Generalized list $T$-colorings of cycles

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Abstract

In this paper, we show that the Alon–Tarski theorem for choosability in graphs has an analogous version for the generalized list $T$-colorings—a concept which captures both the channel assignment problem and the $T$-colorings. We apply this result to cycles for generalized list labelings with a condition at distance two.

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1. Introduction

One of the main issues concerning the efficient use of radio spectra in telecommunication is the assignment of frequencies to transmitters from the shortest possible range while maintaining an adequate quality of the signal. The signal clarity depends on the possible
interference of simultaneous transmission from close sources. The level of interference depends mainly on the distance of the transmitters, but also on other factors, e.g. on the hilliness of the terrain.

The following generalized concept of labelings is introduced by Fiala et al. [4]. A generalized list T-coloring problem is a triple \((G, L, t)\): G is a graph, \(L\) is a function assigning sets of positive integers to the vertices of \(G\), i.e. \(L : V(G) \to 2^\mathbb{N}\), and \(t\) is a function assigning sets of positive integers to the edges of \(G\), i.e. \(t : E(G) \to 2^\mathbb{N}\). The function \(t\) must satisfy \(0 \in t(e)\) for each edge \(e \in E(G)\). The goal of the problem is to find a mapping \(c : V(G) \to \mathbb{N}\) such that \(c(v) \in L(v)\) for each \(v \in V(G)\) and \(|c(u) - c(v)| \notin t(uv)\) for each \(uv \in E(G)\). Such a mapping is called an \(L_t\)-labeling of \(G\).

A generalized list T-coloring problem can be interpreted as follows: The vertices of \(G\) are transmitters, \(L(v)\) is the set of frequencies which can be assigned to a vertex \(v\) and \(t(uv)\) corresponds to interference between transmitters \(u\) and \(v\), which prevents that \(u\) and \(v\) are assigned frequencies with certain special differences.

If \(t(e) = \{0\}\) for all \(e \in E(G)\), then the generalized list T-coloring problem becomes the list-coloring problem [3,11]. If \(L(v) = \{0, \ldots, n\}\) for some fixed \(n\) and for all \(v \in V(G)\), then the problem becomes the channel assignment problem [6]. If both \(w(e) = \{0\}\) for all \(e \in E(G)\) and \(L(v) = \{0, \ldots, n\}\), then the problem becomes the classical proper coloring problem for a graph \(G\). If we restrict \(t\) to be a constant function, then we obtain (list) T-colorings.

Finally, observe that generalized list T-colorings comprise also distance constrained labelings. This notion is defined as follows [5,8–10]: For a sequence \(P = (p_1, \ldots, p_k)\) of positive integers (called distance constraints) and a list-assignment \(L\) of a graph \(G\) we define an \(L_P\)-labeling as a mapping \(c : V(G) \to \mathbb{N}\) such that

1. \(\forall u \in V(G) : c(u) \in L(u)\), and
2. \(|c(u) - c(v)| \geq p_i\) for every distinct \(u, v \in V(G)\) at distance at most \(i \leq k\).

The minimum number \(n\) for which \(G\) has an \(L_P\)-labeling for every list assignment \(L\) with lists of size \(n\) is denoted by \(\chi_P^m(G)\). Similarly, in the non-list version of the problem we denote by \(\chi_P(G)\) the smallest \(n\) for which \(G\) has an \(L_P\)-labeling with all lists being equal to \(\{0, \ldots, n\}\). Observe that in general \(\chi_P(G) \leq \chi_P^m(G) - 1\).

An \(L_P\)-labeling of a graph \(G\) can be interpreted also as a generalized list T-coloring of the graph \(G^{(k)}\). Here \(G^{(k)}\) is the \(k\)th power of \(G\), i.e. the graph that arises from \(G\) by adding edges connecting vertices at distance at most \(k\). To conclude the conversion we keep the same list assignment \(L\) and define \(t : E(G^{(k)}) \to 2^\mathbb{N}\) as \(t(uv) = \{0, p_i - 1\}\) for every two vertices \(u, v\) of distance \(i \in [1, k]\) in \(G\).

In the paper we present an analog of the Alon–Tarsi theorem for the generalized list T-coloring. A similar approach for so called T-colorings was shown in Alon and Zaks [2]. Furthermore, as an example for the application of the extended theorem we provide sharp upper bounds on the list size for the \(L_{(2,1)}\)-labelings of cycles.

2. Alon–Tarsi’s theorem for the generalized list T-colorings

It is well known, that if the vertices of a graph can be ordered such that each vertex is adjacent to at most \(d\) predecessors, then the lists of size \(d + 1\) assure that a suitable coloring
exists. The ordering can be transformed to an acyclic orientation with maximum indegree at most \( d \), where edges are oriented towards successors. Observe that for \( d \)-regular graphs, this approach gives the relatively weak bound of list size \( d + 1 \) at each vertex, which is very close to the maximum degree.

