A new cross-scaling method to deal with the porous flow problem

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Abstract The finite volume method (FVM) and the lattice Boltzmann method (LBM) are coupled to each other to construct a new cross-scaling method to deal with the porous flow problem. To check the effectiveness of our developed cross-scaling LBM–FVM, the above mentioned problem is also solved by the well known LBM–LBM. Based on the data checking of the published data and the results of LBM–FVM and LBM–LBM, good agreement is observed. © 2013 The Chinese Society of Theoretical and Applied Mechanics. [doi:10.1063/2.1303209]

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Last few years, as a mesoscopic method, lattice Boltzmann method (LBM) has experienced a rapid development in the field of fluid flows. LBM shows many advantages compared with conventional methods: it is easy to handle complex geometries, easy to parallelize, easy to program, and easy to obtain stable and accurate results. and LBM has thus been successfully applied to many fields such as multiphase flow, seepage, and magnetic fluids.

The finite volume method (FVM) is known as a control volume integral method, and it is a kind of macroscopic method. In FVM, each term has a clear physical meaning in the integral equations, and each discrete item can be given a certain physical interpretation when the equations are discretized. Mishra et al. used LBM to deal with the complex transient radiation and conduction heat transfer process, where they used FVM to obtain radiation information. However, few other related studies have been carried out based on LBM–FVM.

In the fields of thermal insulation, chemical and petroleum engineering, and food processing, we will frequently encounter the problem of porous flow, therefore, many theoretical and experimental studies have attempted to better explain it. For example, Nield et al. conducted a comprehensive review on this subject. Guo et al. suggested the use of two different particle distribution functions for the velocity and temperature field to solve the problem of a porous flow being confined to an isothermal flow using LBM. Yan et al. used LBM–LBM to simulate the porous flow with variable porosity. However, both of these studies focused on the same scale to solve the nonisothermal flow. Herein, we consider cross-scaling LBM to solve the same problem.

Figure 1 shows the porous flow problem with heat and mass transfer process. Under the assumptions of Boussinesq limit and local thermal equilibrium, considering a homogeneous porous medium, the corresponding macroscopic equations can be expressed as

\[ \nabla \cdot \mathbf{u} = 0, \] (1)

where \( \mathbf{u} \) is the velocity vector, \( \rho \) is the density, \( \nabla \cdot \) denotes the divergence operator, and \( \phi \) is the porosity.

| \( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nabla \cdot (\nu_e \nabla \mathbf{u}) + \mathbf{F}, \) | (2) |
| \( \sigma \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla (\alpha_e \nabla T), \) | (3) |

\( \mathbf{F} \) here, in the above three equations, variables \( \mathbf{u}, p, T, \nu_e, \alpha_e, \) and \( \phi \) represent the velocity, pressure, temperature, effective viscosity, thermal diffusivity, and porosity, respectively. In Eq. (2), the term \( \mathbf{F} \) denotes the total force caused by the porous medium itself and external force field, which can be given by

\[ \mathbf{F} = -\frac{\phi \nu}{K} \mathbf{u} - \frac{\phi F_\phi}{\sqrt{K}} \mathbf{u} \mathbf{u} + \phi G, \] (4)

where variables \( \nu, K, \) and \( F_\phi \) denote the shear viscosity coefficient of the fluid, the permeability, and geometric function of the porous medium, respectively. \( K \) and \( F_\phi \) are relevant with the porosity \( \phi \). Using Ergun's empirical formula, they can be written as

\[ F_\phi = \frac{1.75}{\sqrt{150 \phi}}, \] (5)

here, variable \( d_p \) denotes the particle diameter. \( G \) at the end of Eq. (4) denotes the effective gravity, which

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can be written as

\[
G = -g\beta(T - T_0),
\]

where variables \( g \), \( \beta \) and \( T_0 \) represent the gravitational acceleration, thermal expansion coefficient, and average temperature of the porous flow system, respectively.

We focus on Eq. (3), which governs the fluid and solid heat transfer behavior. Variable \( \sigma \) is the ratio of the heat capacity of the solid phase and the heat capacity of the fluid phase, and it is given by

\[
\sigma = \phi + \frac{(1 - \phi)\rho_s c_{ps}}{\rho_f c_{pf}},
\]

where \( \rho_s \) and \( \rho_f \) represent the density of the solid and fluid phases, respectively, \( c_{ps} \) is the heat capacity of the solid phase, and \( c_{pf} \) is the heat capacity of the fluid phase.

To generalize our numerical results, we adopt parameters \( J \), \( Da \), \( Pr \), and \( Ra \) to characterize Eqs. (1)–(3), which govern the heat transfer behavior of the fluid and solid; the four parameters are named as viscosity ration, Darcy number, Prandtl number, and Rayleigh number, respectively, and they can be defined as

\[
J = \frac{\nu_e}{\nu}, \quad Da = \frac{K}{L^2},
\]

\[
Pr = \frac{\nu}{\alpha_e}, \quad Ra = \frac{g\beta\Delta TL^3}{\nu\alpha_e},
\]

where \( L \) is the medium width, and expression \( \Delta T = T_h - T_c \) denotes the temperature difference. \( T_h \) is the temperature of hot wall, and \( T_c \) is the temperature of the cold wall.

