The number of partially ordered sets with more points than incomparable pairs

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Abstract


Let \( p_{kn} \) denote the number of unlabeled posets with \( n \) points and \( k \) unrelated pairs. We show that for \( k < n \), these numbers satisfy a recursion formula of the form \( p_{kn} = \Sigma_{j=0}^{k} c_j p_{k-j,n-j-1} \), where the coefficients \( c_j \) can be computed if the numbers \( q_{jm} \) of all ordinally indecomposable posets with \( m \) points and \( j \) unrelated pairs are known for \( m - 1 \leq j \leq k \). The crucial lemma for the proof states that \( \lim_{j \to \infty} q_{jm} = 0 \) for \( j < m - 1 \). From the recursion formula it follows that \( p_{kn} \) is a polynomial of degree \( k \) in the variable \( n \) and that \( p_{kn} \approx \binom{n-1}{k} \) with asymptotic equality for fixed \( k \). For small values of \( k \), we determine these polynomials explicitly. At the other end of the scale, we find that \( q_{n-1,n} = 2^{-3} \) for \( n \geq 3 \). Similar results are obtained for the number of labeled posets with a fixed linear extension and a given number of unrelated pairs.

Recently, Culberson and Rawlins [2] have developed a fast algorithm for computing the number \( p_{kn} \) of unlabeled (i.e., isomorphism classes of) posets with a given number \( n \) of points and a fixed number \( k \) of unrelated ('incomparable') pairs, i.e., two-element antichains. Using this algorithm, they have obtained a complete list of these numbers for all \( n \leq 15 \) and \( k \leq 14 \). On account of this numerical material, they have conjectured that for \( k < n \), the numbers \( p_{kn} \) satisfy a formula of the following form:

\[ p_{kn} = \sum_{j=0}^{k} c_j p_{k-j,n-j-1}. \] (*)

where the first coefficients \( c_j \) are

\[ 1, 1, 1, 3, 8, 21, 63, 195, 612, 1971, 6458, 21426, 71905, 243640, 832242, \ldots \]

Since this recursion formula becomes wrong for \( k \geq n \), the above authors have guessed that the validity of the recursion may depend on the fact that any poset
with \( n \) points and less than \( n - 1 \) unrelated pairs must be connected. But as we shall see soon, not the decomposition into (connected) components is relevant for this phenomenon, but the decomposition into ordinal summands. We shall show that in fact an identity of the form (*) exists, although the computation of the coefficients \( c_k \) is rather tedious and requires the knowledge of the numbers \( q_{jm} \) of all unlabeled ordinally indecomposable posets (see below) with \( m \) points and \( j \) incomparable pairs for \( m - 1 \leq j \leq k \).

For a given finite poset \( P \), we denote by

\[ n \text{ or } n_p \text{ the cardinality of the underlying set}, \]
\[ \preceq \text{ or } \preceq_p \text{ the partial order of } P, \]
\[ d \text{ or } d_p \text{ the number of unrelated doubletons (incomparable pairs) in } P. \]

Thus \((*) - d\) is the cardinality of the relation \( \prec \) (considered as a subset of \( P \times P \)).

The ordinal sum \( Q \oplus R \) of two posets \( Q \) and \( R \) is the disjoint union of the underlying sets of \( Q \) and \( R \), respectively, partially ordered by

\[ x \preceq_{Q \oplus R} y \text{ iff } x \preceq_Q y \text{ or } x \preceq_R y \text{ or } (x, y) \in Q \times R. \]

In other words, the ordinal sum is obtained by placing \( R \) above \( Q \). A poset \( P \) is said to be (ordinally) decomposable if \( P \) is empty or \( P = Q \oplus R \) for suitable non-empty posets \( Q \) and \( R \), otherwise (ordinally) indecomposable. By induction one shows easily that each finite poset \( P \) has a unique representation

\[ P = Q_1 \oplus \cdots \oplus Q_m \]

into non-empty ordinally indecomposable summands \( Q_i \). Our first result puts the evident observation that posets with many comparable pairs must be ordinally decomposable into a more precise framework.

