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On the spanning connectivity and spanning laceability of hypercube-like networks

Cheng-Kuan Lin[a], Jimmy J.M. Tan[a], D. Frank Hsu[b], Lih-Hsing Hsu[c,*]

[a] *Department of Computer Science, National Chiao Tung University, Hsinchu, 30010, Taiwan, ROC*
[b] *Department of Computer and Information Science, Fordham University, New York, NY 10023, USA*
[c] *Department of Computer Science and Information Engineering, Providence University, Taichung, 43301, Taiwan, ROC*

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Abstract

Let u and v be any two distinct nodes of an undirected graph G, which is k-connected. For $1 \leq w \leq k$, a w-*container* $C(u, v)$ of a k-connected graph G is a set of w-disjoint paths joining u and v. A w-container $C(u, v)$ of G is a w^*-*container* if it contains all the nodes of G. A graph G is w^*-*connected* if there exists a w^*-container between any two distinct nodes. A bipartite graph G is w^*-*laceable* if there exists a w^*-container between any two nodes from different parts of G. Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two disjoint graphs with $|V_0| = |V_1|$. Let $E = \{(v, \phi(v)) \mid v \in V_0, \phi(v) \in V_1$, and $\phi : V_0 \to V_1$ is a bijection$\}$. Let $G = G_0 \oplus G_1 = (V_0 \cup V_1, E_0 \cup E_1 \cup E)$. The set of n-dimensional hypercube-like graph H'_n is defined recursively as (a) $H'_1 = \{K_2\}$, $K_2 =$ complete graph with two nodes, and (b) if G_0 and G_1 are in H'_n, then $G = G_0 \oplus G_1$ is in H'_{n+1}. Let $B'_n = \{G \in H'_n$ and G is bipartite$\}$ and $N'_n = H'_n \setminus B'_n$. In this paper, we show that every graph in B'_n is w^*-laceable for every w, $1 \leq w \leq n$. It is shown that a constructed N'_n-graph H can not be 4^*-connected. In addition, we show that every graph in N'_n is w^*-connected for every w, $1 \leq w \leq 3$.

Keywords: Hamiltonian; Hamiltonian connected; Hamiltonian laceable; Hypercube networks; Hypercube-like networks; w^*-connected; w^*-laceable; Spanning connectivity; Spanning laceability; Graph container

1. Introduction

1.1. Definitions

For graph definitions and notations we follow [4]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *node set* and E is the *edge set*. We use $n(G)$ to denote $|V|$. Two nodes u and v are *adjacent* if (u, v) is an edge of G. For a node u, $N_G(u)$ denotes the *neighbourhood*

* Corresponding address: Information Engineering Department, Ta Hwa Institute of Technology, Hsinchu, Taiwan, ROC.
E-mail addresses: cklin@cs.nctu.edu.tw (C.-K. Lin), jmtan@cs.nctu.edu.tw (J.J.M. Tan), hsu@trill.cis.fordham.edu (D.F. Hsu), lhhsu@pu.edu.tw, lhhsu@cis.nctu.edu.tw (L.-H. Hsu).

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of u which is the set $\{v \mid (u, v) \in E\}$. For any node u of V, we denote the *degree* of u by $\deg_G(u) = |N_G(u)|$. A graph G is *k-regular* if $\deg_G(u) = k$ for every node u in G. A *path* P between nodes v_1 and v_k is a sequence of adjacent nodes, $\langle v_1, v_2, \ldots, v_k\rangle$, in which the nodes $v_1, v_2, \ldots, v_k$ are distinct except that possibly $v_1 = v_k$. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_1\rangle$. The *length* of P, $l(P)$, is the number of edges in P. We also write the path P as $\langle v_1, v_2, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k\rangle$, where Q is the path $\langle v_i, v_{i+1}, \ldots, v_j\rangle$. Hence, it is possible to write a path as $\langle v_1, v_2, Q, v_2, v_3, \ldots, v_k\rangle$ if $l(Q) = 0$. Let $I(P) = V(P) - \{v_1, v_k\}$ be the set of the internal nodes of P. A set of paths $\{P_1, P_2, \ldots, P_k\}$ are *internally node-disjoint* (abbreviated as *disjoint*) if $I(P_i) \cap I(P_j) = \emptyset$ for any $i \neq j$. A path is a *hamiltonian path* if it contains all nodes of G. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two distinct nodes of G [18]. A *cycle* is a path with at least three nodes such that the first node is the same as the last one. A *hamiltonian cycle* of G is a cycle that traverses every node of G. A graph is *hamiltonian* if it has a hamiltonian cycle. A graph G is *bipartite* if its node set can be partitioned into two subsets V_1 and V_2 such that every edge connects nodes between V_1 and V_2. A bipartite graph G is *hamiltonian laceable* if there is a hamiltonian path of G joining any two nodes from distinct bipartition [20]. A bipartite graph G is *k-edge fault hamiltonian laceable* if $G - F$ is hamiltonian laceable for any edge subset F of G with $|F| \le k$.

A graph G is *k-connected* if there exists a set of k internally disjoint paths $\{P_1, P_2, \ldots, P_k\}$ between any two distinct nodes u and v. A subset S of $V(G)$ is a *cut set* if $G - S$ is disconnected. A *w-container* of G between two distinct nodes u and v is a set of w internally disjoint paths between u and v. The concepts of a container and of a wide distance were proposed by Hsu [12] to evaluate the performance of communication for an interconnection network. The *connectivity* of G, $\kappa(G)$, is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. Hence, a graph G is k-connected if $\kappa(G) \ge k$. It follows from Menger's Theorem [17] that there is a w-container for $w \le k$ between any two distinct nodes of G if G is k-connected.

1.2. w^*-connected graphs and w^*-laceable graphs

In this paper, we are interested in a specific type of container. We say that a w-container $C(u, v)$ is a *w^*-container* if every node of G is on some path in $C(u, v)$. A graph G is said to be *w^*-connected* if there exists a w^*-container between any two distinct nodes u and v. Obviously, we have the following remarks:

Remark 1. (1.*a*) a graph G is 1^*-connected if and only if it is hamiltonian connected [18], (1.*b*) a graph G is 2^*-connected if it is hamiltonian, and (1.*c*) an 1^*-connected graph except K_1 and K_2 is 2^*-connected.

Using our definition of a w^*-connected graph, the globally 3^*-connected graphs proposed by Albert et al. [3] are 3-regular 3^*-connected graphs. Assume that the graph G is w^*-connected where $w \le \kappa(G)$. The *spanning connectivity* of a graph G, $\kappa^*(G)$, is the largest integer k such that G is i^*-connected for every i, $1 \le i \le k$. A graph G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$. In such case, the number $\kappa^*(G) = \kappa(G)$ is called the super spanning connectivity of G. In [13,16,15,21], some families of graphs are proved to be super spanning connected.

Let G be a bipartite graph with bipartition V_1 and V_2 such that $|V_1| \ge |V_2|$. Suppose that there exists a w^*-container $C(u, v) = \{P_1, P_2, \ldots, P_w\}$ in G joining u to v with $u, v \in V_1$. Obviously, the number of nodes in P_i is $2t_i + 1$ for some integer t_i. There are $t_i - 1$ nodes of P_i in V_1 other than u and v, and t_i nodes of P_i in V_2. As a consequence, $|V_1| = \sum_{i=1}^{w}(t_i - 1) + 2$ and $|V_2| = \sum_{i=1}^{w} t_i$. Therefore, any bipartite graph G with $\kappa(G) \ge 3$ is not w^*-connected for any w, $3 \le w \le \kappa(G)$.

For this reason, a bipartite graph is said to be *w^*-laceable* if there exists a w^*-container between any two nodes from different partite sets for some w, $1 \le w \le \kappa(G)$. Obviously, any bipartite w^*-laceable graph with $w \ge 2$ has the equal size of bipartition. We have the following remarks:

Remark 2. (2.*a*) an 1^*-laceable graph is also known as hamiltonian laceable graph [20], (2.*b*) a graph G is 2^*-laceable if and only if it is hamiltonian, and (2.*c*) an 1^*-laceable graph except K_1 and K_2 are 2^*-laceable.

The *spanning laceability* of a bipartite graph G, $\kappa^{*L}(G)$, is the largest integer k such that G is i^*-laceable for every i, $1 \le i \le k$. A graph G is *super spanning laceable* if the number $\kappa^{*L}(G) = \kappa(G)$. Recently, Chang et al. [5] proved that the n-dimensional hypercube Q_n is superspanning laceable for every positive integer n. It was proved in [15] that the n-dimensional star graph S_n is superspanning laceable if and only if $n \neq 3$.

1.3. Hypercube-like graphs H'_n

Graph containers do exist in engineering design information and telecommunication networks or in biological and neural systems ([2,12] and its references). The study of w-container, w-wide distance, and their w^*-versions play a pivotal role in the design and the implementation of parallel routing and efficient information transmission in large-scale networking systems. In bioinformatics and neuroinformatics, the existence as well as the structure of a w^*-container signifies the cascade effect in the signal transduction system and the reaction in a metabolic pathway.