Alon and Tarsi [1] extended this result to special orientations (see Theorem 1). In specific cases a suitable orientation might bring a twice better bound, e.g. in the case of bipartite \( k \)-regular graphs. However, the proof is not constructive. It means, we only know that such list coloring with bounded number of colors exists but it is not known how to find a feasible coloring in polynomial time.

An orientation of a multigraph \( G \) provides a directed multigraph \( \vec{D} \), on the same vertex set \( V \). For each edge \( uv \in E(G) \), the graph \( \vec{D} \) contains one of its two arcs. We denote the indegree and the outdegree of a vertex \( u \in V \) by \( d^-(u) \) and \( d^+(u) \), respectively. A directed multigraph is called Eulerian if for every vertex \( u : d^+(u) = d^-(u) \). The set of all Eulerian subgraphs of a directed multigraph \( \vec{D} \) is denoted by \( E(\vec{D}) \). An Eulerian graph is even (resp. odd) if it has even (resp. odd) number of edges. Denote by \( E_o(\vec{D}) \) and \( E_e(\vec{D}) \) as the sets of odd and even Eulerian subgraphs, respectively. Thus, \( E(\vec{D}) = E_o(\vec{D}) \cup E_e(\vec{D}) \).

One of the most interesting results in the theory of list-colorings is the following result [1].

**Theorem 1** (Alon and Tarsi). Let \( G \) be a graph and \( L \) a list assignment of \( G \). Suppose that an orientation \( \vec{D} \) of \( G \) satisfies \( |E_e(\vec{D})| \neq |E_o(\vec{D})| \) and \( |L(u)| \geq d^+(u) + 1 \) for all vertices \( u \in V(G) \). Then, the graph \( G \) admits an \( L \)-coloring.

The proof of the above theorem bases on the following lemma (see [1, Lemma 2.1]).

**Lemma 2.** Let \( P = (x_1, x_2, \ldots, x_n) \) be a polynomial in \( n \) variables over the ring of integers \( \mathbb{Z} \). Suppose that for each \( 1 \leq i \leq n \) the degree of \( P \) as a polynomial in \( x_i \) is at most \( d_i \) and let \( S_i \subset \mathbb{Z} \) be a set of \( d_i + 1 \) distinct integers. If \( P(x_1, x_2, \ldots, x_n) = 0 \) for all \( n \)-tuples \( (x_1, x_2, \ldots, x_n) \in S_1 \times S_2 \times \cdots \times S_n \) then \( P \equiv 0 \).

For a weight function \( t : E(G) \rightarrow 2^\mathbb{N} \) denote by \( G^{(t)} \) the multigraph with the same vertex set as \( G \) such that every edge \( e \) is replaced by a set of \( 2|t(e)| - 1 \) multiple edges. We extend Theorem 1 to the concept of generalized list \( T \)-colorings. Following the argument from [2], the proof is an easy task but for the sake of completeness we include it.

**Theorem 3.** Let \((G, L, t)\) be a generalized list \( T \)-coloring problem. Suppose that an orientation \( \vec{D} \) of \( G^{(t)} \) satisfies \( |E_e(\vec{D})| \neq |E_o(\vec{D})| \) and \( d^+(u) + 1 \leq |L(u)| \) for every vertex \( u \in V(\vec{D}) \). Then, \( G \) admits an \( L_t \)-labeling.

**Proof.** Assign to every vertex \( u \) a variable \( x_u \) and consider a polynomial in \( \mathbb{N}[V(G)] \):

\[
f^*_G = \prod_{uv \in E(G)} \left[ (x_u - x_v) \prod_{j \in t(uv) \setminus \{0\}} (x_u - x_v - j)(x_u - x_v + j) \right].
\]
In the definition of the polynomial \( f_G^t \), we assume that vertices of \( G \) are linearly ordered, and that \( u \) is a predecessor of \( v \) regarding this ordering. If \( c \) is an \( L \)-labeling of \( G \), then \( c \) is an \( L_t \)-labeling if and only if the polynomial \( f_G^t \) evaluates to a nonzero value at \( x_u = c(u), \ u \in V(G) \) (since every term of the product has a nonzero value in such a case). Let

\[
  r_u(x_u) = x_{uL(u)} - \prod_{s \in L(u)} (x_u - s).
\]

Then the degree of \( r_u \) is at most \( |L(u)| - 1 \) and \( r_u(c(u)) = c(u)^{|L(u)|} \) if \( c(u) \in L(u) \), since \( \prod_{s \in L(u)} (x_u - s) = 0 \) in such a case.

