Discrete Mathematics 52 (1984) 113–121
North-Holland

INDUCTIVE DEFINITION OF TWO RESTRICTED CLASSES OF TRIANGULATIONS*

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Received 31 January 1983
Revised 19 June 1984

The inductive definitions of (i) the class of all triangulations (of the sphere) without vertices of degree 3; and (ii) the class of all triangulations with all vertices of even degree are given. The dual rules give us (i) the class of all 3-connected planar cubic graphs without triangles; and (ii) the class of all 3-connected bipartite planar cubic graphs (related to Barnette's hamiltonicity conjecture).

Steinitz and Rademacher [7, p. 243, exercise 1] and Bowen and Fisk [3] presented an inductive definition of the class of all triangulations (of the sphere). They essentially showed:

Theorem 1. *The inductive class with the base graph B and the generating rule P (see Fig. 1) is equal to the class T of all triangulations (of the sphere).*

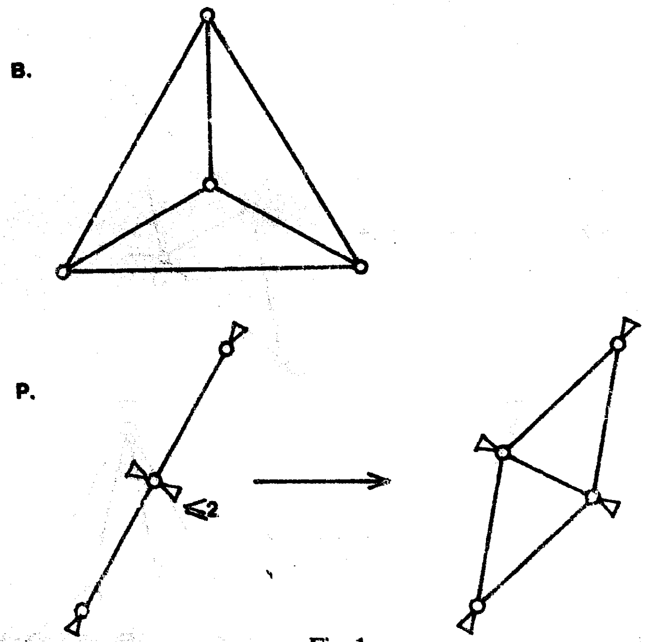

Fig. 1.

* Detailed version of the talk presented at the 3rd Yugoslav Seminar on Graph Theory, Kragujevac, June 3–4, 1982. This work was supported in part by the Boris Kidrič Fund, Slovenia, Yugoslavia.

The small triangles attached to the vertices in the description of the rule denote any number (zero or more) of edges; the condition ≤2 means that on that side there can be no more than 2 edges. The rule should be understood as embedded in the sphere (plane). In the following we shall use the half-edges to indicate that there must be an edge.

The dual inductive definition corresponds to the class of all 3-connected planar cubic graphs [5, 7].

In this paper we shall give the inductive definition of two restricted classes of triangulations:

- the class of all triangulations (of the sphere) without vertices of degree 3,
- the class of all triangulations with all vertices of even degree.

In the paper we shall denote by $\mathrm{Cn}(\mathscr{B}; \mathscr{P})$ the inductive class defined with basic graphs $\mathscr{B} = \{B_i\}$ and generating rules $\mathscr{P} = \{P_i\}$ (see also [1, 4]).

Theorem 2. *The inductive class* $\mathrm{Cn}(O; R, S)$ (*see Fig.* 2) *is equal to the class* $T(>3)$ *of all triangulations* (*of the sphere*) *without vertices of degree* 3.