Among all interconnection networks proposed in the literature, the hypercube Q_n is one of the most popular topologies [5,14]. However, the hypercube does not have the smallest diameter for its resources. Various networks are proposed by twisting some pairs of links in hypercubes [1,8,10,11]. Because of the lack of the unified perspective on these variants, results of one topology are hard to be extended to others. To make a unified study of these variants, Vaidya et al. introduced the class of hypercube-like graphs [22]. We denote there graphs as H'-graphs. The class of H'-graphs, consisting of simple, connected, and undirected graphs, contains most of the hypercube variants.

Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two disjoint graphs with the same number of nodes. A *1–1 connection* between G_0 and G_1 is defined as an edge set $E = \{(v, \phi(v)) \mid v \in V_0, \phi(v) \in V_1$, and $\phi : V_0 \to V_1$ is a bijection$\}$. We use $G_0 \oplus G_1$ to denote $G = (V_0 \cup V_1, E_0 \cup E_1 \cup E)$. The operation "$\oplus$" may generate different graphs depending on the bijection ϕ. There are some studies on the operation "$\oplus$" [6,7]. Let $G = G_0 \oplus G_1$ and let x be any node in G. We use $\bar{x}$ to denote the unique node matched under ϕ.

Now, we can define the set of n-dimensional H'-graph, H'_n, as follows:

(1) $H'_1 = \{K_2\}$, where K_2 is the complete graph with two nodes.
(2) Assume that $G_0, G_1 \in H'_n$. Then $G = G_0 \oplus G_1$ is a graph in H'_{n+1}.

Note that some n-dimensional H'-graphs are bipartite. We can define the set of bipartite n-dimensional H'-graph, B'_n, as follows:

(1) $B'_1 = \{K_2\}$, where K_2 is the complete graph defined on $\{a, b\}$ with bipartition $V_0 = \{a\}$ and $V_1 = \{b\}$.
(2) For $i = 0, 1$, let G_i be a graph in B'_n with bipartition V_0^i and V_1^i. Let ϕ be a bijection between $V_0^0 \cup V_1^0$ and $V_0^1 \cup V_1^1$ such that $\phi(v) \in V_{1-i}^1$ if $v \in V_i^0$. Then $G = G_0 \oplus G_1$ is a graph in B'_{n+1}.

Every graph in H'_n is an n-regular graph with 2^n nodes, and every graph in B'_n contains 2^{n-1} nodes in each bipartition. We use N'_n to denote the set of non-bipartite graphs in H'_n. Clearly, we have $Q_n \in B'_n$.

Let G be a graph in H'_{n+1}. Then $G = G_0 \oplus G_1$ with both G_0 and G_1 in H'_n. Let u be a node in $V(G)$. Then u is a node in $V(G_i)$ for some $i = 0, 1$. We use $\bar{u}$ to denote the node in $V(G_{1-i})$ matched under ϕ. So $u = \bar{v}$ if $\bar{u} = v$.

In the following section, we give some basic properties about H'_n-graphs. In Section 3, we prove that every graph in B'_n is super spanning laceable. In Section 4, we show that every graph in N'_n is w^*-connected for every w, $1 \le w \le 3$, for $n \ge 3$. We also construct an N'_n-graph H and show that H can not be 4^*-connected. In the final section, we give our concluding remark.

2. Preliminaries

Lemma 1. *Assume that G is graph in N'_n. Then $n \ge 3$.*

Theorem 1 (*[19]*). *Let $n \ge 3$. Every graph in N'_n is hamiltonian connected and hamiltonian.*

Theorem 2 (*[19]*). *Every graph in B'_n is hamiltonian laceable and every graph in B'_n is hamiltonian if $n \ge 2$.*

Theorem 3 (*[19]*). *Let $n \ge 2$. Suppose that G is a graph in B'_n with bipartition V_0 and V_1. Suppose that u_1 and u_2 are two distinct nodes in V_i and that v_1 and v_2 are two distinct nodes in V_{1-i} with $i \in \{0, 1\}$. Then there are two disjoint paths P_1 and P_2 of G such that (1) P_1 joins u_1 to v_1, (2) P_2 joins u_2 to v_2, and (3) $P_1 \cup P_2$ spans G.*

Theorem 4. *Let G be a graph in B'_n with bipartition V_0 and V_1 for $n \ge 2$. Suppose that z is a node in V_i and that u and v are two distinct nodes in V_{1-i} with $i \in \{0, 1\}$. Then there is a hamiltonian path of $G - \{z\}$ joining u to v.*

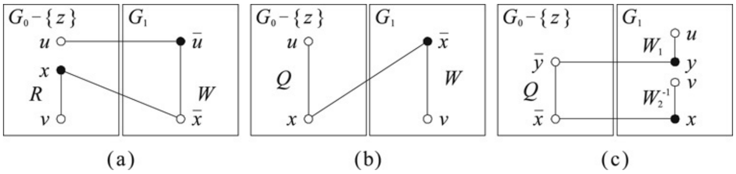

Fig. 1. Illustration for Theorem 4.

Proof. We prove this statement by induction on n. Since Q_2 is the only graph in B'_2, it is easy to check that this statement holds for $n = 2$. Thus, we assume that $G = G_0 \oplus G_1$ in B'_n with $n \geq 3$. We have $G_i \in B'_{n-1}$ for $i = 0, 1$. Let V_0^i and V_1^i be the bipartition of G_i for $i = 0, 1$. Without loss of generality, we assume that $V_0^0 \cup V_0^1$ and $V_1^0 \cup V_1^1$ form the bipartition of G. Let z be any node in $V_1^0 \cup V_1^1$, and let u and v be any two distinct nodes in $V_0^0 \cup V_0^1$. We need to show that there is a hamiltonian path of $G - \{z\}$ joining u to v. Without loss of generality, we assume that $z \in V_1^0$. We have the following cases:

Case 1: $u \in V_0^0$ and $v \in V_0^0$. By induction, there is a hamiltonian path Q in $G_0 - \{z\}$ joining u to v. Without loss of generality, we write Q as $\langle u, x, R, v\rangle$. Since $u \in V_0^0$, $x \in V_1^0$. By Theorem 2, there is a hamiltonian path W of G_1 joining the node $\bar{u} \in V_1^1$ to the node $\bar{x} \in V_0^1$. Then $\langle u, \bar{u}, W, \bar{x}, x, R, v\rangle$ is the hamiltonian path of $G - \{z\}$ joining u to v. See Fig. 1(a) for an illustration.

Case 2: $u \in V_0^0$ and $v \in V_0^1$. Since $n \geq 3$, $|V_0^0| = 2^{n-1} \geq 2$. We can choose a node x in $V_0^0 - \{u\}$. By induction, there is a hamiltonian path Q in $G_0 - \{z\}$ joining u to x. Since $x \in V_0^0$, $\bar{x} \in V_1^1$. By Theorem 2, there is a hamiltonian path W of G_1 joining $\bar{x}$ to v. Then $\langle u, Q, x, \bar{x}, W, v\rangle$ is the hamiltonian path of $G - \{z\}$ joining u to v. See Fig. 1(b) for an illustration.

Case 3: $u \in V_0^1$ and $v \in V_0^1$. We can choose a node x in V_1^1. By Theorem 2, there is a hamiltonian path W in G_1 joining u to x. Without loss of generality, we write W as $\langle u, W_1, y, v, W_2, x\rangle$. Since $v \in V_0^1$, $y \in V_1^1$. By induction, there is a hamiltonian path Q in $G_0 - \{z\}$ joining the node $\bar{y} \in V_0^0$ to the node $\bar{x} \in V_0^0$. Then $\langle u, W_1, y, \bar{y}, Q, \bar{x}, x, W_2^{-1}, v\rangle$ is the hamiltonian path of $G - \{z\}$ joining u to v. See Fig. 1(c) for an illustration. □

3. Every B'_n-graph is super spanning laceable

Let n be any positive integer. To prove that every graph in B'_n is w^*-laceable for every w, $1 \leq w \leq n$, we need the concept of spanning fan. We note that there is another Menger-type Theorem. Let u be a node of G and $S = \{v_1, v_2, \ldots, v_k\}$ be a subset of $V(G)$ not including u. An (u, S)-fan is a set of disjoint paths $\{P_1, P_2, \ldots, P_k\}$ of G such that P_i joins u and v_i [9]. It is proved that a graph G is k-connected if and only if there exists an (u, S)-fan between any node u and any k-subset S of $V(G)$ such that $u \notin S$. With this observation, we define a *spanning fan* is a fan that spans G. Naturally, we can study $\kappa^*_{fan}(G)$ as the largest integer k such that there exists a spanning (u, S)-fan between any node u and any k-node subset S with $u \notin S$. However, we defer such a study for the following reasons:

First, let S be a cut set of a graph G. Let u be any node of $V(G) - S$. It is easy to see that there is no spanning (u, S)-fan in G. Thus, $\kappa^*_{fan}(G) < \kappa(G)$ if G is not a complete graph.