**Lemma 1.** If a finite poset \( P \) is ordinally indecomposable then:

1. for any two maximal elements \( x, y \in P \), one of the subposets \( P - \{x\} \) and \( P - \{y\} \) is ordinally indecomposable,
2. \( d_p \geq n_p - 1 \).

**Proof.** Since every poset has a linear extension, we may assume that \( P = (\mathbb{N}, \preceq_p) \), that \( x \preceq_p y \) implies \( x \preceq y \) in the natural order of \( \mathbb{N} = \{1, \ldots, n\} \), and that the elements \( n - 1 \) and \( n \) are maximal. Of course, \( P - \{x\} \) denotes the subset \( \mathbb{N} - \{x\} \) together with the order induced from \( \preceq_p \).

1. Assume \( P - \{n\} \) and \( P - \{n - 1\} \) are ordinally decomposable, say

\begin{align*}
P - \{n\} &= \{1, 2, \ldots, k\} \oplus \{k + 1, \ldots, n - 1\}, \\
P - \{n - 1\} &= \{1, 2, \ldots, m\} \oplus \{m + 1, \ldots, n - 2, n\}.
\end{align*}

(Notice that in any ordinal decomposition \( P - \{x\} = Q \oplus R \), we must have \( q \preceq r \) for all \( q \in Q \) and \( r \in R \).)
If \( k \leq m \) then \( k < n \) and consequently \( P = \{1, \ldots, k\} \oplus \{k + 1, \ldots, n\} \); otherwise, \( m < k < n - 1 \) and consequently \( P = \{1, \ldots, m\} \oplus \{m + 1, \ldots, n\} \).

In both cases, we obtain a contradiction to the indecomposability of \( P \). Hence, either for \( x = n \) or for \( x = n - 1 \), the subposet \( P - \{x\} \) is indecomposable. By duality, \( P - \{1\} \) or \( P - \{2\} \) is indecomposable, too.

(2) This follows from (1) by induction on \( n_p \): if \( n > 1 \), choose a maximal \( x \in P \) such that \( P - \{x\} \) is indecomposable. Then the induction hypothesis yields \( d_{P - \{x\}} \geq n_p - 2 \), and since \( x \) cannot be the greatest element of \( P \) (otherwise \( P \) would be decomposable), \( d_p \geq d_{P - \{x\}} + 1 \geq n_p - 1 \).

Of course, the lower bound \( n - 1 \) for the number \( d \) of unrelated pairs in an indecomposable poset is the best possible, since the disjoint union of a singleton and an \( (n - 1) \)-element chain satisfies \( d = n - 1 \).

Observing that every finite poset \( P \) has a unique representation \( P = Q \oplus R \) where \( Q \) is indecomposable, and that

\[
d_{Q \oplus R} = d_Q + d_R,
\]

we obtain immediately a recursion formula for the numbers \( p_{kn} \) in terms of the corresponding numbers \( q_{kn} \) of unlabeled indecomposable posets with \( n \) points and \( k \) unrelated pairs. Thus, \( p_{00} = 1 \) but \( q_{00} = q_{k0} = p_{k0} = 0 \) for \( k > 0 \).

**Lemma 2.** For all natural numbers \( n > 0 \) and \( k \geq 0 \),

\[
p_{kn} = q_{kn} + \sum_{m=1}^{n-1} \sum_{j=m}^{k} q_{jm} p_{k-j,n-m} = \sum_{m=1}^{n-1} \sum_{j=m-1}^{k} q_{jm} p_{k-j,n-m}
\]

and

\[
q_{kn} = 0 \quad \text{if } k < n - 1 \quad \text{or } k > \binom{n}{2}.
\]

Using the generating functions

\[
p(x, y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p_{kn} x^k y^n \quad \text{and} \quad q(x, y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} q_{kn} x^k y^n,
\]

we may write the recursion in Lemma 2 as a functional equation for these formal power series (cf. Stanley [4]): setting \( p_0(x, y) = p(x, y) - 1 \), we get

\[
p_0(x, y) = q(x, y) p(x, y),
\]

whence

\[
p(x, y) = (1 - q(x, y))^{-1} = 1 + q(x, y) + q(x, y)^2 + q(x, y)^3 + \cdots
\]

and

\[
q(x, y) = 1 - p(x, y)^{-1} = p_0(x, y) - p_0(x, y)^2 + p_0(x, y)^3 - \cdots.
\]
Comparing coefficients leads to the following explicit expression for the numbers \( q_{kn} \) in terms of the numbers \( p_{kn} \) and vice versa.