We again focus on the governing equations (1)–(3). It should be noted that if \( \phi \) tends to 1, these equations can be used to describe the general fluid.

The determination of the mesh distribution is a key point in our present LBM–FVM model. To achieve information transfer between the macroscopic and the mesoscopic level, it is necessary to construct a correct mesh distribution that complies with the LBM and FVM concepts. We validated different meshing methods and obtained a suitable mesh distribution, as shown in Fig. 2.

Here, Eqs. (1) and (2) are solved by LBM, and detailed information about LBM can be found in Ref. 12. Equation (3) is solved by FVM. We move the convection term \( \nabla \cdot (uT) \) and obtain

\[
\frac{\partial(\sigma T)}{\partial t} = \nabla \cdot (\alpha_e \nabla T) - \nabla \cdot (uT).
\]

Integrating the left side of Eq. (9) over the control volume \( \Delta V \) and the time interval \( t \) to \( t + dt \), we obtain

\[
\int_{t}^{t+dt} \left[ \int_{\Delta V} \frac{\partial(\sigma T)}{\partial t} dV \right] dt = \int_{\Delta V} \left[ \int_{t}^{t+dt} \frac{\partial(\sigma T)}{\partial t} dV \right] dt.
\]

where the term \( \partial T / \partial t \) can be expressed as

\[
\frac{\partial T}{\partial t} \approx \frac{T_P - T_0}{\Delta t},
\]

where variables \( T_0 \) and \( T_P \) are the temperature of point \( P \) at time \( t \) and time \( t + \Delta t \), respectively. We obtain

\[
\int_{\Delta V} \left[ \int_{t}^{t+\Delta t} \frac{\partial(\sigma T)}{\partial t} dV \right] dt \approx \sigma \Delta V \int_{t}^{t+\Delta t} \frac{T_P - T_0}{\Delta t} dt = \sigma \Delta V (T_P - T_0). \tag{12}
\]

Integrating the right side of Eq. (12) over the control volume \( \Delta V \) and the time interval \( t \) to \( t + dt \) and adopting the fully implicit scheme, we obtain

\[
\int_{t}^{t+\Delta t} \left\{ \int_{\Delta V} \left[ \nabla \cdot (\alpha_e \nabla T) - \nabla \cdot (uT) \right] dV \right\} dt = \Delta t \int_{\Delta V} \left[ \nabla \cdot (\alpha_e \nabla T) - \nabla \cdot (uT) \right] dV. \tag{13}
\]

Here, we adopt the divergence theorem for a vector \( \mathbf{a} \), which can be expressed as

\[
\int_{\Delta V} (\nabla \cdot \mathbf{a}) dV = \int_{A} (\mathbf{n} \cdot \mathbf{a}) dA,
\]

where variable \( A \) is the surface area of the control volume. Combining Eqs. (12) and (13), we obtain

\[
\sigma \Delta V (T_P - T_0) = \Delta t \int_{A} \alpha_e \frac{dT}{dx} dA + \Delta t \int_{A} \alpha_e \frac{dT}{dy} dA - \Delta t \int_{A} uT dA. \tag{15}
\]

Using the concept of FVM and supposing the continuity of the flow, the energy equation can be discreted and it is given by

\[
a_P T_P = a_W T_W + a_F T_E + a_N T_N + a_S T_S + a_h T_h + S_u,
\]

where \( a_P \) is the heat production, \( a_W \), \( a_F \), and \( a_N \) are the heat transfer coefficients, \( a_S \), \( a_h \), and \( S_u \) are the heat sources, and \( T_W \), \( T_E \), \( T_N \), \( T_S \), \( T_h \), and \( S_u \) are the temperatures of the walls, edges, nodes, solid, hot, and the source, respectively.
where $a_W$, $a_E$, $a_N$, $a_S$, and $a_P$ are the field variable coefficients for each mesh grid, $S_P$ and $S_u$ are related to the source term, and $a_P^0$ is derived from the fully implicit scheme. The relevant parameters are given by

$$
\begin{align*}
    a_W &= \alpha_e \frac{u_W}{\Delta x} + \frac{u_W}{2}, \\
    a_E &= \alpha_e \frac{u_E}{\Delta x} - \frac{u_E}{2}, \\
    a_N &= \alpha_e \frac{v_N}{\Delta x} - \frac{v_N}{2}, \\
    a_S &= \alpha_e \frac{v_S}{\Delta x} + \frac{v_S}{2}, \\
    a_P &= a_W + a_E + a_N + a_S + a_P^0 - S_P, \\
    a_P^0 &= \frac{\sigma \Delta x}{\Delta t}, \\n    S_P &= 0, \quad S_u = 0.
\end{align*}
$$

(17)

For a certain mesh grid in the computational domain, we can obtain the corresponding discrete equation; however, for the boundary mesh grids, we should modify the relevant parameters $a_W$, $a_E$, $a_N$, $a_S$, $a_P$, $a_P^0$, $S_P$, and $S_u$ according to the boundary conditions.