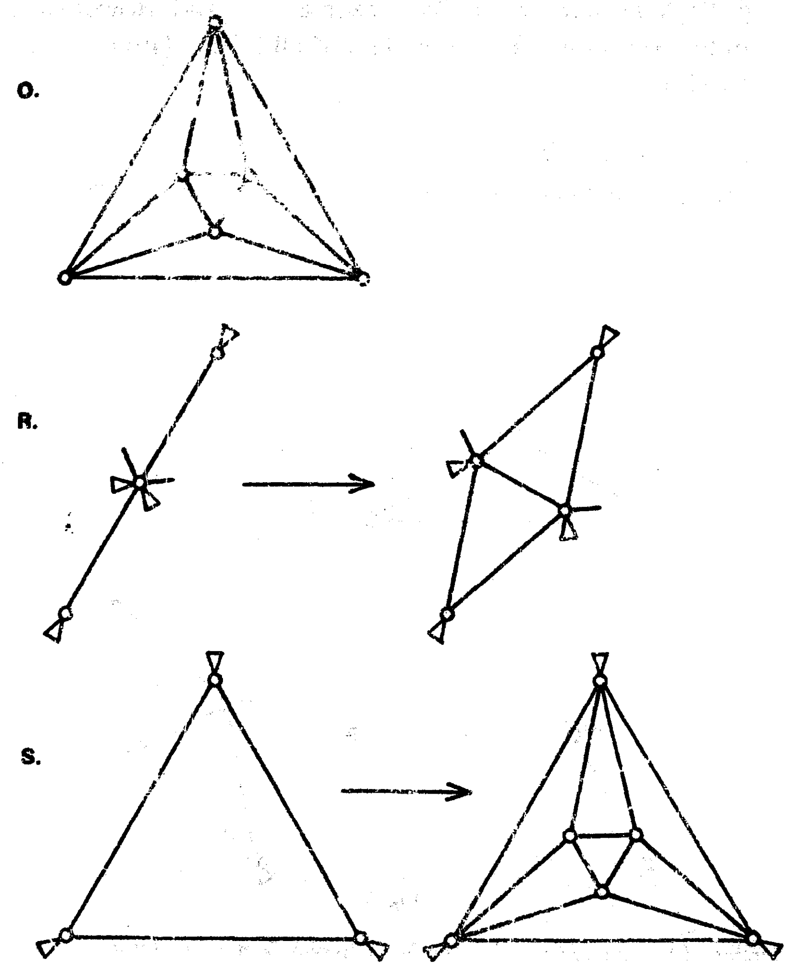

Fig. 2.

Proof. Because the base graph $O \in T(>3)$ and the rule R or S produces from a graph belonging to $T(>3)$ a new graph also belonging to $T(>3)$, by inductive generalization, the inductive class $\mathrm{Cn}(O; R, S) \subseteq T(>3)$.

To prove that also $T(>3) \subseteq \mathrm{Cn}(O; R, S)$, we must show that every triangulation G without vertices of degree 3, different from O, can be reduced by inverse rules R^- and S^- to the triangulation of the same type.

Usually we rely the inductive proofs of facts about the planar graphs on the property that in the planar graph there always exists a vertex of degree less than 6. In this proof we shall use an opposite approach. Let $x \in V(G)$ be a vertex of maximal degree, i.e., $\deg(x) = \Delta(G)$.

In the case $\deg(x) = 4$ we have [8, p. 52] $G = O$. O is the unique triangulation with all vertices of degree 4. Otherwise $\deg(x) \geq 5$.

Let us say that in two triangles with a common edge the pair of vertices, which do not belong to this edge, are *opposite*.

In the case $\deg(x) \geq 5$ we have to consider the following possibilities:

Case A. There exists a vertex y opposite to x with $\deg(y) \geq 5$. This can happen in two ways:

*Case A*1. The vertex x is not adjacent to vertex y (see Fig. 3(a): the circle around x represents the link of x). In this case we can apply the rule R^-.

*Case A*2. The vertex x is adjacent to vertex y (see Fig. 3(b)). In this case there must be at least one vertex (different from z and y) on the path a; otherwise $a = b$ (there are no parallel edges) and therefore $\deg(z) = 3$, which contradicts the assumption $G \in T(>3)$. We can apply the rule R^-.

