which is complete and an injection \( i \) of \( R \) into \( L \) such that every element \( f > 0 \) of \( L \) is a supremum of elements of \( iR \).

**Proof.** Let \( \tilde{R} \) be the cut extension of \( R \) and let \( L \) be the extension of \( \tilde{R} \) with respect to some maximal orthogonal set \( \mathfrak{R} \) in \( \tilde{R} \) as described in Section 2. Let \( i \) be the composition of the injection \( i_1 \) of \( R \) in \( \tilde{R} \) and the injection \( i_2 \) of \( \tilde{R} \) in \( L \).

\( L \) is projectable and contains a weak unit by the results of Section 2. From Proposition 2.5 every element \( f > 0 \) in \( L \) is a supremum of elements in \( i_2 \tilde{R} \), and by [5, Chapter V] every \( f > 0 \) in \( \tilde{R} \) is the supremum of elements in \( i_1 \tilde{R} \). Thus if \( f \in L \),

\[
f= \bigvee_{u \in D} i_2(f(u)) = \bigvee_{u \in D} [\tilde{u}]
\]

where \( D = D(\mathfrak{R}) \) and

\[
f(u) = \bigvee_{\beta} i_1(f_\beta)
\]

where \( \beta \) ranges over a set \( \mathcal{B} \) and \( f_\beta \in R \). From [5, Theorem 7.6] one can prove

\[
i_2(\bigvee_{\beta} i_1(f_\beta)) = \bigvee_{\beta} i_2 i_1(f_\beta).
\]

Therefore,

\[
f= \bigvee_{u \in D} i_2(f(u)) = \bigvee_{u \in D} \bigvee_{\beta} i_2 i_1(f_\beta) = \bigvee_{u \in D, \beta \in \mathcal{B}} i_2 i_1(f_\beta),
\]

and hence every element of \( L \) is a supremum of elements of \( iR \).
Theorem 7.6 of [5] can be employed again to show that $L$ is a complete vector lattice.

The vector lattice $L$ of Theorem 3.3 is a complete $F$-space and hence by [3, Corollary 3.7] can be embedded as an order-convex \(^3\) subspace of a complete regular $F$-ring. Therefore we have

**Corollary 3.2.** Every archimedean vector lattice $R$ can be embedded in a complete regular $F$-ring $K$ so that every element $f \geq 0$ of $K$ is a supremum of elements of $R$.

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\(^3\) A subset $A$ of a vector lattice $V$ is order-convex if $g, h \in A$ and $g \leq k \leq h$ imply that $k \in A$.

**REFERENCES**