**Corollary 3.**

\[
p_{kn} = \sum_{j_1 + \cdots + j_r = k, m_1 + \cdots + m_r = n} \prod_{r=1}^{\infty} q_{j_1, m_1} \quad q_{kn} = - \sum_{j_1 + \cdots + j_r = k, m_1 + \cdots + m_r = n} \prod_{r=1}^{\infty} (-p_{j_r, m_r}).
\]

Using Lemma 2 and the numerical tables for the coefficients \( p_{kn} \) given in [2], we can compute backwards the coefficients \( q_{kn} \) for \( k \leq 14 \): see Table 1. Evidently, the diagonal elements are powers of 2. Indeed, Lemma 1 provides an easy proof for this observation.

**Proposition 4.** \( q_{n, 1, n} = [2^{n-3}] \).

Here, as usual, \([x]\) denotes the least integer upper bound of \( x \). The proof is similar to that of Proposition 4', which will be given later on. More involved is the computation of the diagonal elements \( q_{nn} \). A careful distinction of several cases, combined with an iterated application of Lemma 1, leads to (cf. [3]):

**Proposition 5.** \( q_{nn} = [2^{n-5}(3n - 7)] \).

Now consider the ‘shifted’ generating function

\[
\tilde{q}(x, y) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} q_{k, k-i+1} x^k y^i.
\]

The usual method of comparing coefficients yields the following result.
Lemma 6. There is a unique formal power series \( c(x) = \sum_{k=0}^{\infty} c_k x^k \) solving the fixpoint equation
\[
\hat{q}(x, c(x)) = c(x).
\]
The coefficients \( c_k \) are obtained recursively by the following identities:
\[
c_k = \sum_{i=0}^{k} \sum_{s=0}^{k-i} c_s^{(i)} q_{k-s, k-i-s+1},
\]
\[
c_s^{(0)} = 1, \quad c_s^{(0)} = 0 \quad \text{for} \ s > 0 \quad \text{and} \quad c_s^{(i+1)} = \sum_{r=0}^{s} c_r c_{s-r}^{(i)}.
\]
Apparently, the numbers \( c_s^{(i)} \) are just the coefficients of the \( i \)th power of \( c(x) \):
\[
c(x)^i = \sum_{s=0}^{\infty} c_s^{(i)} x^s.
\]
Furthermore, the power series \( q(x, y) \) and \( \hat{q}(x, y) \) are related by the self-inverse transformations
\[
q(x, y) = \hat{q}(xy, y^{-1})y \quad \text{and} \quad \hat{q}(x, y) = q(xy, y^{-1})y.
\]
Now we can show that the coefficients \( c_k \) satisfy the desired identity (*)

Theorem 7. \( p_{kn} = \sum_{j=0}^{k} c_j p_{k-j,n-j-1} \) for \( k < n \). Thus
\[
p_{0n} = p_{0,n-1} = 1
\]
\[
p_{1n} = p_{1,n-1} + p_{0,n-2}
\]
\[
p_{2n} = p_{2,n-1} + p_{1,n-2} + p_{0,n-3}
\]
\[
p_{3n} = p_{3,n-1} + p_{2,n-2} + p_{1,n-3} + 3p_{0,n-4}
\]
\[
p_{4n} = p_{4,n-1} + p_{3,n-2} + p_{2,n-3} + 3p_{1,n-4} + 8p_{0,n-5}
\]
\[\vdots\]
Proof. Define recursively numbers \( b_{jm}^{(i)} \) by
\[
b_{jm}^{(0)} = q_{jm}, \quad b_{jm}^{(i+1)} = \sum_{s=0}^{m-i-1} c_s b_{j-s,m-s}^{(i)},
\]
and put
\[
d_{jm}^{(i)} = \sum_{r=0}^{s} b_{j-r-1}^{(r)} \quad \text{(with} \ b_{j-r}^{(r)} = 0 \ \text{for} \ r > j).\]
First one proves by a straightforward induction on \( m \) the identity
\[
b_{jm}^{(i+1)} = \sum_{s=0}^{i-1} c_s^{(m)} b_{j-s,m-s}^{(i+1)} \quad (1 \leq m \leq i+1).
Now the special choice \( m = i + 1 \) yields for \( j \leq k \) (see Lemma 6):

