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On the Lines–Planes Inequality for Matroids

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DEDICATED TO THE MEMORY OF GIAN-CARLO ROTA

1. THE UNIMODALITY CONJECTURE

Although Gian-Carlo Rota did not publish much in matroid theory, his influence on the subject is pervasive (see [1, 7]). Among the many conjectures bearing his name in matroid theory, the unimodality conjecture is perhaps the most intractable.

Let G be a combinatorial geometry (or simple matroid). The *ith Whitney number* W_i (*of the second kind*) is the number of rank-i flats in G. Thus, W_1 is the number of points, W_2 is the number of lines, and W_3 is the number of planes.

Rota's Unimodality Conjecture. Let G be a rank-n geometry. Then, the sequence $W_0, W_1, W_2, ..., W_n$ is unimodal, that is, there is a rank s such that

$$W_0 \leqslant W_1 \leqslant W_2 \leqslant \cdots \leqslant W_s \qquad \text{and} \qquad W_{s+1} \geqslant W_{s+2} \geqslant \cdots \geqslant W_n.$$

One of Rota's motivation is that the Minkowski mixed volumes of a convex set form a unimodal sequence (see [4, 7]). There might be a way to use methods or ideas from convexity theory to prove the unimodality conjecture.

In this paper, we present a partial result about the case $n=5$ of the unimodality conjecture.

1.1. THEOREM. *Let G be a geometry of rank at least 5 in which all the lines have the same number of points. Then*

$$W_2 \leqslant W_3.$$

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There are many geometries in which all the lines have the same number of points. Among them are the affine binary geometries, that is, subgeometries of $AG(n-1, 2)$. In these geometries, every line has exactly two points.

2. RADON TRANSFORMS

Let $L(G)$ be the lattice of flats of the geometry G and let $L_k(G)$ be the set of rank-k flats in $L(G)$. A function f defined from $L(G)$ to the rational numbers $\mathbb{Q}$ is *supported on* a set J of flats in $L(G)$ if $f(X)$ is zero unless X is in J. The *Radon transform* T is the linear transformation on the vector space of rational-valued functions on $L(G)$ defined by

$$Tf(X) = \sum_{Y:\, Y \leqslant X} f(Y).$$

The *mass* of a function f is the value of Tf at the maximum flat $\hat{1}$ of $L(G)$, that is,

$$\text{mass}(f) = Tf(\hat{1}) = \sum_{Y:\, Y \in L(G)} f(Y).$$

A function $f: L(G) \to \mathbb{Q}$ is said to be *reconstructible* from its Radon transform Tf restricted to a set M of flats if f is uniquely determined by the table of values of Tf on M.

If every function supported on the set J is reconstructible from its Radon transform restricted to the set M, then, by comparing dimensions of subspaces, $|J| \leqslant |M|$. Hence, Theorem 1.1 follows from the following reconstruction theorem.

2.1. THEOREM. *Let G be a geometry of rank at least 5 in which all the lines have the same number of points and let $f: L(G) \to \mathbb{Q}$ be a function supported on the lines of G. Then f is reconstructible from its Radon transform Tf restricted to the planes.*

The idea behind our proof of Theorem 2.1 is the following special case of a result of Dowling and Wilson [3] (see also [5, 6]).

2.2. THEOREM. *Let G be a rank-5 geometry and let $f: L(G) \to \mathbb{Q}$ be a function supported on $L_0(G) \cup L_1(G) \cup L_2(G)$. Then f is reconstructible from its Radon transform Tf restricted to $L_3(G) \cup L_4(G) \cup L_5(G)$.*

By Theorem 2.2 and the fact that geometric lattices can be truncated, we can proved Theorem 2.1 by showing that the restriction of Tf to planes gives sufficient information to reconstruct the Radon transform of f on the rank-4 flats or *spaces* and the mass of f. In Section 3, we will show how the space transforms can be reconstructed. The mass, however, cannot be reconstructed directly. To get around this, we use the "missing mass method," which requires us to follow through an explicit reconstruction algorithm given in [5, 6].

3. RECONSTRUCTING THE SPACE TRANSFORMS

We begin the proof of Theorem 2.1 by reconstructing the space transform from the plane transform.