Second, let G be a bipartite graph with bipartition V_0 and V_1 and $|V_0| = |V_1|$. Let u be a vertex in V_i, $S = \{v_1, v_2, \ldots, v_k\}$ be a subset of G not containing u, and $k \leq \kappa(G)$. Suppose that $|S \cap V_{1-i}| = r$. Without loss of generality, we assume that $\{v_1, v_2, \ldots, v_r\} \subset V_{1-i}$. Let $\{P_1, P_2, \ldots, P_k\}$ be any spanning (u, S)-fan of G. Then $l(P_j)$ is odd if $j \leq r$, and $l(P_j)$ is even if $r < j \leq k$. Let $l(P_j) = 2t_j + 1$ if $j \leq r$ and $l(P_j) = 2t_j$ if $j > r$. For $j \leq r$, there are $t_j - 1$ nodes of P_j in V_i other than u and there are t_j nodes of P_j in V_{1-i}. For $j > r$, there are t_j nodes of P_j in V_i other than u and there are t_j nodes of P_j in V_{1-i}. Thus, $|V_i| = 1 - r + \sum_{j=1}^{k} t_j$ and $|V_{1-i}| = \sum_{j=1}^{k} t_j$. Since $|V_i| = |V_{1-i}|$, $r = 1$. Thus, $r = 1$ is a fact requirement as we study the spanning fan of bipartite graphs with equal size of bipartition.

Theorem 5. *Let n and k be any two positive integer with $k \leq n$. Let G be a graph in B'_n with bipartition V_0 and V_1. There exists a spanning (u, S)-fan in G for any node u in V_i and any node subset S with $|S| \leq n$ such that $|S \cap V_{1-i}| = 1$ with $i \in \{0, 1\}$.*

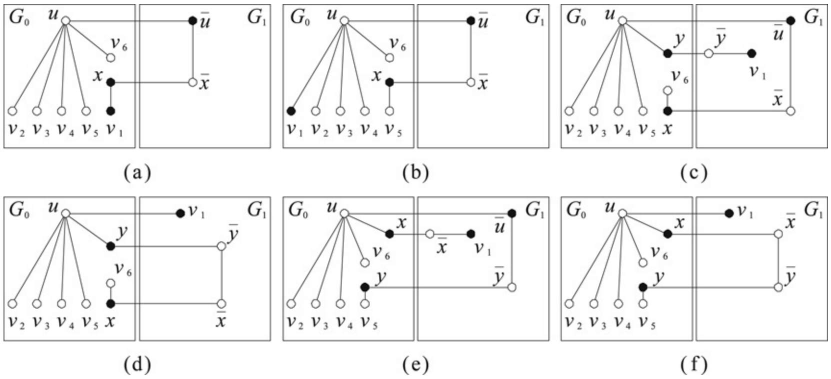

Fig. 2. Illustration for Case 1 of Theorem 5.

Proof. We prove this statement by induction on n. Let $G = G_0 \oplus G_1$ in B'_n such that V_0^i and V_1^i be the bipartition of G_i for every $i = 0, 1$. Without loss of generality, we assume that $V_0^0 \cup V_0^1$ and $V_1^0 \cup V_1^1$ form the bipartition of G. Let u be any node in $V_0^0 \cup V_0^1$ and $S = \{v_1, v_2, \ldots, v_k\}$ be any node subset in $G - \{u\}$ with v_1 being the unique node in $(V_1^0 \cup V_1^1) \cap S$. Without loss of generality, we assume that $u \in V_0^0$. By Theorem 2, this statement holds for $k = 1$. Thus, we assume that $k = 2$ and $n \geq 2$. By Theorem 2, there is a hamiltonian path P of G joining v_1 to v_2. Without loss of generality, we write P as $\langle v_1, P_1, u, P_2, v_2 \rangle$. Then $\{P_1, P_2\}$ forms the spanning (u, S)-fan of G. Thus, this statement holds for $k = 2$. Moreover, this statement holds for $n = 2$. We assume that $3 \leq k \leq n$. Suppose that this statement holds for B'_{n-1}, and $G_i \in B'_{n-1}$ for $i = 0$ and 1. Without loss of generality, we assume that $u \in G_0$. Let $T = S - \{v_1\}$. We have the following cases:

Case 1: $|T \cap V_0^0| = |T|$. Then $v_i \in V_0^0$ for every i, $2 \leq i \leq k$.

Case 1.1: $v_1 \in V_1^0$. Let $H = S - \{v_k\}$. Obviously, $H \subset G_0$, $|H \cap V_1^0| = 1$, and $|H| = k - 1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of G_0. Without loss of generality, we assume that P_i is joining u to v_i for every i, $1 \leq i \leq k - 1$.

Suppose that $v_k \in V(P_1)$. Without loss of generality, we write P_1 as $\langle u, Q_1, v_k, x, Q_2, v_1 \rangle$. Since $v_k \in V_0^0$, $x \in V_1^0$. (Note that $x = v_1$ if $l(Q_2) = 0$.) By Theorem 2, there is a hamiltonian path R of G_1 joining node $\bar{u} \in V_1^1$ to node $\bar{x} \in V_0^1$. We set $W_1 = \langle u, \bar{u}, R, \bar{x}, x, Q_2, v_1 \rangle$, $W_i = P_i$ for every i, $2 \leq i \leq k - 1$, and $W_k = \langle u, Q_1, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 2(a) for an illustration where $k = 6$.

Suppose that $v_k \in V(P_i)$ for some $2 \leq i \leq k - 1$. Without loss of generality, we assume that $v_k \in V(P_{k-1})$ and we write P_{k-1} as $\langle u, Q_1, v_k, x, Q_2, v_{k-1} \rangle$. Since $v_k \in V_0^0$, $x \in V_1^0$. By Theorem 2, there is a hamiltonian path R of G_1 joining node $\bar{u} \in V_1^1$ to node $\bar{x} \in V_0^1$. We set $W_i = P_i$ for every $i \in \langle k - 2 \rangle$, $W_{k-1} = \langle u, \bar{u}, R, \bar{x}, x, Q_2, v_{k-1} \rangle$, and $W_k = \langle u, Q_1, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 2(b) for an illustration where $k = 6$.

Case 1.2: $v_1 \in V_1^1$. We choose a node x in V_1^0. Let $H = (T \cup \{x\}) - \{v_k\}$. So $H \subset G_0$, $|H \cap V_1^0| = 1$, and $|H| = k - 1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of G_0. Without loss of generality, we assume that P_1 is joining u to x and P_i is joining u to v_i for every $2 \leq i \leq k - 1$. We have $\bar{u} \in V_1^1$ and $\bar{x} \in V_0^1$.

Case 1.2.1: $v_k \in V(P_1)$. Without loss of generality, we write P_1 as $\langle u, Q_1, y, v_k, Q_2, x \rangle$. Since $v_k \in V_0^0$, $y \in V_1^0$ and $\bar{y} \in V_0^1$.

Suppose that $v_1 \neq \bar{u}$. By Theorem 3, there are two disjoint paths R_1 and R_2 in G_1 such that (1) R_1 joins $\bar{y}$ to v_1, (2) R_2 joins $\bar{u}$ to $\bar{x}$, and (3) $R_1 \cup R_2$ spans G_1. We set $W_1 = \langle u, Q_1, y, \bar{y}, R_1, v_1 \rangle$, $W_i = P_i$ for every $2 \leq i \leq k - 1$, and $W_k = \langle u, \bar{u}, R_2, \bar{x}, x, Q_2^{-1}, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 2(c) for an illustration where $k = 6$.

Suppose that $v_1 = \bar{u}$. By Theorem 4, there is a hamiltonian path R of $G_1 - \{v_1\}$ joining $\bar{y}$ to $\bar{x}$. We set $W_1 = \langle u, \bar{u} = v_1 \rangle$, $W_i = P_i$ for every $2 \leq i \leq k - 1$, and $W_k = \langle u, Q_1, y, \bar{y}, R, \bar{x}, x, Q_2^{-1}, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 2(d) for an illustration where $k = 6$.

Case 1.2.2: $v_k \in V(P_i)$ for some $2 \leq i \leq k - 1$. Without loss of generality, we assume that $v_k \in V(P_{k-1})$ and we write P_{k-1} as $\langle u, Q_1, v_k, y, Q_2, v_{k-1} \rangle$. Since $v_k \in V_0^0$, $y \in V_1^0$ and $\bar{y} \in V_0^1$.

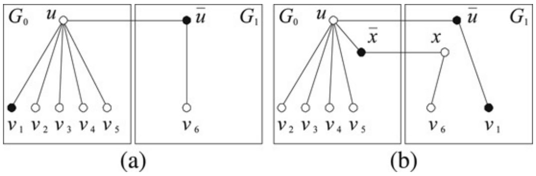

Fig. 3. Illustration for Case 2 of Theorem 5.

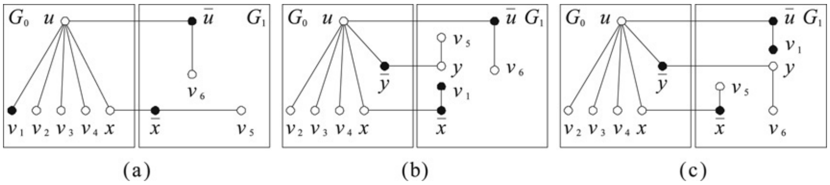

Fig. 4. Illustration for Case 3 of Theorem 5.