We expand \( f_G^t \) into a linear combination of monomials and recursively replace every occurrence of \( x_{uL(u)} \) by \( r_u(x_u) \), until the degree of every variable \( x_u \) in every monomial of the modified polynomial \( h_G^t \) is at most \( |L(u)| - 1 \). Observe that the new polynomial \( h_G^t \) retains the value of \( f_G^t \) for all selections \( x_u \in L(u) \).

Now consider the multigraph \( G(t) \), its orientation \( \vec{D} \) and a polynomial

\[
g_{\vec{D}} = \prod_{uv \in E(\vec{D})} (x_u - x_v),
\]

where the term \( (x_u - x_v) \) appears in \( g_{\vec{D}} \) as many times as the arc \( uv \) in \( E(\vec{D}) \). Observe that for different orientations \( \vec{D} \) of \( G(t) \) the polynomial \( g_{\vec{D}} \) is the same or multiplied by the \((-1)\) factor. It follows from the construction of \( G(t) \) that every monomial with a nonzero coefficient in \( g_{\vec{D}} \) appears also with the same coefficient in \( f_G^t \) (upto \((-1)\) factor).

Corollary 2.3. of [1] shows that in the polynomial \( g_{\vec{D}} \), the coefficient of the monomial \( M = \prod_{u \in V} x_{uL(u)} \) is equal to \( |E_o(\vec{D})| - |E_e(\vec{D})| \), perhaps with the multiplicative \((-1)\) factor. By our assumptions, \( M \) has nonzero coefficients in all polynomials \( g_{\vec{D}}, f_G^t, \) and \( h_G^t \), since

(1) \( M \) was not reduced, because it contains no \( x_{uL(u)} \), and
(2) the monomial \( M \) cannot be obtained by such reduction from another monomial \( \prod_{u} x_{da} \), because the reduction decreases the value \( \sum d_a \leq |E(G(t))| \) and the sum of the degrees in \( M \) attains the upper bound \( |E(G(t))| \).

We may summarize that in this moment we know that \( h_G^t \) contains a nonzero coefficient by monomial \( M \), and the degree of every variable in every monomial is at most \( d^{+}(u) \leq |L(u)| - 1 \).

If we assume that the value of \( h_G^t \) is zero for all possible selections \( x_u \in L(u) \), we get a contradiction with Lemma 2, since such polynomial \( h_G^t \equiv 0 \). □

3. Counting the Eulerian subgraphs of \( C_n^{(p,1)} \)

In this section we derive recursive formulas for counting the difference in the number of odd and even Eulerian subgraphs of the clockwise orientation of the graph \( C_n^{(p,1)} \).
as a possible necessary step for proving the existence of list $L(p,1)$-labelings of cycles.

In the sequel the symbols $C_n$ and $P_n$ denote the cycle and the path of length $n$. For a pair of distance constraints $(p, 1)$ and a graph $G$ (usually for $C_n$ or $P_n$) we denote by $G^{(p,1)}$ the multigraph with the same vertex set as $G$ such that every two vertices adjacent in $G$ are in $G^{(p,1)}$ connected with a cluster of $2p - 1$ multiple edges, and every two vertices at distance two in $G$ are joined by an edge in $G^{(p,1)}$. An edge $uv$ of $G^{(p,1)}$ is called short if $u$ and $v$ are adjacent in $G$; otherwise $uv$ is a long edge.

We always assume $C_n^{(p,1)}$ being clockwise oriented. See Fig. 2 for the clockwise orientation of $C_n^{(2,1)}$. Similarly the graph $P_n^{(p,1)}$ on vertices $v_0, v_1, \ldots, v_n$ is oriented such that every edge points towards the higher numbered vertex.

For both $C_n^{(p,1)}$ and $P_n^{(p,1)}$ we call a simple cut the set of all short and long edges incoming into some vertex $v_i$, together with the long edge “passing above” the vertex $v_i$, if it exists (usually $v_{i-1}v_{i+1}$). An example of a simple cut in paths and cycles is depicted by the dotted edges in Figs. 1 and 2, respectively. Observe that simple cuts of $P_n^{(p,1)}$ intersect each path between $v_0$ and $v_n$, whereas simple cuts of $C_n^{(p,1)}$ intersect each directed cycle.
A subgraph of $P_n^{(p,1)}$ such that each simple cut of $F_n^{(p,1)}$ contains exactly $k$ of its edges is called a $k$-flow. In other words, in a $k$-flow the vertex $v_0$ has outdegree $k$, $v_n$ has indegree $k$, and for every inner vertex of $P_n^{(p,1)}$, its indegree is equal to its outdegree. It is easy to see that in a $k$-flow of $P_n^{(p,1)}$ each inner vertex has outdegree/indegree $k-1$ or $k$, regarding whether the edge $v_{i-1}v_{i+1}$ belongs to the flow. (See Fig. 1 for an example.)