\[
d_j^{(k)} = d_j^{(j)} = a_{j,j+1} + \sum_{i=0}^{j-1} b_{j,j-i} = a_{j,j+1} + \sum_{i=0}^{j-1} \sum_{s=0}^{j-i-1} c_s^{(i+1)} a_{j-s,j-i-s} = \sum_{i=0}^{j-i} \sum_{s=0}^{j-i-1} c_s^{(i+1)} a_{j-s,j-i-s+1} = c_j.
\]

By induction on \( i \) and \( n > k \), we shall prove the equations

\[
\text{(E}_i) \quad p_{kn} = \sum_{j=0}^{k} d_j^{(i)} p_{k-j,n-j-1} + \sum_{m=1}^{n-1} \sum_{j=0}^{k} b_{jm}^{(i)} p_{k-j,n-m-i}.
\]

including, for \( i = k \), the claimed equation in Theorem 7.

For this, we shall use the induction hypothesis

\[
p_{rt} = \sum_{s=0}^{r} c_s p_{r-s,t-s-1} \quad \text{for } r < t < n.
\]

By Lemma 2, we have for \( k < n \) the equation

\[
\text{(E}_0) \quad p_{kn} = \sum_{j=0}^{k} q_{j,j+1} p_{k-j,n-j-1} + \sum_{m=1}^{n-1} \sum_{j=0}^{k} q_{jm} p_{k-j,n-m-i}.
\]

Now, assuming \( \text{(E}_i) \) for some \( i \), we conclude that

\[
p_{kn} = \sum_{j=0}^{k} d_j^{(i)} p_{k-j,n-j-1} + \sum_{m=1}^{n-1} \sum_{j=0}^{k} b_{jm}^{(i)} p_{k-j,n-m-i} = \sum_{j=0}^{k} \left( d_j^{(i)} + \sum_{s=0}^{j-i-1} c_s b_{j-s,j-i-s} \right) p_{k-j,n-j-1} + \sum_{m=1}^{n-1} \sum_{j=0}^{k} \sum_{s=0}^{m-1} c_s b_{j-s,m-s} p_{k-j,n-m-i-1} = \sum_{j=0}^{k} \left( d_j^{(i)} + b_{j,j+1}^{(i+1)} \right) p_{k-j,n-j-1} + \sum_{m=1}^{n-1} \sum_{j=0}^{k} b_{jm}^{(i+1)} p_{k-j,n-m-i-1},
\]

which gives \( \text{(E}_{i+1}) \). \( \square \)

In terms of generating functions, the proof would become a bit more elegant, but this would require a few extra definitions and arguments.

Notice that

\[
c_k = p_{k,k+1} - \sum_{j=0}^{k-1} c_j p_{k-j,k+j}.
\]
From Theorem 7 it follows by induction that for \( k \leq n \), the function \( p_{kn} \) is a polynomial of degree \( k \) in the variable \( n \). Moreover, concrete computation shows that \( p_{kn} \) is a sum of binomial coefficients multiplied by certain integers:

\[
p_{0n} = \binom{n-1}{0}
\]

\[
p_{1n} = \binom{n-1}{1}
\]

\[
p_{2n} = \binom{n-1}{2}
\]

\[
p_{3n} = \binom{n-1}{3} + 2\binom{n-3}{1} + \binom{n-4}{0}
\]

\[
p_{4n} = \binom{n-1}{4} + 4\binom{n-3}{2} + 4\binom{n-4}{1} + 3\binom{n-5}{0}
\]

\[
p_{5n} = \binom{n-1}{5} + 6\binom{n-3}{3} + 7\binom{n-4}{2} + 11\binom{n-5}{1} + 10\binom{n-6}{0}
\]

\[\vdots\]

\[
p_{kn} = \sum_{j=0}^{k} a_{kj} \binom{n-j-1}{k-j}
\]

where the coefficients \( a_{kj} \) can be determined recursively from the 'diagonal sequence' \( p_{kk} \) and the numbers \( c_k \) (see [3]):

\[
a_{kj} = c_{j+1} + \sum_{i=1}^{j} c_i (a_{k-i,j-i+1} - a_{k-i,j-i}) \quad \text{for} \ j < k, \ a_{kk} = p_{kk}.
\]