Case B. All vertices opposite to x are of degree 4. Again we have to consider several possibilities:

*Case B*1. All vertices on the link are of degree at least 5—we shall say that there is a *crown around* x. Let us show that in this case there always exists on the link a vertex z which is not adjacent with more than 2 vertices on the link. If there exist chords, i.e., edges not lying on the link which connect two vertices lying on the link (see Fig. 4), we take one among the shortest (length = # of internal vertices in the shortest part of the link connecting these two vertices).

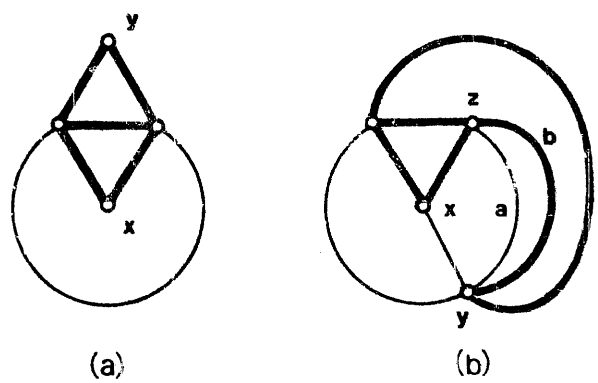

Fig. 3.

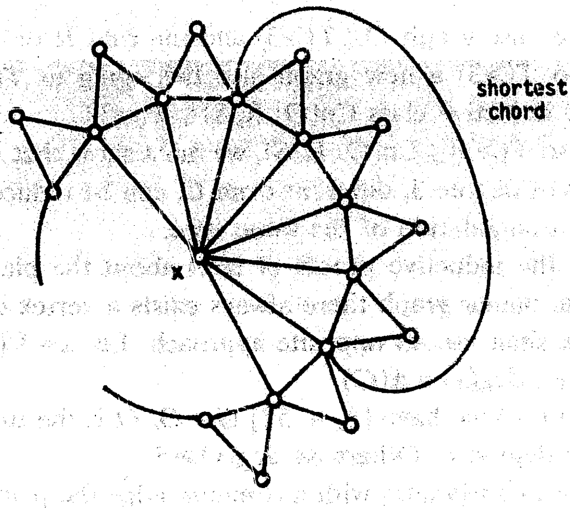

Fig. 4.

Because there are no parallel edges in G, the length of any chord is at least 1. Any internal vertex of the part of the link corresponding to the shortest chord can be taken as a vertex z, because of the minimality of the chord and the planarity of the graph G it can not lie on any chord.

That means that in the case of a crown around x we can always apply the rule R^- (see Fig. 5).

Case B2. There exists on the link a vertex of degree 4 with both neighbours (on the link) of degree at least 5 (see Fig. 6). Again, we can apply the rule R^-.

Case B3. There exists on the link a pair of adjacent vertices u and v of degree 4. Let the vertex y be the opposite vertex to x with respect to the edge uv and let p and q be the neighbours on the link of vertices u and v. Because deg(y) = 4 the vertices p and q have to be adjacent (see Fig. 7(a)). Therefore they are of degree at least 5. This is the last case to be considered, because on the link there can not

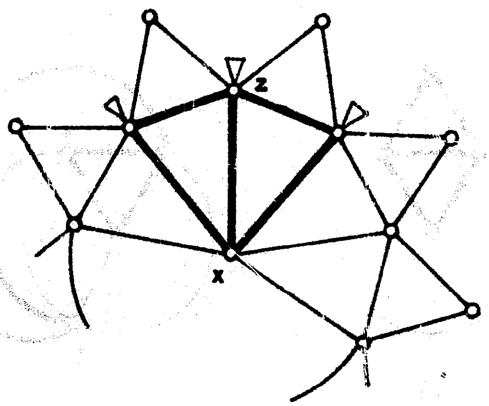

Fig. 5.