Suppose that $v_1 \neq \bar{u}$. By Theorem 3, there are two disjoint paths R_1 and R_2 in G_1 such that (1) R_1 joins $\bar{x}$ to v_1, (2) R_2 joins $\bar{u}$ to $\bar{y}$, and (3) $R_1 \cup R_2$ spans G_1. We set $W_1 = \langle u, P_1, x, \bar{x}, R_1, v_1 \rangle$, $W_i = P_i$ for every $2 \le i \le k-2$, $W_{k-1} = \langle u, \bar{u}, R_2, \bar{y}, y, Q_2, v_{k-1} \rangle$, and $W_k = \langle u, Q_1, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 2(e) for an illustration where $k = 6$.

Suppose that $v_1 = \bar{u}$. By Theorem 4, there is a hamiltonian path R of $G_1 - \{v_1\}$ joining $\bar{x}$ to $\bar{y}$. We set $W_1 = \langle u, \bar{u} = v_1 \rangle$, $W_i = P_i$ for every $2 \le i \le k-2$, $W_{k-1} = \langle u, P_1, x, \bar{x}, R, \bar{y}, y, Q_2, v_{k-1} \rangle$, and $W_k = \langle u, Q_1, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 2(f) for an illustration where $k = 6$.

Case 2: $|T \cap V_0^1| = 1$. Without loss of generality, we assume that $v_k \in V_0^1$. We have $\bar{u} \in V_1^1$.

Case 2.1: $v_1 \in V_1^0$. Let $H = S - \{v_k\}$. Obviously, $H \subset G_0$, $|H \cap V_1^0| = 1$, and $|H| = k-1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of G_0. Without loss of generality, we assume that P_i is joining u to v_i for every $1 \le i \le k-1$. By Theorem 2, there is a hamiltonian path R of G_1 joining $\bar{u}$ to v_k. We set $P_k = \langle u, \bar{u}, R, v_k \rangle$. Then $\{P_1, P_2, \ldots, P_k\}$ is the spanning (u, S)-fan of G. See Fig. 3(a) for an illustration where $k = 6$.

Case 2.2: $v_1 \in V_1^1$. By Theorem 2, there is a hamiltonian path R of G_1 joining v_1 to v_k. Without loss of generality, we write R as $\langle v_1, R_1, \bar{u}, x, R_2, v_k \rangle$. (Note that $v_1 = \bar{u}$ if $l(R_1) = 0$ and $x = v_k$ if $l(R_2) = 0$.) Since $\bar{u} \in V_1^1$, $x \in V_0^1$ and $\bar{x} \in V_1^0$. Let $H = (T \cup \{\bar{x}\}) - \{v_k\}$. Obviously, $H \subset G_0$, $|H \cap V_1^0| = 1$, and $|H| = k-1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of G_0. Without loss of generality, we assume that P_1 is joining u to $\bar{x}$ and P_i is joining u to v_i for every $2 \le i \le k-1$. We set $W_1 = \langle u, \bar{u}, R_1^{-1}, v_1 \rangle$, $W_i = P_i$ for every $2 \le i \le k-1$, and $W_k = \langle u, P_1, \bar{x}, x, R_2, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the (u, S)-fan of G. See Fig. 3(b) for an illustration where $k = 6$. **Case 3:** $|T \cap V_0^1| = 2$. Without loss of generality, we assume that $\{v_{k-1}, v_k\} \subset V_0^1$. We have $|V_0^0| \ge n \ge k$. We can choose a node x in $V_0^0 - \{u, v_2, v_3, \ldots, v_{k-2}\}$. Obviously, $\{\bar{x}, \bar{u}\} \subset V_1^1$ with $\bar{x} \neq \bar{u}$. By Theorem 3, there are two disjoint paths R_1 and R_2 in G_1 such that (1) R_1 joins $\bar{x}$ to v_{k-1}, (2) R_2 joins $\bar{u}$ to v_k, and (3) $R_1 \cup R_2$ spans G_1.

Case 3.1: $v_1 \in V_1^0$. Let $H = (S \cup \{x\}) - \{v_{k-1}, v_k\}$. Obviously, $H \subset G_0$, $|H \cap V_1^0| = 1$, and $|H| = k-1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of G_0. Without loss of generality, we assume that P_i is joining u to v_i for every $1 \le i \le k-2$ and P_{k-1} is joining u to x. We set $W_i = P_i$ for every $1 \le i \le k-2$, $W_{k-1} = \langle u, P_{k-1}, x, \bar{x}, R_1, v_{k-1} \rangle$, and $W_k = \langle u, \bar{u}, R_2, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 4(a) for an illustration where $k = 6$.

Case 3.2: $v_1 \in V_1^1$ and $v_1 \in V(R_1)$. Without loss of generality, we write R_1 as $\langle \bar{x}, Q_1, v_1, y, Q_2, v_{k-1} \rangle$. Since $v_1 \in V_1^1$, $y \in V_0^1$ and $\bar{y} \in V_1^0$. Let $H = (T \cup \{x, \bar{y}\}) - \{v_{k-1}, v_k\}$. Obviously, $H \subset G_0$, $|H \cap V_1^0| = 1$, and $|H| = k-1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of G_0. Without loss of generality, we assume that P_1 is joining u to x, P_i is joining u to v_i for every $i \in \langle k-2 \rangle$, and P_{k-1} is joining u to $\bar{y}$. We set $W_1 = \langle u, P_1, x, \bar{x}, Q_1, v_1 \rangle$, $W_i = P_i$ for every $2 \le i \le k-2$, $W_{k-1} = \langle u, P_{k-1}, \bar{y}, y, Q_2, v_{k-1} \rangle$, and $W_k = \langle u, \bar{u}, R_2, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 4(b) for an illustration where $k = 6$.

Case 3.3: $v_1 \in V_1^1$ and $v_1 \in V(R_2)$. Without loss of generality, we write R_2 as $\langle \bar{u}, Q_1, v_1, y, Q_2, v_k \rangle$. Since $v_1 \in V_1^1$, $y \in V_0^1$ and $\bar{y} \in V_1^0$. Let $H = (T \cup \{x, \bar{y}\}) - \{v_{k-1}, v_k\}$. Obviously, $H \subset G_0$, $|H \cap V_1^0| = 1$, and $|H| = k-1$. By induction, there is a spanning (u, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of G_0. Without loss of generality, we assume that P_1

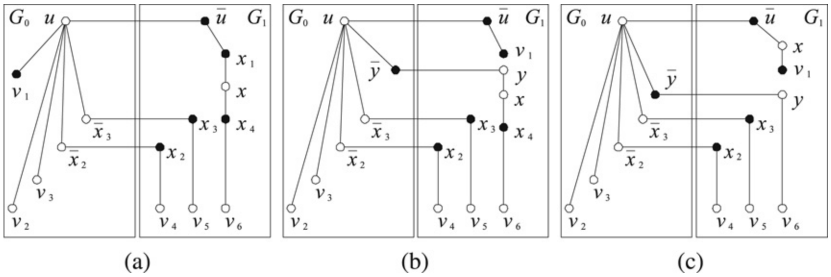

Fig. 5. Illustration for Case 4 of Theorem 5.

is joining u to x, P_i is joining u to v_i for every $2 \le i \le k-2$, and P_{k-1} is joining u to $\bar{y}$. We set $W_1 = \langle u, \bar{u}, Q_1, v_1\rangle$, $W_i = P_i$ for every $2 \le i \le k-2$, $W_{k-1} = \langle u, P_1, x, \bar{x}, R_1, v_{k-1}\rangle$, and $W_k = \langle u, P_{k-1}, \bar{y}, y, Q_2, v_k\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 4(c) for an illustration where $k = 6$.

Case 4: $|T \cap V_0^1| \ge 3$ and $|T \cap V_0^0| \ge 1$. We have $n \ge k = |S| \ge 5$. Without loss of generality, we assume that $A = T \cap V_0^0 = \{v_2, v_3, \ldots, v_t\}$ and $B = T \cap V_0^1 = \{v_{t+1}, v_{t+2}, \ldots, v_k\}$ for some $2 \le t \le k-3$. Since $t \le k-3$ and $k \le n$, $|A| = t-1 \le n-4$ and $|B| \le n-2$. Since $n \ge 5$, $(n-1)|A|+|B| \le (n-1)(n-4)+(n-2) < 2^{n-2} = |V_0^1|$. Thus, we can choose a node x in $V_0^1 - B$ such that $\bar{v}_i \notin N_{G_1}(x)$ for every $2 \le i \le t$. Since $2 \le t \le k-3$ and $k \le n$, $k-t+1 \le n-1$. Let $H = B \cup \{\bar{u}\}$. Obviously, $H \subset G_1$, $|H \cap V_1^1| = 1$, and $|H| = k-t+1$. By induction, there is a spanning (x, H)-fan $\{P_1, P_2, \ldots, P_{k-t+1}\}$ of G_1. Without loss of generality, we assume that P_1 is joining x to $\bar{u}$ and P_i is joining x to v_{t+i-1} for every $2 \le i \le k-t+1$. Moreover, we write $P_1 = \langle x, x_1, R_1, \bar{u}\rangle$ and $P_i = \langle x, x_i, R_i, v_{t+i-1}\rangle$ for every $2 \le i \le k-t+1$. Since $x \in V_0^1$, $x_i \in V_1^1$ and $\bar{x}_i \in V_0^0$ for every $1 \le i \le k-t+1$. We set $C = \{\bar{x}_2, \bar{x}_3, \ldots, \bar{x}_{k-t}\}$.