Table 2
The coefficients \( a_{kj} \) for \( j \leq k \leq 13 \)

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Explicitly, one obtains the following identities (see Table 2):

\[
\begin{align*}
  a_{k0} &= \binom{k-1}{0} = 1 \\
  a_{k1} &= \binom{k-1}{1} = 0 \\
  a_{k2} &= 2\binom{k-2}{1} + 0\binom{k-2}{0} = 2k - 4 \\
  a_{k3} &= 3\binom{k-3}{1} + 1\binom{k-3}{0} = 3k - 8 \\
  a_{k4} &= 4\binom{k-4}{2} + 8\binom{k-4}{1} + 3\binom{k-4}{0} = 2k^2 - 10k + 11 \\
  a_{k5} &= 12\binom{k-5}{2} + 27\binom{k-5}{1} + 10\binom{k-5}{0} = 6k^3 - 39k + 55 \\
  a_{k6} &= 8\binom{k-6}{3} + 44\binom{k-6}{2} + 82\binom{k-6}{1} + 44\binom{k-6}{0} = \frac{3}{4}k^3 - \frac{3}{2}k^2 + \frac{1}{2}k + 41 \\
  a_{k7} &= 36\binom{k-7}{3} + 152\binom{k-7}{2} + 265\binom{k-7}{1} + 168\binom{k-7}{0} = 6k^4 - 68k^2 + 271k - 455 \\
  a_{k8} &= 16\binom{k-8}{4} + 150\binom{k-8}{3} + 534\binom{k-8}{2} + 896\binom{k-8}{1} + 629\binom{k-8}{0} = \frac{3}{4}k^4 - \frac{3}{2}k^3 - \frac{1}{2}k^2 + \frac{45}{4}k - 35 \\
  a_{kj} &= \sum_{m=j/2}^\infty b_{mj}\binom{k-j}{m} \quad \text{for nonnegative integers } b_{mj} \text{ (see [3]).}
\end{align*}
\]

For \( j = 2m \), \( a_{kj} \) is a polynomial in \( k \) of degree \( m = j/2 \), with leading term \((2k)^m/m!\).

It is easy to see that

\[ q_{kn} \leq q_{k-1,n-1}, \]

and with the help of Lemma 6, it follows that the sequence \( \{c_k\} \) is monotone increasing. This together with Theorem 7 gives the inequality

\[ p_{kn} \geq p_{k,n-1} + p_{k-1,n-1}. \]

Thus we obtain the following.

**Corollary 8.** \( p_{kn} \geq \binom{n-1}{k} \) for \( k < n \). Hence the number of unlabeled posets with \( n \) points and less than \( n \) incomparable pairs is at least \( 2^{n-1} \).

The actual numbers are

\[ 1, 2, 4, 11, 32, 96, 311, 1043, 3567, \ldots \]

and this sequence seems to increase faster than \( 3^{n-2} \).

Now let us turn to a slightly different situation and consider all natural orders on \( g = \{1, \ldots, n\} \), that is, all posets \( P = (g, \leq_P) \) such that \( x \leq_P y \) implies \( x \leq y \) in
the usual order for natural numbers (cf. Avann [1]). The number of (labeled) naturally ordered sets with \( n \) points and \( k \) unrelated pairs is denoted \( P_{nk} \), and similarly \( Q_{nk} \) denotes the corresponding number of ordinally indecomposable naturally ordered sets. A straightforward inspection shows that all previous results on the numbers \( p_{kn} \) and \( q_{kn} \) have strict analogues for the numbers \( P_{kn} \) and \( Q_{kn} \), respectively. For example, the same arguments as for Lemma 2 show the following.