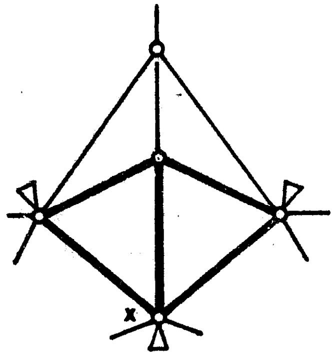

Fig. 6.

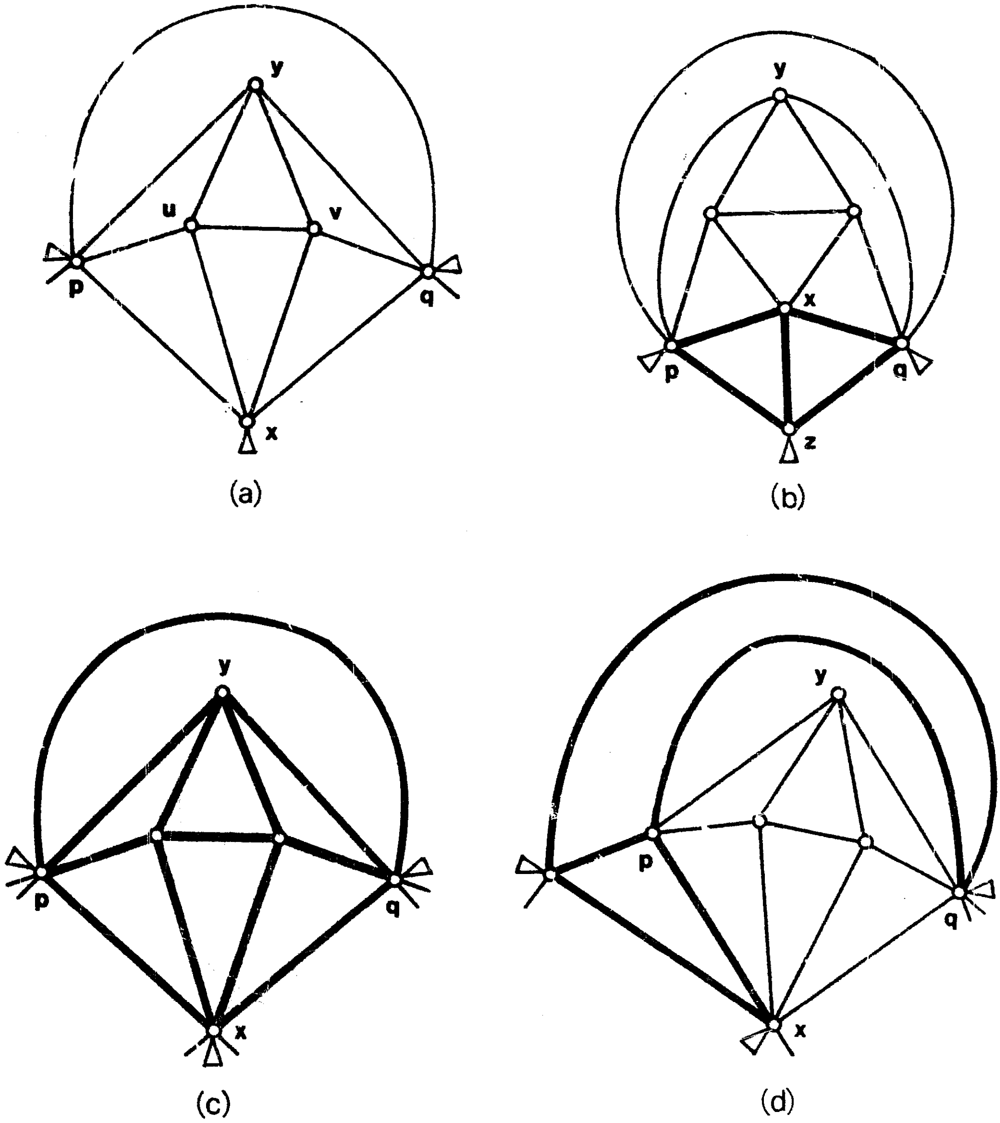

Fig. 7.

be more than two consecutive vertices of degree 4. Two subcases arise in the analysis of this case:

*Case B*3.1. $\deg(x)=5$; because the vertex z (see Fig. 7(b)) is not an extreme vertex of a chord we can in this case apply the rule R^-.