To check the accuracy of our developed LBM–FVM, the porous flow problem is studied by both LBM–FVM and LBM–LBM. The obtained numerical results are compared with those of previous studies in Refs. 14 and 15. In our validation, the computation parameters are given as $\phi = 0.6$, $Pr = 1.0$, $J = 1.0$, $\sigma = 1.0$, $Da = 10^{-4}–10^{-2}$, and $Ra = 10^5–10^7$. In all computations in this study, we considered that if the maximum variation of the velocity field between the two iterations does not exceed $10^{-9}$, then the relevant computation converges. A $300 \times 300$ lattice was adopted in LBM–FVM and LBM–LBM, respectively. Table 1 shows data checking of the average Nusselt number ($\overline{Nu}$) on the hot wall among LBM–LBM, LBM–FVM, and published data. It is observed that the LBM–FVM results show good agreement with both the LBM–LBM results and the published data. This suggests that the LBM–FVM model is sufficiently reliable and effective.

For the sake of data checking, the above mentioned porous flow problem was also solved by LBM–LBM. Detailed information about the LBM–LBM can be found in Ref. 12.
Table 2. Data checking of average Nusselt number on the hot wall \((Ra = 10^6, \phi = 0.6)\).

<table>
<thead>
<tr>
<th>(Da)</th>
<th>LBM–FVM</th>
<th>LBM–LBM</th>
</tr>
</thead>
<tbody>
<tr>
<td>10(^{-2})</td>
<td>7.122</td>
<td>7.120</td>
</tr>
<tr>
<td>10(^{-3})</td>
<td>5.866</td>
<td>5.860</td>
</tr>
<tr>
<td>10(^{-4})</td>
<td>2.712</td>
<td>2.706</td>
</tr>
<tr>
<td>10(^{-5})</td>
<td>1.074</td>
<td>1.075</td>
</tr>
</tbody>
</table>

Table 3. Data checking of average Nusselt number on the hot wall \((Da = 10^{-3}, \phi = 0.6)\).

<table>
<thead>
<tr>
<th>(Ra)</th>
<th>LBM–FVM</th>
<th>LBM–LBM</th>
</tr>
</thead>
<tbody>
<tr>
<td>10(^4)</td>
<td>1.046</td>
<td>1.046</td>
</tr>
<tr>
<td>10(^5)</td>
<td>2.190</td>
<td>2.192</td>
</tr>
<tr>
<td>10(^6)</td>
<td>5.866</td>
<td>5.860</td>
</tr>
<tr>
<td>10(^7)</td>
<td>12.339</td>
<td>12.179</td>
</tr>
</tbody>
</table>

Table 2 shows data checking of the average \(Nu\) on the hot wall of the porous medium until the velocity field converges between LBM–FVM and LBM–LBM for different values of \(Da\), which ranges from \(10^{-2}\) to \(10^{-5}\), with \(Ra = 10^6\) and \(\phi = 0.6\). The average \(Nu\) for LBM–FVM differs slightly from that for LBM–LBM. Therefore, LBM can be coupled to FVM, the two methods are compatible with each other, and the obtained numerical results are effective and reliable.

Figure 3 shows data checking between LBM–FVM and LBM–LBM in terms of the vertical velocity and temperature distribution at the mid-height of the porous cavity for \(Da\) values of \(10^{-2}\), \(10^{-3}\), \(10^{-4}\), and \(10^{-5}\) with \(Ra = 10^6\) and \(\phi = 0.6\). The obtained numerical results are in good agreement.

Table 3 shows data checking between LBM–FVM and LBM–LBM in terms of the effects of \(Ra\) values of \(10^4\), \(10^5\), \(10^6\), and \(10^7\) with \(Da = 10^{-3}\) and \(\phi = 0.6\). The average \(Nu\) for LBM–FVM and LBM–LBM is similar. Therefore, LBM and FVM are compatible with each other, and the obtained numerical results are reliable. Figure 4 shows data checking between LBM–FVM and LBM–LBM in terms of the vertical velocity and temperature distribution for \(Ra\) values of \(10^4\), \(10^5\), \(10^6\), and \(10^7\) with \(Da = 10^{-3}\) and \(\phi = 0.6\) at the half height of the porous medium. To investigate the influence of, both LBM–FVM and LBM–LBM, values were used. The obtained numerical results are in good agreement.

In this study, we adopt LBM–FVM to solve the porous flow problem. LBM was used to obtain information about the velocity field and FVM, to solve the energy equation. We derived the discrete form of the energy equation, dealt with the corresponding boundary conditions, and coupled LBM to FVM through the buoyancy force. To achieve this coupling, we also constructed an effective mesh distribution that enabled information transfer between the macroscopic and the mesoscopic level.

For the sake of a comparison, the above mentioned problem was also solved by LBM–LBM. Results of both LBM–FVM and LBM–LBM showed good agreement with the published studies in all cases, and the two also showed good agreement with each other for a series of parameters. Based on the results of this study, we can derive the following conclusions:

1. LBM and FVM are compatible with each other, and the numerical results obtained using LBM–FVM are effective and reliable.

2. We constructed a reliable and effective cross-scaling coupling method based on FVM and LBM.

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