Case 4.1: $v_1 \in V_1^0$. Let $H' = A \cup C \cup \{v_1\}$. Obviously, $H' \subset G_0$, $|H' \cap V_1^0| = 1$, and $|H'| = k-1$. By induction, there is a spanning (u, H')-fan $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ of G_0. Without loss of generality, we assume that Q_i is joining u to v_i for every $1 \le i \le t$ and Q_j is joining u to $\bar{x}_{j-t+2}$ for every $t+1 \le j \le k-1$. We set $W_i = \langle u, Q_i, v_i\rangle$ for every $1 \le i \le t$, $W_j = \langle u, Q_j, \bar{x}_{i-t+2}, x_{i-t+2}, R_{i-t+2}, v_j\rangle$ for every $t+1 \le j \le k-1$, and $W_k = \langle u, \bar{u}, P_1^{-1}, x, P_{k-t+1}, v_k\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 5(a) for an illustration where $k = 6$ and $t = 3$.

Case 4.2: $v_1 \in V_1^1$ and $v_1 \in V(P_1)$. Without loss of generality, we write P_1 as $\langle x, Z_1, y, v_1, Z_2, \bar{u}\rangle$. Since $v_1 \in V_1^1$, $y \in V_0^1$ and $\bar{y} \in V_1^0$. Let $H' = A \cup C \cap \{\bar{y}\}$. Obviously, $H' \subset G_0$, $|H' \cap V_1^0| = 1$, and $|H'| = k-1$. By induction, there is a spanning (u, H')-fan $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ of G_0. Without loss of generality, we assume that Q_1 is joining u to $\bar{y}$, Q_i is joining u to v_i for every $2 \le i \le t$, and Q_j is joining u to $\bar{x}_{j-t+2}$ for every $t+1 \le j \le k-1$. We set $W_1 = \langle u, \bar{u}, Z_2^{-1}, v_1\rangle$, $W_i = \langle u, Q_i, v_i\rangle$ for every $2 \le i \le t$, $W_j = \langle u, Q_j, \bar{x}_{i-t+2}, x_{i-t+2}, R_{i-t+2}, v_j\rangle$ for every $t+1 \le j \le k-1$, and $W_k = \langle u, Q_1, \bar{y}, y, Z_1^{-1}, x, P_{k-t+1}, v_k\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 5(b) for an illustration where $k = 6$ and $t = 3$.

Case 4.3: $v_1 \in V_1^1$ and $v_1 \in V(P_i)$ for some $2 \le i \le k-t+1$. Without loss of generality, we assume that $v_1 \in V(P_{k-t+1})$ and we write P_{k-t+1} as $\langle x, Z_1, v_1, y, Z_2, v_k\rangle$. Since $v_1 \in V_1^1$, $y \in V_0^1$ and $\bar{y} \in V_1^0$. Let $H' = A \cup C \cup \{\bar{y}\}$. Obviously, $H' \subset G_0$, $|H' \cap V_1^0| = 1$, and $|H'| = k-1$. By induction, there is a spanning (u, H')-fan $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ of G_0. Without loss of generality, we assume that Q_1 is joining u to $\bar{y}$, Q_i is joining u to v_i for every $2 \le i \le t$, and Q_j is joining u to $\bar{x}_{j-t+2}$ for every $t+1 \le j \le k-1$. We set $W_1 = \langle u, \bar{u}, P_1^{-1}, x, Z_1, v_1\rangle$, $W_i = \langle u, Q_i, v_i\rangle$ for every $2 \le i \le t$, $W_j = \langle u, Q_j, \bar{x}_{i-t+2}, x_{i-t+2}, R_{i-t+2}, v_j\rangle$ for every $t+1 \le j \le k-1$, and $W_k = \langle u, Q_1, \bar{y}, y, Z_2, v_k\rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ forms the spanning (u, S)-fan of G. See Fig. 5(c) for an illustration where $k = 6$ and $t = 3$.

Case 5: $|T \cap V_0^1| = |T| \ge 3$. Let $H = (T \cup \{\bar{u}\}) - \{v_k\}$. Obviously, $H \subset G_1$, $|H \cap V_1^1| = 1$, and $|H| = k-1$. By induction, there is a spanning (v_k, H)-fan $\{P_1, P_2, \ldots, P_{k-1}\}$ of G_1. Without loss of generality, we assume that P_1 is joining v_k to $\bar{u}$ and P_i is joining v_k to v_i for every $2 \le i \le k-1$. Without loss of generality, we write $P_1 = \langle v_k, x_1, R_1, \bar{u}\rangle$ and write $P_i = \langle v_k, x_i, R_i, v_i\rangle$ for every $2 \le i \le k-1$. Since $v_k \in V_0^1$, $x_i \in V_1^1$ and $\bar{x}_i \in V_0^0$ for every $1 \le i \le k-1$. We set $C = \{\bar{x}_2, \bar{x}_3, \ldots, \bar{x}_{k-1}\}$.

Case 5.1: $v_1 \in V_1^0$. Let $H' = C \cup \{v_1\}$. Obviously, $H' \subset G_0$, $|H' \cap V_1^0| = 1$, and $|H'| = k-1$. By induction, there is a spanning (u, H')-fan $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ of G_0. Without loss of generality, we assume that Q_1 is joining

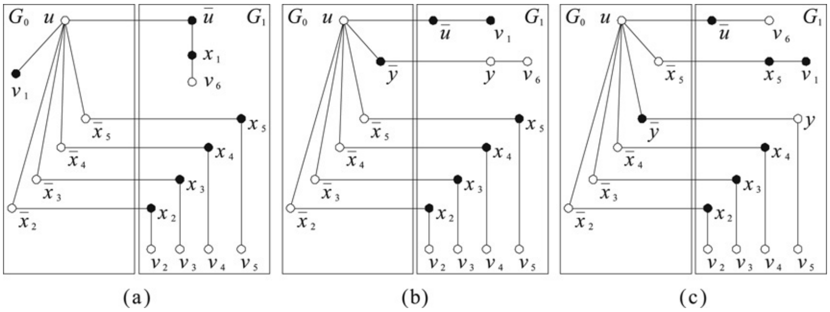

Fig. 6. Illustration for Case 5 of Theorem 5.

u to v_1 and Q_i is joining u to $\bar{x}_i$ for every $2 \le i \le k-1$. We set $W_1 = Q_1$, $W_i = \langle u, Q_i, \bar{x}_i, x_i, R_i, v_i \rangle$ for every $2 \le i \le k-1$, and $W_k = \langle u, \bar{u}, P_1^{-1}, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 6(a) for an illustration where $k = 6$.

Case 5.2: $v_1 \in V_1^1$ and $v_1 \in V(P_1)$. Without loss of generality, we write $P_1 = \langle v_k, Z_1, y, v_1, Z_2, \bar{u} \rangle$. Since $v_1 \in V_1^1$, $y \in V_0^1$ and $\bar{y} \in V_1^0$. Let $H' = C \cup \{\bar{y}\}$. Obviously, $H' \subset G_0$, $|H' \cap V_0^1| = 1$, and $|H'| = k-1$. By induction, there is a spanning (u, H')-fan $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ of G_0. Without loss of generality, we assume that Q_1 is joining u to $\bar{y}$ and Q_i is joining u to $\bar{x}_i$ for every $2 \le i \le k-1$. We set $W_1 = \langle u, \bar{u}, Z_2^{-1}, v_1 \rangle$, $W_i = \langle u, Q_i, \bar{x}_i, x_i, R_i, v_i \rangle$ for every $2 \le i \le k-1$, and $W_k = \langle u, Q_1, \bar{y}, y, Z_1^{-1}, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 6(b) for an illustration where $k = 6$.