*Case B*3.2. $\deg(x)\geqslant 6$; if degrees of vertices p and q are both at least 6, we can apply the rule S^- (see Fig. 7(c)); otherwise for a vertex, let us say, p $\deg(p)=5$. In this case we have a configuration represented in Fig. 7(d); we can apply the rule R^-.

This completes the proof of Theorem 2. □

Theorem 3. *The inductive class* $\mathrm{Cn}(O, Q, S)$ *(see Figs.* 2 *and* 8) *is equal to the class* T_2 *of all triangulations (of the sphere) with all vertices of even degree.*

Proof. The proof is similar to the proof of Theorem 2. For this reason we only give the sketch of the second part of the proof.

From Euler's formula it follows that in each $G\in T_2$ there exist at least 6 vertices of degree 4.

We can easily verify that either we can apply the rule Q^- without the danger of producing a pair of parallel edges; or we have in graph G a configuration represented in Fig. 9(a).

The question marks in Fig. 9(a) mean that in those regions there can be vertices of graph G. Now we can apply the rule S^- or we cut out one of the non-empty regions together with its boundary obtaining a new triangulation $H\neq O$ (see Fig. 9(b)).

The internal vertices of H are all of even degree. But, for vertices u, y and w there are still two possibilities:

(a) all three vertices u, y and w are of even degree;

(b) two among the vertices u, y and w are of odd degree.

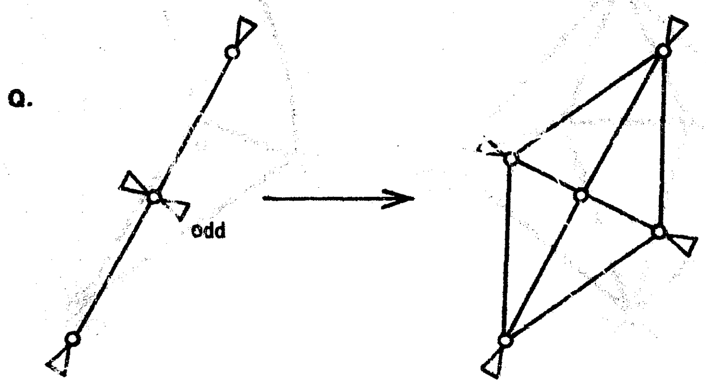

Fig. 8.