**Lemma 2'. For all natural numbers \( n > 0 \) and \( k \geq 0 \),**

\[
P_{kn} = Q_{kn} + \sum_{m=1}^{n-1} \sum_{j=m-1}^{k} Q_{jm} P_{k-j,n-m}
\]

and

\[
Q_{kn} = 0 \quad \text{if} \quad k < n - 1 \quad \text{or} \quad k > \binom{n}{2}.
\]

Again, this formula in connection with the table for the coefficients \( P_{kn} \) presented in [2] yields explicit values of the numbers \( Q_{kn} \) for \( k \leq 12 \) and all \( n \), see Table 3.

The diagonal of Table 3 suggests the following observation.

**Proposition 4'.** \( Q_{n-1,n} = 3^{n-2} \) for \( n > 1 \).

Avann [1] claimed to have a ‘lengthy but not complicated’ proof for this formula. Lemma 1 provides a rather succinct argument: Let \( \mathcal{S}(k, n) \) denote the set of all indecomposable naturally ordered sets with \( n \) points and \( k \) unrelated pairs. By Lemma 1, each \( P \in \mathcal{S}(n - 1, n) \) has exactly two maximal elements, \( m \) and \( n \). One of them is related to all points but one, and the other to exactly

**Table 3**

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\( n - 2 \) points (otherwise, \( P \) would be decomposable, or \( d_{P(x)} = d_{P(2)} = n - 3 < n_{P(x)} - 1 \) for \( x = m \) or \( x = n \) contradicting Lemma 1). Thus, for \( n \geq 3 \), we can construct all posets in \( \mathcal{P}(n - 1, n) \) from those in \( \mathcal{P}(n - 2, n - 1) \) as follows:

1. To each \( Q \in \mathcal{P}(n - 2, n - 1) \) add \( n \) as a new maximal element, putting it above all points of \( Q \) except one (which must be maximal). Since \( Q \) has exactly two maximal elements, this produces \( 2Q_{n-2,n-1} \) members of \( \mathcal{P}(n - 1, n) \).

2. Replace in each poset \( Q \in \mathcal{P}(n - 2, n - 1) \) the point \( n - 1 \) by \( n \), and then join \( n - 1 \) with all points except \( n \). This contributes a further amount of \( Q_{n-2,n-1} \) posets to \( \mathcal{P}(n - 1, n) \).

In this way, each poset in \( \mathcal{P}(n - 1, n) \) is obtained exactly once, either by (1) (namely if \( n \) dominates all but one point) or by (2) (namely if \( n - 1 \) dominates all points except \( n \)). Hence \( Q_{n-1,n} = 3Q_{n-2,n-1} \), and the obvious identity \( Q(1, 2) = 1 \) concludes the proof. In the same way, but with more effort, one can show the following.

**Proposition 5'**. \( Q_{nn} = [3^{n-5}(25n - 67)] \).

In complete analogy to Theorem 7, we have the following result.

**Theorem 7'**. For \( k < n \), the numbers \( P_{kn} \) satisfy an identity

\[
P_{kn} = \sum_{i=0}^{k} C_i P_{k-i,n-j-1},
\]

where the coefficients \( C_k \) define a generating function \( C(x) = \sum_{k=0}^{\infty} C_k x^k \) which is the unique solution of the functional fixpoint equation

\[
\hat{Q}(x, C(x)) = C(x).
\]

Here, as in the unlabeled case, we set

\[
\hat{Q}(x, y) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} Q_{k,k-i-1} x^i y^i,
\]

and as before, it is possible to compute recursively the coefficients \( C_k \) from the numbers \( Q_{kn} \). The first values of \( C_k \) are

1, 1, 3, 10, 39, 159, 685, 3042, 13860, 64393, 303949, 1453428, 7025982, \ldots

in accordance with the values obtained by Culberson and Rawlins [2] via a backwards computation from their table for \( P_{kn} \). In particular,

\[
\begin{align*}
P_{0n} &= P_{0,n-1} = 1 \\
P_{1n} &= P_{1,n-1} + P_{0,n-2} \\
P_{2n} &= P_{2,n-1} + P_{1,n-2} + 3P_{0,n-2} \\
P_{3n} &= P_{3,n-1} + P_{2,n-2} + 3P_{1,n-2} + 10P_{0,n-3} \\
&\vdots
\end{align*}
\]
As in the unlabeled case, for fixed \( k \), the numbers \( P_{kn} \) are polynomials of degree \( k \) in the variable \( n \) \((n \geq k)\) and may be written as certain sums of binomial coefficients multiplied by polynomials in \( k \). For example,