Case 5.3: $v_1 \in V_1^1$ and $v_1 \in V(P_i)$ for some $2 \le i \le k-1$. Without loss of generality, we assume that $v_1 \in V(P_{k-1})$ and write $P_{k-1} = \langle v_k, x_{k-1}, Z_1, v_1, y, Z_2, v_{k-1} \rangle$. Since $v_1 \in V_1^1$, $y \in V_0^1$ and $\bar{y} \in V_1^0$. Let $H' = C \cup \{\bar{y}\}$. Obviously, $H' \subset G_0$, $|H' \cap V_1^0| = 1$, and $|H'| = k-1$. By induction, there is a (u, H')-fan $\{Q_1, Q_2, \ldots, Q_{k-1}\}$ of G_0. Without loss of generality, we assume that Q_1 is joining u to $\bar{y}$ and Q_i is joining u to $\bar{x}_i$ for every $2 \le i \le k-1$. We set $W_1 = \langle u, Q_{k-1}, \bar{x}_{k-1}, x_{k-1}, Z_1, v_1 \rangle$, $W_i = \langle u, Q_i, \bar{x}_i, x_i, R_i, v_i \rangle$ for every $2 \le i \le k-2$, $W_{k-1} = \langle u, Q_1, \bar{y}, y, Z_2, v_{k-1} \rangle$, and $W_k = \langle u, \bar{u}, P_1^{-1}, v_k \rangle$. Then $\{W_1, W_2, \ldots, W_k\}$ is the spanning (u, S)-fan of G. See Fig. 6(c) for an illustration where $k = 6$. □

Theorem 6. *Every graph in B_n' is super spanning laceable for $n \ge 1$.*

Proof. Suppose that $G = G_0 \oplus G_1$ in B_n' with bipartition V_0 and V_1. Let u be any node in V_0 and v be any node in V_1. We need to show there is a k^*-container of G between u and v for every positive integer k with $k \le n$. By Theorem 2, there is a 1^*-container of G joining u to v. Thus, we assume that $k \ge 2$ and $n \ge 2$. Since $k \le n$ and $|N_G(v)| = n$, we can choose $(k-1)$ distinct nodes $x_1, x_2, \ldots, x_{k-1}$ in $N_G(v) - \{u\}$. Since v is in V_1, x_i is in $V_0 - \{u\}$ for $i = 1$ to $k-1$. We set $S = \{v, x_1, x_2, \ldots, x_{k-1}\}$. By Theorem 5, there is a spanning (u, S)-fan $\{R_1, R_2, \ldots, R_k\}$ of G. Without loss of generality, we assume that R_1 is joining u to v and R_i is joining u to x_{i-1} for every $2 \le i \le k$. We set $P_1 = R_1$ and $P_i = \langle u, R_i, x_{i-1}, v \rangle$ for every $2 \le i \le k$. Then $\{P_1, P_2, \ldots, P_k\}$ is the k^*-container of G between u and v. □

4. On the w^*-connectedness of N_n'-graphs

4.1. Every graph in N_n' is 3^*-connected

Lemma 2. *According to isomorphism, there is only one graph in N_3'. Moreover, this graph is 3^*-connected.*

Proof. By brute force, we can check the graph T in Fig. 7 is the only graph in N_3'.

Let x and y be two distinct nodes of T. By the symmetry of T, we can assume that $x = 0$ and $y \in \{1, 2, 3, 4\}$. The 3^*-containers $\{P_1, P_2, P_3\}$ of T between x and y are listed below:

$y = 1$	$\{P_1 = \langle 0, 1 \rangle, P_2 = \langle 0, 4, 3, 2, 1 \rangle, P_3 = \langle 0, 7, 6, 5, 1 \rangle\}$
$y = 2$	$\{P_1 = \langle 0, 1, 2 \rangle, P_2 = \langle 0, 7, 3, 2 \rangle, P_3 = \langle 0, 4, 5, 6, 2 \rangle\}$
$y = 3$	$\{P_1 = \langle 0, 4, 3 \rangle, P_2 = \langle 0, 7, 3 \rangle, P_3 = \langle 0, 1, 5, 6, 2, 3 \rangle\}$
$y = 4$	$\{P_1 = \langle 0, 4 \rangle, P_2 = \langle 0, 1, 2, 3, 4 \rangle, P_3 = \langle 0, 7, 6, 5, 4 \rangle\}$

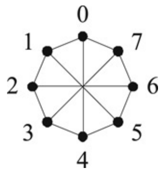

Fig. 7. The only graph T in N'_3.

Thus, T is 3^*-connected. □

Let $n \geq 3$. Let $G = G_0 \oplus G_1 \in N'_{n+1}$ with $G_0 \in H'_n$ and $G_1 \in H'_n$. Depending on G_0 and G_1 is bipartite or not, we prove that $G = G_0 \oplus G_1$ is 3^*-connected with the following lemmas.

Lemma 3. *Let $n \geq 3$. Assume that $G = G_0 \oplus G_1$ in N'_{n+1} with both G_0 and G_1 in N'_n. Then G is 3^*-connected.*

Proof. Let u and v be any two distinct nodes of G. We need to construct a 3^*-container of G between u and v.

Case 1: $u, v \in G_0$. By Theorem 1, there is a 2^*-container $\{P_1, P_2\}$ of G_0 between u and v. By Theorem 1 again, there is a hamiltonian path P of G_1 joining $\bar{u}$ to $\bar{v}$. We set P_3 as $\langle u, \bar{u}, P, \bar{v}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 2: $u \in G_0$ and $v \in G_1$ with $\bar{u} = v$. Since there are 2^n nodes in G_0 and $2^n > 3$ for $n \geq 3$, we can choose two distinct nodes x and y in $G_0 - \{u\}$. By Theorem 1, there is a hamiltonian path R of G_0 joining x to y. Again, there is a hamiltonian path W of G_1 joining $\bar{x}$ to $\bar{y}$. We write $R = \langle x, R_1, u, R_2, y\rangle$ and $W = \langle \bar{x}, W_1, v, W_2, \bar{y}\rangle$. We set $P_1 = \langle u, R_1^{-1}, x, \bar{x}, W_1, v\rangle$, $P_2 = \langle u, R_2, y, \bar{y}, W_2^{-1}, v\rangle$, and $P_3 = \langle u, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 3: $u \in G_0$ and $v \in G_1$ with $\bar{u} \neq v$. Since there are 2^n nodes in G_0, we choose a node x in $G_0 - \{u, \bar{v}\}$. By Theorem 1, there is a hamiltonian path R of G_0 joining x to $\bar{v}$. Again, there is a hamiltonian path W of G_1 joining $\bar{x}$ to $\bar{u}$. We write $R = \langle x, R_1, u, R_2, \bar{v}\rangle$ and $W = \langle \bar{x}, W_1, v, W_2, \bar{u}\rangle$. We set $P_1 = \langle u, \bar{u}, W_2^{-1}, v\rangle$, $P_2 = \langle u, R_1^{-1}, x, \bar{x}, W_1, v\rangle$, and $P_3 = \langle u, R_2, \bar{v}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v. □

Lemma 4. *Let $n \geq 3$. Assume that $G = G_0 \oplus G_1$ in N'_{n+1} with G_0 in B'_n and G_1 in N'_n. Then G is 3^*-connected.*

Proof. Let V_0 and V_1 be the bipartition of G_0. Let u and v be any two distinct nodes of G. We need to construct a 3^*-container of G between u and v.

Case 1: $u, v \in G_0$. By Theorem 2, there is a 2^*-container $\{P_1, P_2\}$ of G_0 between u and v. By Theorem 1, there is a hamiltonian path P of G_1 joining $\bar{u}$ to $\bar{v}$. We set $P_3 = \langle u, \bar{u}, P, \bar{v}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 2: $u, v \in G_1$. Without loss of generality, we assume that $\bar{u} \in V_0$.

Case 2.1: $\bar{v} \in V_0$. Since there are 2^{n-1} nodes in V_1 and $2^{n-1} \geq 4$ for $n \geq 3$, we can choose two distinct nodes x and y in V_1. By Theorem 1, there is a hamiltonian path R of G_1 joining $\bar{x}$ to $\bar{y}$. Without loss of generality, we write $R = \langle \bar{x}, R_1, u, R_2, v, R_3, \bar{y}\rangle$. By Theorem 3, there are two disjoint paths T_1 and T_2 of G_0 such that (1) T_1 joins $\bar{u}$ to y, (2) T_2 joins x to $\bar{v}$, and (3) $T_1 \cup T_2$ spans G_1. We set $P_1 = \langle u, R_2, v\rangle$, $P_2 = \langle u, R_1^{-1}, \bar{x}, x, T_2, \bar{v}, v\rangle$, and $P_3 = \langle u, \bar{u}, T_1, y, \bar{y}, R_3^{-1}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 2.2: $\bar{v} \in V_1$. By Theorem 1, there is a 2^*-container $\{P_1, P_2\}$ of G_1 between u and v. By Theorem 2, there is a hamiltonian path P of G_0 joining $\bar{u}$ to $\bar{v}$. We set $P_3 = \langle u, \bar{u}, P, \bar{v}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 3: $u \in G_0$ and $v \in G_1$ with $\bar{u} \neq v$. By Theorem 2, there is a hamiltonian cycle C of G_0. Without loss of generality, we write $C = \langle u, R_1, \bar{v}, x, R_2, u\rangle$. By Theorem 1, there is a hamiltonian path T of G_1 joining $\bar{u}$ to $\bar{x}$. Without loss of generality, we write $T = \langle \bar{u}, T_1, v, T_2, \bar{x}\rangle$. We set $P_1 = \langle u, R_1, \bar{v}, v\rangle$, $P_2 = \langle u, \bar{u}, T_1, v\rangle$, and $P_3 = \langle u, R_2^{-1}, x, \bar{x}, T_2^{-1}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 4: $u \in G_0$ and $v \in G_1$ with $\bar{u} = v$. Without loss of generality, we assume that $u \in V_0$. We can choose a node x in $V_0 - \{u\}$ and a node y in V_1. By Theorem 2, there is a hamiltonian path R of G_0 joining x to y. By Theorem 1, there is a hamiltonian path T of G_1 joining $\bar{x}$ to $\bar{y}$. Without loss of generality, we write $R = \langle x, R_1, u, R_2, y\rangle$ and $T = \langle \bar{x}, T_1, v, T_2, \bar{y}\rangle$. We set $P_1 = \langle u, v\rangle$, $P_2 = \langle u, R_1^{-1}, x, \bar{x}, T_1, v\rangle$, and $P_3 = \langle u, R_2, y, \bar{y}, T_2^{-1}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v. □

Lemma 5. *Assume that $G = G_0 \oplus G_1$ in N'_{n+1} with both G_0 and G_1 in B'_n for $n \geq 2$. Then G is 3^*-connected.*

Proof. Let V_0^i and V_1^i be the bipartition of G_i for $i = 0, 1$. Let u and v be two distinct nodes of G. Without loss of generality, we assume that $u \in V_0^0$ and $\bar{u} \in V_1^1$. We need to construct a 3^*-container of G between u and v.