We call a $k$-flow odd if it has an odd number of edges, while the remaining $k$-flows are called even. The next lemma counts the difference $F_{k,n}$ between the numbers of odd and even $k$-flows in $P_n^{(p,1)}$.

We remark here that in the paper, we always assume that $\binom{n}{k} = 0$ if $k < 0$ or $n < k$.

**Lemma 4.** For any $p \geq 1$, $n \geq 0$ and $k = 0, \ldots, 2p$ the difference $F_{k,n}$ between the numbers of odd and even $k$-flows in $P_n^{(p,1)}$ is determined by the following recursive formula:

$$F_{k,0} = -1, \quad F_{k,1} = (-1)^{k+1} \binom{2p-1}{k} \quad \text{and for } n \geq 2:\n$$

$$F_{k,n} = (-1)^k F_{k,n-1} \left[ \left( \binom{2p-1}{k} - \binom{2p-1}{k-2} \right) + F_{k,n-2} \left( \binom{2p-1}{k} \left( \binom{2p-1}{k-2} - \binom{2p-1}{k-1} \right) \right) \right].$$

**Proof.** When $n = 0$ or $k = 0$ the only $k$-flow has no edges, i.e. it is even, which justifies the formulas $F_{0,n} = -1$ and $F_{k,0} = -1$.

For $n = 1$, we have $F_{k,1} = (-1)^{k+1} \binom{2p-1}{k}$ because out of $2p-1$ edges connecting $v_0$ and $v_1$ we have to select $k$ edges to form a $k$-flow, and the parity of the number of the edges of such a $k$-flow is just the parity of $k$. From these $2p-1$ edges it is impossible to form a $2p$-flow, hence $F_{2p,1} = (-1)^{2p-1} \binom{2p-1}{2p} = 0$ since $\binom{2p-1}{2p} = 0$. Also notice that for $k = 0$ this formula gives $F_{0,1} = -1$.

In what follows we assume that $n \geq 2$. If $k = 0$ then the formula gives $F_{0,n} = F_{0,n-1}$ and it coincides with the fact that $F_{0,n} = -1$ for each $n$. We consider cases $k = 1$ and $k = 2p$, separately. So, suppose first that $k = 1$. Since $\binom{2p-1}{2p} = 0$, we have to show that

$$F_{1,n} = -(2p-1) F_{1,n-1} - F_{1,n-2}$$

for $n \geq 2$. The first term of the formula corresponds to the case when the 1-flow ends by the short edge $v_{n-1}v_n$. We construct such a flow from a 1-flow between $v_0$ and $v_{n-1}$ by composing with one of $2p-1$ edges between $v_{n-1}$ and $v_n$. This composition changes the parity so we switch the sign. The other summand of that formula corresponds to the case when the 1-flow ends by the long edge $v_{n-2}v_n$.

Suppose now that $k = 2p$. We argue similarly as in the previous case. Since $\binom{2p-1}{2p} = 0$, we have to show that

$$F_{2p,n} = -(2p-1) F_{2p,n-1} - F_{2p,n-2}$$
for \( n \geq 2 \). Notice that any \( 2p \)-flow of length \( n \) can be obtained in two ways regarding whether the long edge \( v_{n-3}v_{n-1} \) is present or not. It can be composed as an extension of a \( 2p \)-flow of length \( n - 2 \) with the only \( 2p \)-flow of length two. In this case we increase the number of edges by \( 4p - 1 \) and we must switch the parity. In the other case we take a \( 2p \)-flow between \( v_0 \) and \( v_{n-1} \) and extend it by all edges incoming to the vertex \( v_n \). Finally, to obtain a valid \( 2p \)-flow we must remove one of the edges between vertices \( v_{n-2} \) and \( v_{n-1} \), what can be done in \( 2p - 1 \) ways. The number of edges increases by \( 2p - 1 \) so we also change the sign.

Suppose now that \( 2 \leq k \leq 2p - 1 \). In what follows, we show that

\[
F_{k,n} = (-1)^k F_{k,n-1} \binom{2p-1}{k} - F_{k,n-2} \binom{2p-1}{k-1}^2 \\
+ (-1)^{k-1} \left[ F_{k,n-1} - (-1)^k F_{k,n-2} \binom{2p-1}{k} \right] \binom{2p-1}{k-2}
\]

and afterwards the proof follows easily. In order to prove the above equality, we distinguish two cases according to the presence of the long edge \( e \) or not exist for \( n \), more precisely we have increased the number of edges \( k \)-flow and we switch it, so we impose the factor \((-1)^k\). This explains the first summand in the above formula.