3.1. LEMMA. *Let H be a rank-4 geometry on the set S in which all the lines have the same number c of points and let $f: L(H) \to \mathbb{Q}$ be a function supported on the lines. Then the mass of f can be reconstructed from its Radon transform Tf restricted to the planes.*

Proof. Let τ_i be the sum $\sum_\ell f(\ell)$ over all the lines ℓ in H contained in exactly i planes. Then

$$\mathrm{mass}(f) = \sum_{i \geqslant 2} \tau_i.$$

When we sum $Tf(U)$ over all the planes U in H, the lines on exactly i planes contribute i times to the sum. Hence,

$$\begin{aligned} \sum_U Tf(U) &= \sum_{i \geqslant 2} i\tau_i \\ &= \sum_{i \geqslant 2} (i-1)\, \tau_i + \mathrm{mass}(f). \end{aligned} \tag{1}$$

Next, consider the sum $\sum_U |U|\; Tf(U)$ over all planes. If $U_1, U_2, ..., U_i$ are all the planes containing the line ℓ, then the function value $f(\ell)$ occurs with coefficient $|U_1| + |U_2| + \cdots + |U_i|$ in this sum. Because the sets $U_1 \backslash \ell, U_2 \backslash \ell, ..., U_i \backslash \ell$ partition the set $S \backslash \ell$ of points not in ℓ (by the exchange property for closure),

$$|U_1| + |U_2| + \cdots + |U_i| = |S| + (i-1)\; |\ell|.$$

Hence, by the hypothesis that every line contains exactly c points,

$$\sum_U |U|\, Tf(U) = \sum_{i \geqslant 2} [\,|S| + (i-1)\, c\,]\, \tau_i \tag{2}$$

$$= |S| \operatorname{mass}(f) + c \sum_{i \geqslant 2} (i-1)\, \tau_i.$$

Combining Eqs. (1) and (2), we obtain the reconstruction formula

$$\operatorname{mass}(f) = \frac{1}{|S| - c} \left(\sum_U (|U| - c)\, Tf(U) \right).$$

We remark that the proof of Lemma 3.1 makes no use of the fact that lines have rank 2. The lemma extends to functions supported on rank-k flats, provided, of course, those flats all contain the same number of points.

Lemma 3.1 does not hold if the hypothesis on the lines of H is dropped. A simple example is the direct sum $U_{2,m} \oplus U_{2,m}$ of two m-point lines.

4. MISSING MASS

To finish the reconstruction, we first reconstruct f from the given plane transform, the reconstructed space transforms, and a "variable" mass, using an algorithm given in [5, 6]. When this is done, we will derive the value of the mass from the fact that $f(\varnothing) = 0$.

We use the following Möbius function identity which first appeared in [2, 3] (see [6] for a simple combinatorial proof).

4.1. Lemma. *Let $f: L \to \mathbb{Q}$ be a rational-valued function on a lattice L and let X be a flat in L. Then*

$$\sum_{Y: X \leqslant Y \leqslant \hat{1}} \mu(Y, \hat{1})\, Tf(Y) = \sum_{Z: Z \vee X = \hat{1}} f(Z), \tag{3}$$

where μ is the Möbius function of L.

Let L be $L(G; 5)$, the *truncation* of $L(G)$ to rank 5, obtained from $L(G)$ by identifying all the flats in $L(G)$ of rank at least 5. Choose the flat X in Eq. (3) to be a line ℓ. By the submodular inequality, all the flats Z on the right hand side in Eq. (3) have rank greater than 2. Hence, as f is supported on lines, $f(Z)$ equals 0 for all such flats and the right hand sum is zero.

Since the plane and space transforms are known and $Tf(\ell) = f(\ell)$, we can solve for $f(\ell)$ in Eq. (3) to obtain

$$f(\ell) = -\frac{1}{\mu(\ell, \hat{1})}\left(M + \sum_{U:\ell<U<\hat{1}} \mu(U, \hat{1})\, Tf(U)\right), \tag{4}$$

where M is a variable standing for the unknown mass of f. Note that we can divide by $\mu(L, \hat{1})$ since it is not zero by a theorem of Rota [8, p. 357].

To finish the proof, we use Eq. (3) with X the empty flat $\emptyset$. The maximum flat $\hat{1}$ is the only flat contributing to the right hand sum, and so the right hand sum is again zero. Because $Tf(\emptyset) = 0$, $Tf(p) = 0$ for any point, and $Tf(\ell) = f(\ell)$ for any line ℓ, we have

$$\left(\sum_{\ell} \mu(\ell, \hat{1})\, f(\ell)\right) + B + M = 0,$$

where B is a linear combination of plane or space Radon transform values. Substituting in the values of $f(\ell)$ given in Eq. (4) and simplifying, we obtain

$$M = \frac{1}{W_2 - 1}\left(-B - \sum_{\ell}\left(\sum_{U:\ell<U<\hat{1}} \mu(U, \hat{1})\, Tf(U)\right)\right).$$

Note that W_2, the number of lines in G, is an integer greater than 1. We can now calculate M and finish the reconstruction. This completes the proof of Theorem 2.1.

Rephrasing Theorem 2.1 in terms of matrices and using a standard determinant argument (see [5]), we obtained the following corollary.

4.2. COROLLARY. *Let G be a geometry of rank at least 5 in which every line has the same number of points. Then, the line–plane incidence matrix has maximum rank equal to the number of lines. In particular, there exists an injection σ from the lines to the planes such that $\sigma(\ell) > \ell$ for all lines ℓ.*

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