Case 1: $v \in V_0^0 \cup V_1^0$ and $\bar{v} \in V_0^1$. By Theorem 2, there is a 2^*-container $\{P_1, P_2\}$ of G_0 between u and v. By Theorem 2, there is a hamiltonian path P of G_1 joining $\bar{u}$ to $\bar{v}$. We set $P_3 = \langle u, \bar{u}, P, \bar{v}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 2: $v \in V_0^0$ and $\bar{v} \in V_1^1$. Since $u \in V_0^0$, $\bar{u} \in V_1^1$, $v \in V_0^0$, and $\bar{v} \in V_1^1$, we can choose a node x in V_1^0 such that $\bar{x} \in V_0^1$ and choose a node y in V_0^0 such that $\bar{y} \in V_0^1$. By Theorem 2, there is a hamiltonian path R of G_0 joining x to y. Without loss of generality, we write $R = \langle x, R_1, p, R_2, q, R_3, y\rangle$ where $\{p, q\} = \{u, v\}$. Without loss of generality, we assume that $p = u$ and $q = v$. By Theorem 3, there are two disjoint paths T_1 and T_2 of G_1 such that (1) T_1 joins $\bar{x}$ to $\bar{v}$, (2) T_2 joins $\bar{u}$ to $\bar{y}$, and (3) $T_1 \cup T_2$ spans G_1. We set $P_1 = \langle u, R_2, v\rangle$, $P_2 = \langle u, R_1^{-1}, x, \bar{x}, T_1, \bar{v}, v\rangle$, $P_3 = \langle u, \bar{u}, T_2, \bar{y}, y, R_3^{-1}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 3: $v \in V_1^0$ and $\bar{v} \in V_1^1$. Since $u \in V_0^0$ and $\bar{u} \in V_1^1$, $v \in V_1^0$, and $\bar{v} \in V_1^1$, we can choose a node x in V_1^0 such that $\bar{x} \in V_0^1$ and choose a node y in V_0^0 such that $\bar{y} \in V_0^1$. By Theorem 2, there is a hamiltonian path R of G_0 joining x to y. Without loss of generality, we write $R = \langle x, R_1, p, R_2, q, R_3, y\rangle$ where $\{p, q\} = \{u, v\}$. Without loss of generality, we assume that $p = u$ and $q = v$. By Theorem 3, there are two disjoint paths T_1 and T_2 of G_1 such that (1) T_1 joins $\bar{x}$ to $\bar{v}$, (2) T_2 joins $\bar{u}$ to $\bar{y}$, and (3) $T_1 \cup T_2$ spans G_1. We set $P_1 = \langle u, R_2, v\rangle$, $P_2 = \langle u, R_1^{-1}, x, \bar{x}, T_1, \bar{v}, v\rangle$, $P_3 = \langle u, \bar{u}, T_2, \bar{y}, y, R_3^{-1}, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 4: $v \in V_0^1 \cup V_1^1$ and $\bar{u} \neq v$.

Case 4.1: $\bar{v} \in V_0^0$. Since $u \in V_0^0$, $\bar{u} \in V_1^1$, and $\bar{v} \in V_0^0$, we can choose a node $x \in V_1^0$ such that $\bar{x} \in V_0^1$. By Theorem 2, there is a hamiltonian path R of G_0 joining x to $\bar{v}$. Again, by Theorem 2, there is a hamiltonian path T of G_1 joining $\bar{x}$ to $\bar{u}$. Write $R = \langle x, R_1, u, R_2, \bar{v}\rangle$ and $T = \langle \bar{x}, T_1, v, T_2, \bar{u}\rangle$. We set $P_1 = \langle u, \bar{u}, T_2^{-1}, v\rangle$, $P_2 = \langle u, R_2, \bar{v}, v\rangle$, and $P_3 = \langle u, R_1^{-1}, x, \bar{x}, T_1, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 4.2: $\bar{v} \in V_1^0$ and $v \in V_0^1$. Since $u \in V_0^0$, $\bar{u} \in V_1^1$, $v \in V_0^1$, and $\bar{v} \in V_1^0$, we can choose a node $x \in V_0^0$ such that $\bar{x} \in V_0^1$. By Theorem 2, there is a hamiltonian path R of G_0 joining x to $\bar{v}$, and there is a hamiltonian path T of G_1 joining $\bar{x}$ to $\bar{u}$. We write $R = \langle x, R_1, u, R_2, \bar{v}\rangle$ and $T = \langle \bar{x}, T_1, v, T_2, \bar{u}\rangle$. We set $P_1 = \langle u, \bar{u}, T_2^{-1}, v\rangle$, $P_2 = \langle u, R_2, \bar{v}, v\rangle$, and $P_3 = \langle u, R_1^{-1}, x, \bar{x}, T_1, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 4.3: $\bar{v} \in V_1^0$ and $v \in V_1^1$. Since $u \in V_0^0$, $\bar{u} \in V_1^1$, and $v \in V_1^1$, we can choose a node $x \in V_0^0$ such that $\bar{x} \in V_0^1$. By Theorem 2, there is a hamiltonian path R of G_0 joining x to $\bar{v}$, and there is a hamiltonian path T of G_1 joining $\bar{x}$ to $\bar{u}$. We write $R = \langle x, R_1, u, R_2, \bar{v}\rangle$ and $T = \langle \bar{x}, T_1, v, T_2, \bar{u}\rangle$. We set $P_1 = \langle u, \bar{u}, T_2^{-1}, v\rangle$, $P_2 = \langle u, R_2, \bar{v}, v\rangle$, and $P_3 = \langle u, R_1^{-1}, x, \bar{x}, T_1, v\rangle$. Then $\{P_1, P_2, P_3\}$ is the 3^*-container of G between u and v.

Case 5: $v = \bar{u}$. Since $u \in V_0^0$ and $\bar{u} \in V_1^1$, we can choose a node $x \in V_0^0$ such that $\bar{x} \in V_0^1$ and choose a node $y \in V_1^0$ such that $\bar{y} \in V_1^1$. By Theorem 2, there is a hamiltonian path R of G_0 joining x to y, and there is a hamiltonian path T of G_1 joining $\bar{x}$ to $\bar{y}$. Without loss of generality, we write that $R = \langle x, R_1, u, R_2, y\rangle$ and $T = \langle \bar{x}, T_1, v, T_2, \bar{y}\rangle$. We set $P_1 = \langle u, v\rangle$, $P_2 = \langle u, R_1^{-1}, x, \bar{x}, T_1, v\rangle$, and $P_is = \langle u, R_2, y, \bar{y}, T_2^{-1}, v\rangle$. Then $\{P_1, P_2, P_3\}$ forms the 3^*-container of G between u and v. □

With Lemmas 2–5, we have the following theorem:

Theorem 7. *Every graph in N'_n is 3^*-connected.*

4.2. *An N'_n-graph H is not 4^*-connected*

We say that $u = u_n u_{n-1} \dots u_2 u_1$ is an n-bit binary string if $u_i \in \{0, 1\}$ for every $1 \leq i \leq n$. For $1 \leq i \leq n$, we use $(u)^i$ to denote the binary string, $v_n v_{n-1} \dots v_2 v_1$, such that $v_i = 1 - u_i$ and $v_j = u_j$ for every $j \neq i$. Moreover, we use $(u)_i$ to denote u_i. The *Hamming weight* of an n-bit binary strings $u = u_n u_{n-1} \dots u_2 u_1$, $w(u)$, is $\sum_{i=1}^{n} u_i$. The *n-dimensional hypercube*, Q_n, consists of all n-bit binary strings as its nodes. Two nodes $u = u_n u_{n-1} \dots u_2 u_1$ and $v = v_n v_{n-1} \dots v_2 v_1$ of Q_n are adjacent if and only if $v = (u)^i$ for some $i \in \{1, 2, \dots, n\}$. Note that Q_n is a bipartite graph with bipartition $\{u \mid w(u) \text{ is even}\}$ and $\{u \mid w(u) \text{ is odd}\}$. Let Q_n^i be the subgraph of Q_n induced by

$\{u \in V(Q_n) \mid (u)_n = i\}$ for $i \in \{0, 1\}$. Then Q_n^i is isomorphic to Q_{n-1}. By the definition of Q_n, $Q_n \in B'_n$. Let $n \geq 4$ and let $e = \underbrace{00\ldots0}_{n}$ be a node in Q_n. We set $v = (e)^1$, $p = (e)^n$, and $q = ((e)^1)^n$.