The other two summands describe the case when \( e \) takes part in such a \( k \)-flow and we again consider two subcases: If \( e' = v_{n-3}v_{n-1} \) does not appear in the \( k \)-flow (also if it does not exist for \( n = 2 \)), then we compose any \( k \)-flow between \( v_0 \) and \( v_{n-2} \) with \( (k-1) \)-flow between \( v_{n-2} \) and \( v_n \) and the edge \( e \). Note that in this case we always add an odd number of edges to the \( k \)-flow between \( v_0 \) and \( v_{n-2} \), so we must switch the sign to obtain the second summand.

The last summand describes the case when both \( e \) and \( e' \) take part in the flow. The difference in odd and even number of \( k \)-flows between \( v_0 \) and \( v_{n-1} \) containing \( e' \) is \( F_{k,n-1} - (-1)^k F_{k,n-2} \binom{2p-1}{k} \). Observe that this term cancels to zero for \( n = 2 \). When \( n \geq 3 \) we replace in such a flow some short edge between \( v_{n-2} \) and \( v_{n-1} \) by the long edge \( e \) and combine with \( k-1 \) short edges between \( v_{n-1} \) and \( v_n \). The overall computation shows the two factors \( \binom{2p-1}{k-1} \) cancel each other while only one factor \( \binom{2p-1}{k-1} \) remains. Again, the parity of the number of edges depends on \( k \), more precisely we have increased the number of edges by \( k-1 \). In this way, we obtain the third term of the above formula. Thus, the proof is established.

We continue with the case of the cycles. For \( k = 0, \ldots, 2p+1 \) we define a \( k \)-whirl of \( C_n^{(p,1)} \) as its subgraph which contains exactly \( k \) edges from each simple cut. Observe, that the set of \( k \)-whirls for \( k = 0, \ldots, 2p+1 \) is exactly the set of all Eulerian subgraphs of \( C_n^{(p,1)} \). Moreover, each vertex of a \( k \)-whirl has indegree \( k \) or \( k-1 \). Denote by \( W_{k,n} \) the difference between the numbers of odd and even \( k \)-whirls of \( C_n^{(p,1)} \). The next lemma determines this difference \( W_{k,n} \) for \( n \geq 5 \) and all feasible \( k \).
Lemma 5. For any \( p \geq 1, n \geq 5 \) and \( k = 0, \ldots, 2p + 1 \) the difference \( W_{k,n} \) between the numbers of odd and even \( k \)-whirls in \( C_n^{(p,1)} \) is determined recursively by the following formulas:

- \( W_{0,n} = W_{2p+1,n} = -1 \),
- for \( k = 1, 2, \ldots, p \):

\[
W_{k,n} = W_{2p-k+1,n}
= F_{k,n} \left[ 1 - \frac{(k-1)^2}{(2p-k+1)^2} \right] + F_{k,n-1}(-1)^k \left[ -2 \binom{2p-1}{k-2} \right]
+ 2 \binom{2p-1}{k} \binom{2p-1}{k-2} \binom{2p-1}{k-1}^{-2}
+ F_{k,n-2} \left[ -\binom{2p-1}{k-1}^2 + 2 \binom{2p-1}{k} \binom{2p-1}{k-2} \right]
- \binom{2p-1}{k}^2 \binom{2p-1}{k-2} \binom{2p-1}{k-1}^{-2}.
\]

Proof. The complement of a \( k \)-whirl takes from each simple cut exactly \( 2p+1-k \) edges and hence it is a \((2p-k+1)\)-whirl. The total number of edges in \( C_n^{(p,1)} \) is even (namely \( 2pn \)) so the numbers of odd \( k \)-whirls and odd \((2p-k+1)\)-whirls are equal. The same holds for even whirls. Therefore, \( W_{k,n} = W_{2p-k+1,n} \) for all \( k = 0, \ldots, p \) and we may restrict ourselves to only such values of \( k \).

Observe that for \( k = 0 \), the subgraph of \( C_n^{(p,1)} \) with no edges is the only 0-whirl, and therefore \( W_{0,n} = -1 \).

For \( k \geq 1 \), we distinguish two cases, depending on the fact whether the long edge \( v_{n-1}v_1 \) appears in such a \( k \)-whirl. In both cases, we remove \( v_{n-1}v_1 \) from \( C_n^{(p,1)} \) and cut the vertex \( v_n \) into two new vertices \( v_n \) and \( v_0 \), so that the first one is incident with all incoming edges and the second one is incident with all outgoing edges. Notice that in this way, we obtain \( P_n^{(p,1)} \).