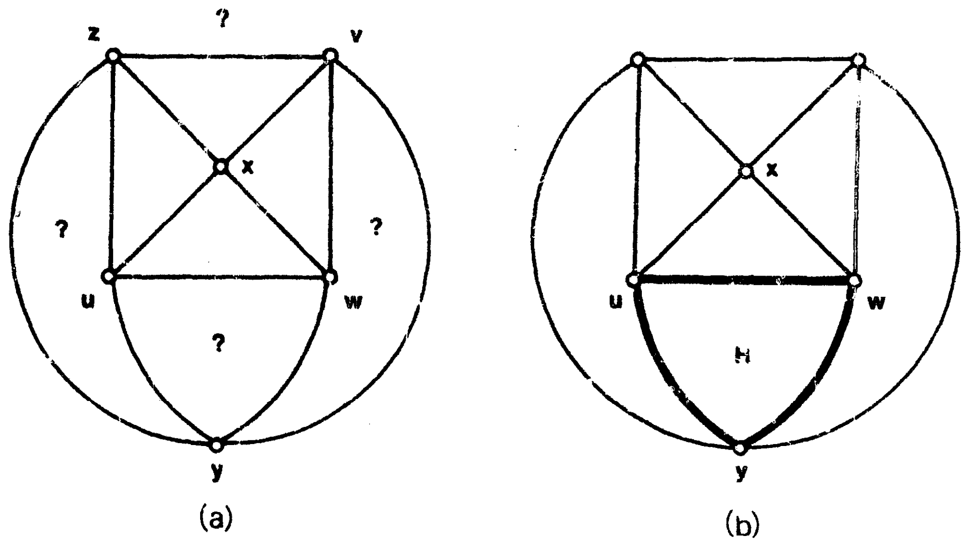

Fig. 9.

To avoid a direct proof of the fact that case (b) is impossible we refer to the result of Fisk [6, p. 311] which says: in the triangulation with exactly two odd vertices these two vertices can not be adjacent.

Therefore the triangulation $H \in T_2$ and we can repeat the described reduction procedure on it. □

In the dual form we can express Theorems 2 and 3 as follows (see Fig. 10):

Theorem 2'. *The inductive class* Cn(c; r, s) *is equal to the class* CP3(>3) *of all 3-connected planar cubic graphs without triangles.*

Theorem 3'. *The inductive class* Cn(c; q, s) *is equal to the class* CBP3 *of all 3-connected planar cubic bipartite graphs.*

The rules r and q in Theorems 2' and 3' can be equivalently replaced by the 'local' rules represented in Fig. 11.

The rules $r1$ and $q1$ are the 'expanding' rules; while the rules $r2$ and $q2$ play the role of 'moving' (around the face boundary) rules.

Therefore we have:

Theorem 2''. $\mathrm{Cn}(c; r1, r2, s) = \mathrm{CP3}(>3)$.

Theorem 3''. $\mathrm{Cn}(c; q1, q2, s) = \mathrm{CPB3}$.