Let H be the graph with $V(H) = V(Q_n)$ and $E(H) = (E(Q_n) - \{(e, p), (v, q)\}) \cup \{(e, q), (v, p)\}$. Obviously, $H - \{(e, q), (v, p)\}$ is a bipartite graph with bipartition $A = \{x \mid w(x)$ is even$\}$ and $B = \{x \mid w(x)$ is odd$\}$. Moreover, H is in N'_n and $H = Q_n^0 \oplus Q_n^1$ for some 1–1 connection ϕ. We will show that H is not k^*-connected for $k \geq 4$.

Suppose that there is a k^*-container $C = \{P_1, P_2, \ldots, P_k\}$ of H between e and q for some $k \geq 4$. We have the following cases:

Case 1: $(e, q) \in \cup_{i=1}^k P_i$ and $(v, p) \in \cup_{i=1}^k P_i$. Without loss of generality, we assume that $(e, q) \in P_1$. Thus, $P_1 = \langle e, q\rangle$. Again, we can assume without loss of generality that $(v, p) \in P_2$. Obviously, the number of nodes in P_2 is $2t_2$ for some integer t_2 and the number of nodes in P_i is $2t_i + 1$ for some integer t_i for every $3 \leq i \leq k$. Therefore, there are t_2 nodes of $V(P_2) \cap B$ and $(t_2 - 2)$ nodes of $V(P_2) \cap A$ other than e and q, and there are t_i nodes of $V(P_i) \cap B$ and $(t_i - 1)$ nodes of $V(P_i) \cap A$ other than e and q for every $3 \leq i \leq k$. As a consequence, $|A| = \sum_{i=2}^k t_i + 2 - k$ and $|B| = \sum_{i=2}^k t_i$. Thus, $|A| \neq |B|$.

Case 2: $(e, q) \in \cup_{i=1}^k P_i$ and $(v, p) \notin \cup_{i=1}^k P_i$. Without loss of generality, we assume that $(e, q) \in P_1$. Obviously, the number of nodes in P_i is $(2t_i + 1)$ for some integer t_i for every $2 \leq i \leq k$. Moreover, there are t_i nodes of $V(P_i) \cap B$, and $(t_i - 1)$ nodes of $V(P_i) \cap A$ other than e and q for every $2 \leq i \leq k$. As a consequence, $|A| = \sum_{i=2}^k t_i + 3 - k$ and $|B| = \sum_{i=2}^k t_i$. Thus, $|A| \neq |B|$.

Case 3: $(e, q) \notin \cup_{i=1}^k P_i$ and $(v, p) \in \cup_{i=1}^k P_i$. Without loss of generality, we assume that $(v, p) \in P_1$. Obviously, the number of nodes in P_1 is $2t_1$ for some integer t_1, and the number of nodes in P_i is $(2t_i + 1)$ for some integer t_i for every $2 \leq i \leq k$. Moreover, there are t_1 nodes of $V(P_1) \cap B$ and $(t_1 - 2)$ nodes of $V(P_1) \cap A$ other than e and q, and there are t_i nodes of $V(P_i) \cap B$ and $(t_i - 1)$ nodes of $V(P_i) \cap A$ other than e and q for every $2 \leq i \leq k$. As a consequence, $|A| = \sum_{i=1}^k t_i + 1 - k$ and $|B| = \sum_{i=1}^k t_i$. Thus, $|A| \neq |B|$.

Case 4: $(e, q) \notin \cup_{i=1}^k P_i$ and $(v, p) \notin \cup_{i=1}^k P_i$. Obviously, the number of nodes in P_i is $(2t_i + 1)$ for some integer t_i for every $1 \leq i \leq k$. Moreover, there are t_i nodes of $V(P_i) \cap B$, and $(t_i - 1)$ nodes of $V(P_i) \cap A$ other than e and q for every $1 \leq i \leq k$. As a consequence, $|A| = \sum_{i=1}^k t_i + 2 - k$ and $|B| = \sum_{i=1}^k t_i$. Thus, $|A| \neq |B|$.

With Case 1, Case 2, Case 3, and Case 4, C is not a k^*-container of H between e and q. Thus, H is not k^*-connected for any k, $4 \leq k \leq n$.

5. Concluding remark

In this paper, we have shown that every B'_n graph is super spanning laceable. With this result, we believe that there should exist more super spanning laceable graphs than we expected. Similarly, there are more superspanning connected graphs to be discussed. We have also shown that every N'_n-graph is w^*-connected for every w, $1 \leq w \leq 3$. It would be interesting to characterize those graphs being superspanning connected or superspanning laceable.

Finally, we prove that there exists a spanning (x, S)-fan in any B'_n graph G with bipartition V_0 and V_1, for any node x in V_i with $i \in \{0, 1\}$, and any node subset S with $|S| \leq n$ such that $|S \cap V_{1-i}| = 1$. We believe that there are other bipartite graphs with such a nice property.

We also think that there exists a spanning (x, S)-fan in some incomplete graph G with $\kappa(G) = k$ for any vertex x and any node subset S such that S is not a cut set with $|S| \leq k$. We can easily prove that G is superspanning connected once the above property holds.

References

[1] S. Abraham, K. Padmanabhan, The twisted cube topology for multiprocessors: a study in network asymmetry, Journal of Parallel and Distributed Computing 13 (1991) 104–110.

[2] S.B. Akers, B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Transactions on Computers 38 (1989) 555–566.

[3] M. Albert, R.E.L. Aldred, D. Holton, On 3*-connected graphs, Australasian Journal of Combinatorics 24 (2001) 193–207.

[4] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, North Holland, New York, 1980.

[5] C.H. Chang, C.K. Lin, H.M. Huang, L.H. Hsu, The super laceability of the hypercube, Information Processing Letters 92 (2004) 15–21.

[6] Y.C. Chen, J.M. Tan, L.H. Hsu, S.S. Kao, Super-connectivity and super-edge-connectivity for some interconnection networks, Applied Mathematics and Computation 140 (2003) 245–254.

[7] Y.C. Chen, C.H. Tsai, L.H. Hsu, J.M. Tan, On some super fault-tolerant hamiltonian graphs, Applied Mathematics and Computation 148 (2004) 729–741.
[8] P. Cull, S.M. Larson, The Möbius cubes, IEEE Transactions on Computers 44 (1995) 647–659.
[9] G.A. Dirac, In abstrakten graphen vorhandene vollstandige 4-graphen und ihre unterteilungen, Mathematische Nachrichten 22 (1960) 61–85.
[10] K. Efe, A variation on the hypercube with lower diameter, IEEE Transactions on Computers 40 (1991) 1312–1316.
[11] K. Efe, The crossed cube architecture for parallel computing, IEEE Transactions on Parallel and Distributed Systems 3 (1992) 513–524.
[12] D.F. Hsu, On container width and length in graphs, groups, and networks, IEICE Transactions Fundamentals E77-A (1994) 668–680.
[13] H.C. Hsu, C.K. Lin, H.M. Huang, L.H. Hsu, The spanning connectivity of the (n, k)-star graphs, International Journal of Foundations of Computer Science 17 (2006) 415–434.
[14] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays · Trees · Hypercubes, Morgan Kaufmann, San Mateo, CA, 1992.
[15] C.K. Lin, H.M. Huang, L.H. Hsu, The super connectivity of the pancake graphs and the super laceability of the star graphs, Theoretical Computer Science 339 (2005) 257–271.
[16] C.K. Lin, H.M. Huang, L.H. Hsu, On the spanning connectivity of graphs, Discrete Mathematics 307 (2007) 285–289.
[17] K. Menger, Zur allgemeinen Kurventheorie, Fundamentale Mathematik 10 (1927) 95–115.
[18] O. Ore, Hamiltonian connected graphs, Journal of Mathematic Pures Application 42 (1963) 21–27.
[19] C.D. Park, K.Y. Chwa, Hamiltonian properties on the class of hypercube-like networks, Information Processing Letters 91 (2004) 11–17.
[20] G.J. Simmons, Almost all n-dimensional rectangular lattices are hamilton-laceable, in: Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, 1978 pp. 649–661.
[21] C.H. Tsai, J.J.M. Tan, L.H. Hsu, The super connected property of recursive circulant graphs, Information Processing Letters 91 (2004) 293–298.
[22] A.S. Vaidya, P.S.N. Rao, S.R. Shankar, A class of hypercube-like networks, in: Proceedings of the Fifth IEEE Symposium on Parallel and Distributed Processing, IEEE Computer Society Press, Los Alamitos, CA, 1993, pp. 800–803.