As in the previous lemma, we consider the case \( k = 1 \) separately. Since \( (2p-1) = 0 \), we have to show that for \( n \geq 2 \)

\[
W_{1,n} = F_{1,n} - F_{1,n-2}.
\]

Notice that \( F_{1,n} \) corresponds to those 1-whirls, which do not contain the long edge \( v_{n-1}v_1 \), the term \( F_{1,n-2} \) is for those whirls which contain \( v_{n-1}v_1 \) and \( v_n \) is isolated. We subtracted this term since the insertion of \( e \) changes the parity of the number of edges.
In what follows we prove the following equality for the case \( k \geq 2 \), which establish the lemma

\[
W_{k,n} = F_{k,n} - F_{k,n-2} \left( \frac{2p - 1}{k - 1} \right)^2 
- 2 \left[ (2p - 1) \right] \left( \frac{2p - 1}{k - 2} \right)
- \left[ F_{k,n} - 2(-1)^k F_{k,n-1} \left( \frac{2p - 1}{k} \right) \right]
+ F_{k,n-2} \left( \frac{2p - 1}{k} \right)^2 \left( \frac{2p - 1}{k - 2} \right)^2 \left( \frac{2p - 1}{k - 1} \right)^{-2}.
\]

The difference in the numbers of odd and even \( k \)-whirls without \( e = v_{n-1} v_1 \) is exactly \( F_{k,n} \), since each such a \( k \)-whirl corresponds to a unique \( k \)-flow of \( P_n(p,1) \). This gives the first term of the above formula. So, it remains to calculate the difference for those \( k \)-whirls containing \( e \). Note that in this case the vertex \( v_n \) has indegree and outdegree \( k - 1 \). Denote by \( e' \) and \( e'' \) the two long edges \( v_0 v_2 \) and \( v_{n-2} v_n \), respectively.

Each \( k \)-whirl which contains neither \( e' \) nor \( e'' \) can be constructed as an extension of a \( k \)-flow between \( v_1 \) and \( v_{n-1} \) by the edge \( e \) and by \( k - 1 \) edges between \( v_1 \) and \( v_n \), and by another \( k - 1 \) edges between \( v_{n-1} \) and \( v_n \). This gives the second monomial \( F_{k,n-2} \left( \frac{2p - 1}{k - 1} \right)^2 \). The minus sign in the above formula is due to the opposite parity of the whirl from that of the appropriate \( k \)-flow.

The third summand counts the difference between the numbers of odd and even \( k \)-whirls, which contain precisely one of the edges \( e' \), \( e'' \). We first construct all \( k \)-whirls with \( e' \) and without \( e'' \) in the following manner: We start with a \( k \)-flow between \( v_0 \), \( v_{n-1} \) which contains the edge \( e' \). The difference in the numbers of such flows is \( F_{k,n-1} - (-1)^k F_{k,n-2} \left( \frac{2p - 1}{k} \right) \).

In each such flow we remove the \( k - 1 \) edges between \( v_0 \) and \( v_1 \) and introduce the edge \( e \) together with \( k - 2 \) edges between \( v_0 \) and \( v_1 \) and \( k - 1 \) edges between \( v_{n-1} \) and \( v_n \). This edge replacement yields the term \( \left( \frac{2p - 1}{k - 1} \right) \left( \frac{2p - 1}{k - 2} \right) \). The number of edges increases by \( k - 1 \), so we multiply the last expression also by \( (-1)^{k-1} \). To encounter the \( k \)-whirls with \( e'' \) but without \( e' \) we use symmetry of this and the previous case and insert the multiplicative factor 2 into the expression.

The remaining term calculates the difference between the numbers of odd and even \( k \)-whirls, which contain all the three long edges \( e', e' \) and \( e'' \). Such \( k \)-whirls are constructed as follows: In each of the \( k \)-flows between \( v_0 \), \( v_n \) which contains both edges \( e', e'' \), we reduce the set of \( k - 1 \) edges between vertices \( v_0 \) and \( v_1 \) to \( k - 2 \) (possibly different) edges, then we do the same with the edges joining \( v_{n-1} \) and \( v_n \), and finally we insert the long edge \( e \). The difference in the numbers of odd and even such \( k \)-flows is

\[
F_{k,n} - 2(-1)^k F_{k,n-1} \left( \frac{2p - 1}{k} \right) + F_{k,n-2} \left( \frac{2p - 1}{k} \right)^2.
\]
In order to obtain the above expression, we apply inclusion–exclusion principle: From all \( k \)-flows between \( v_0 \) and \( v_n \) subtract those which do not contain \( e' \), afterwards those which do not contain \( e'' \), and at the end add the number of \( k \)-flows between \( v_0 \) and \( v_n \) that contains neither \( e' \) nor \( e'' \). We again have to involve the parity term \((-1)^k\).