\[
P_{kn} = \binom{n-1}{0}
\]

\[
P_{kn} = \binom{n-1}{1}
\]

\[
P_{2n} = \binom{n-1}{2} + 2\binom{n-2}{1}
\]

\[
P_{3n} = \binom{n-1}{3} + 4\binom{n-2}{2} + 5\binom{n-3}{1} + \binom{n-4}{0}
\]

\[
P_{4n} = \binom{n-1}{4} + 6\binom{n-2}{3} + 14\binom{n-3}{2} + 19\binom{n-4}{1} + 11\binom{n-5}{0}
\]

\[
P_{5n} = \binom{n-1}{5} + 8\binom{n-2}{4} + 27\binom{n-3}{3} + 57\binom{n-4}{2} + 80\binom{n-5}{1} + 70\binom{n-6}{0}
\]

\[
P_{6n} = \binom{n-1}{6} + 10\binom{n-2}{5} + 44\binom{n-3}{4} + 123\binom{n-4}{3} + 246\binom{n-5}{2} + 374\binom{n-6}{1} + 423\binom{n-7}{0}
\]

\[\vdots\]

\[
P_{kn} = \sum_{j=0}^{k} A_{kj} \binom{n-j-1}{k-j}
\]

where \( A_{kj} \) is a polynomial of degree \( j \) in the variable \( k \) (see Table 4):

\[
A_{k0} = 1 \binom{k-0}{0}
\]

\[
A_{k1} = 2 \binom{k-1}{1} + 0 \binom{k-1}{0}
\]

\[
A_{k2} = 4 \binom{k-2}{2} + 5 \binom{k-2}{1} + 0 \binom{k-2}{0}
\]

\[
A_{k3} = 8 \binom{k-3}{3} + 20 \binom{k-3}{2} + 18 \binom{k-3}{1} + 1 \binom{k-3}{0}
\]

\[
A_{k4} = 16 \binom{k-4}{4} + 60 \binom{k-4}{3} + 97 \binom{k-4}{2} + 69 \binom{k-4}{1} + 11 \binom{k-4}{0}
\]

\[
A_{k5} = 32 \binom{k-5}{5} + 160 \binom{k-5}{4} + 366 \binom{k-5}{3} + 452 \binom{k-5}{2} + 304 \binom{k-5}{1} + 70 \binom{k-5}{0}
\]

\[\vdots\]

\[
A_{kj} = \sum_{m=0}^{j} B_{m,j} \binom{k-j}{m} \quad \text{with} \quad B_{jj} = 2^j \quad \text{and} \quad B_{j-1,j} = 5(j-1)2^{j-2} \quad (\text{cf. [3]}).
\]

A few simple manipulations of the recursion in Theorem 7' yield another recursion formula which was conjectured by Avann [1]:
Corollary 9. With the understanding that $P_{kn} = 0$ for $k < 0$,

$$P_{kn} = P_{k-6,n-1} + \sum_{j=0}^{4} P_{k-j,n-1} + \sum_{j=2}^{4} P_{k-j,n} + P_{k-4,n+1} \quad \text{for } k \leq 6 \text{ and } n > k.$$  

Unfortunately, this recursion no longer holds if $k > 6$. However, a few more computations, using Theorem 7', show how the above formula has to be adapted for larger values of $k$.

Corollary 10. With the same proviso as in Corollary 9, there exists a recursion of the following form for all $n > k$:

$$P_{kn} = P_{k-6,n-1} + \sum_{j=0}^{4} P_{k-j,n-1} + \sum_{j=2}^{4} P_{k-j,n} + P_{k-4,n+1} - \sum_{j=7}^{k} E_j P_{k-j,n-1},$$

where

$$E_7 = 3, \quad E_8 = 14, \quad E_9 = 11, \quad E_{10} = 60, \quad E_{11} = 107, \quad E_{12} = 185, \ldots.$$  

References