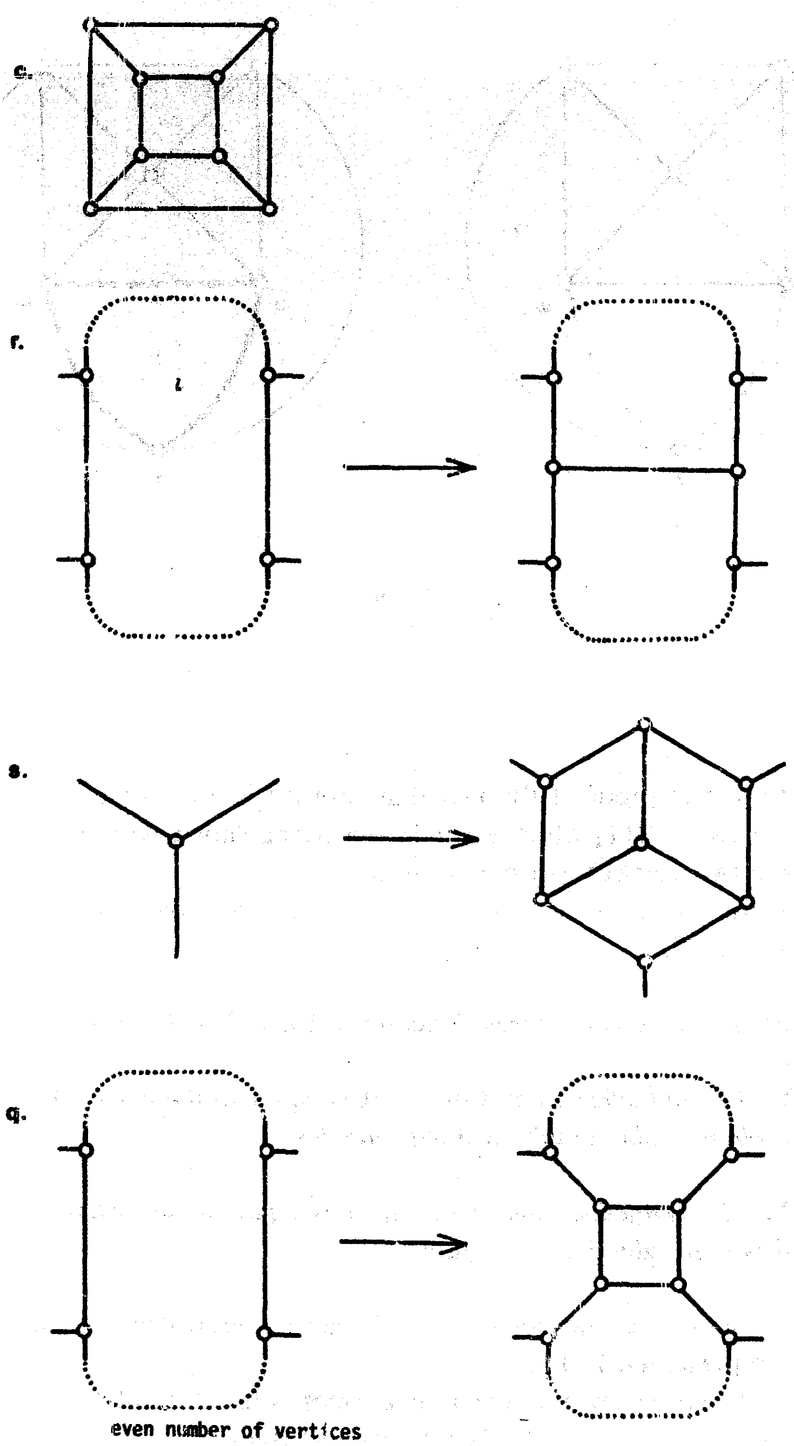

Fig. 10.

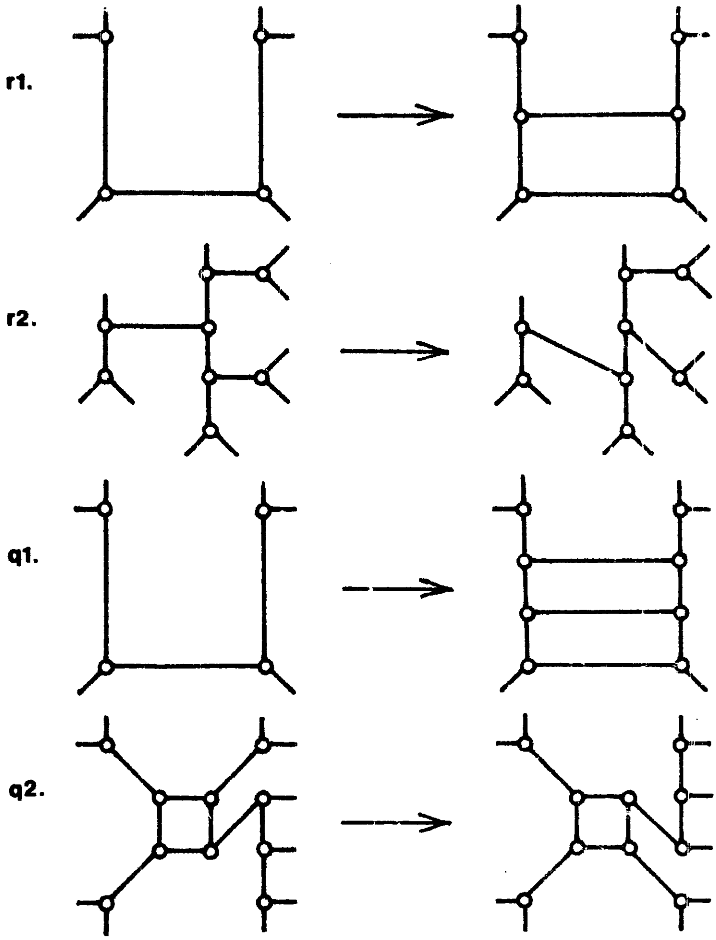

Fig. 11.

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