Both reductions of \( k - 1 \) edges to \( k - 2 \) can be performed as a replacement of the set of \( k - 1 \) edges with a set of size \( k - 2 \). This extends the above expression by the factor \((\frac{2p-1}{k-2})^2(\frac{2p-1}{k-1})^{-2}\). In total we have decreased the number of edges by one, which changes the parity of the difference, so we multiply the expression with \(-1\). □

**Theorem 6.** The difference in the number of odd and even Eulerian subgraphs of \( C_n^{(p,1)} \) is

\[
|\mathcal{E}_o(C_n^{(p,1)})| - |\mathcal{E}_e(C_n^{(p,1)})| = 2 \sum_{k=0}^{p} W_{k,n}. 
\]

**Proof.** We have already mentioned that each Eulerian subgraph of \( C_n^{(p,1)} \) is a \( k \)-whirl for some \( k \in \{0, \ldots, 2p + 1\} \) and vice versa. Thus, the total difference in the number of odd and even Eulerian subgraphs of \( C_n^{(p,1)} \) is

\[
|\mathcal{E}_o(C_n^{(p,1)})| - |\mathcal{E}_e(C_n^{(p,1)})| = \sum_{k=0}^{2p+1} W_{k,n} = 2 \sum_{k=0}^{p} W_{k,n}. \quad \square
\]

4. Generalized \( L(2,1) \)-labelings of cycles

In the sequel we utilize the calculations from the previous section to prove the existence of \( L(2,1) \)-labelings of cycles for lists of size at least 5.

**Theorem 7.** Let \((C_n^{(2,1)}, L, t)\) be a generalized list \( T \)-coloring problem such that \( n \geq 5 \), and for every edge \( e \) of \( C_n^{(2,1)} \)

\[
|t(e)| = \begin{cases} 1, & \text{if } e \text{ is a long edge;} \\ 2, & \text{if } e \text{ is a short edge.} \end{cases}
\]

If \( |L(v)| \geq 5 \) for every vertex of \( C_n^{(2,1)} \), then \( C_n^{(2,1)} \) admits an \( L_t \)-labeling.

Notice that \( t(e) = \{0\} \) for every long edge \( e \) due to the definition of the generalized list \( T \)-colorings.

**Proof.** Let \( C_n^{(2,1)} \) be clockwise oriented as depicted in Fig. 2. In what follows, we show that the difference \( |\mathcal{E}_o| - |\mathcal{E}_e| = |\mathcal{E}_o(C_n^{(2,1)})| - |\mathcal{E}_e(C_n^{(2,1)})| \) is distinct from 0 for every \( n \geq 5 \) and then the claim is established by Theorem 3. For \( n = 5, \ldots, 10 \) the values of the difference \( |\mathcal{E}_o| - |\mathcal{E}_e| \) are summarized in Table 1. So we may assume that \( n \geq 11 \).
Table 1
The difference $|\mathcal{E}_o| - |\mathcal{E}_e|$ for small $n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\mathcal{E}_o</td>
<td>-</td>
<td>\mathcal{E}_e</td>
<td>$</td>
<td>-60</td>
<td>-1350</td>
</tr>
</tbody>
</table>

By Lemmata 4 and 5 with $p = 2$, the recursive formulas for $F_{k,n}$ and $W_{k,n}$ read as follows:

- $k = 1$: $F_{1,0} = -1$, $F_{1,1} = 3$ and $F_{1,n} = -3F_{1,n-1} - F_{1,n-2}$ for $n \geq 2$.
  Now, we easily infer that
  
  $F_{1,n} = \left(\frac{3}{2\sqrt{5}} - \frac{1}{2}\right)\left(-3 + \sqrt{5}\right)^n - \left(\frac{3}{2\sqrt{5}} + \frac{1}{2}\right)\left(-3 - \sqrt{5}\right)^n$

  and

  $W_{1,n} = F_{1,n} - F_{1,n-2}$

  
  $= -\left(\frac{-3 + \sqrt{5}}{2}\right)^n - \left(\frac{-3 - \sqrt{5}}{2}\right)^n$.

- $k = 2$: $F_{2,0} = -1$, $F_{2,1} = -3$, and $F_{2,n} = 2F_{2,n-1} - 6F_{2,n-2}$ for $n \geq 2$.
  From here it follows that:

  $F_{2,n} = \left(\frac{i}{\sqrt{5}} - \frac{1}{2}\right)(1 + i\sqrt{5})^n - \left(\frac{i}{\sqrt{5}} + \frac{1}{2}\right)(1 - i\sqrt{5})^n$ and

  $W_{2,n} = \frac{8}{9}F_{2,n} - \frac{4}{3}F_{2,n-1} - F_{2,n-2}$

  $= \frac{14}{9}F_{2,n} - \frac{8}{3}F_{2,n-1}$

  $= -\left(1 + i\sqrt{5}\right)^n - \left(1 - i\sqrt{5}\right)^n$.

For $n \geq 11$, the absolute value of the term $\left(\frac{-3 - \sqrt{5}}{2}\right)^n$ outweighs the other terms, so by Theorem 6 it follows that

$|\mathcal{E}_o| - |\mathcal{E}_e| = 2\left[-1 - \left(\frac{-3 - \sqrt{5}}{2}\right)^n\right] - \left(\frac{-3 + \sqrt{5}}{2}\right)^n$

$- (1 - i\sqrt{5})^n - (1 + i\sqrt{5})^n$,

$||\mathcal{E}_o| - |\mathcal{E}_e|| \geq 2\left[\frac{-3 - \sqrt{5}}{2}\right]^n - \left[2\left|\frac{-3 + \sqrt{5}}{2}\right|^n\right] - 2|1 - i\sqrt{5}|^n - 2|1 + i\sqrt{5}|^n - 2$

$\geq 2 \ast 2.618^n - 2 \ast 0.383^n - 4 \ast 2.450^n - 2$

$\geq 2.618^{n+1} - (2 \ast 2.618^n - 2 \ast 0.383^n - 4 \ast 2.450^n) - 2.001 > 0$.

This argument closes the proof of Theorem 7. □

In the above theorem we consider graphs $C_n$ with $n \geq 5$. The same result for $C_3$ and $C_4$ follows by the Brooks-type theorem for the generalized list $T$-colorings [4]. Since
Let \( L = \{0, 1, 2, 3, 4\} \) and \( L_0 = \{0, 1, 3, 4, 5\} \).

**Fig. 3.** A graph satisfying the strict inequality \( \lambda_{(2,1)}(G) + 1 < \ell_{(2,1)}(G) \).

\( \lambda_{(2,1)}(C_n) = 4 \), the length 5 of the lists is best possible. Now, as a special case (i.e. the mapping \( t \) assigns the same set to the short edges), we obtain the following corollary. This result below was obtained first by Jonas [7] with a lengthy but constructive proof.

**Corollary 8.** If \( C_n \) is a cycle on \( n \geq 3 \) vertices, then \( \chi_{(2,1)}^\ell(C_n) = 5 \).

The above result says that \( \lambda_{(2,1)}(G) + 1 = \chi_{(2,1)}^\ell(G) \) holds for all cycles. It is known that such an equality holds also for all paths (see [7]). Observe that the acyclic orientation of \( P_n^{(p,1)} \) we used in the previous section allows only one Eulerian subgraph. Hence one can easily show that for any distance constraints \( P \) we have \( \lambda_P(P_n) + 1 = \chi_P^\ell(P_n) \) whenever the length of the path exceeds the number of distance constraints.

In the case of cycles we have evaluated the recursive formulas for the difference between the number of odd and even Eulerian subgraphs of \( C_n^{(p,1)} \) for \( p = 3, \ldots, 20 \). In these cases it is not possible to bound the powers of the complex conjugates by the same method as in the case of \( L_{(2,1)} \)-labelings. However, we would like to express our belief that cycles satisfy a similar equality as paths at least for some special values of \( P \) in the following conjecture.

**Conjecture 9.** For all \( p > 0 \) and \( n \geq 3 \), it holds that \( \lambda_{(p,1)}(C_n) + 1 = \chi_{(p,1)}^\ell(C_n) \).

In [3] are shown constructions of graphs satisfying the strict inequality for the standard graph coloring and choosability, i.e. \( P = (1) \). The easiest example of such graph is the complete bipartite graph \( K_{k,k} \) satisfying \( \chi(K_{k,k}) = 2 < k < \chi^\ell(K_{k,k}) \). To conclude our study we mention that the strict inequality \( \lambda_P(G) + 1 < \chi_P^\ell(G) \) can be expected for a variety of graphs \( G \) and distance constraints \( P \). A simple example of a graph satisfying this inequality for \( P = (2, 1) \) is depicted in **Fig. 3.** A feasible labeling showing \( \lambda_{(2,1)}(G) = 4 \) is indicated by the numbers by the vertices, while for the indicated lists it is impossible to find a feasible \( L_{(2,1)} \)-labeling. Therefore, \( \chi_{(2,1)}^\ell(G) > 5 = 1 + \lambda_{(2,1)}(G) \).

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