Discrete Mathematics 100 (1992) 229–233
North-Holland

On total covers of graphs

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Received 13 December 1990
Revised 19 August 1991

Abstract

Alavi, Y., J. Liu, J. Wang and Z. Zhang, On total covers of graphs, Discrete Mathematics 100 (1992) 229–233.

A *total cover* of a graph G is a subset of $V(G) \cup E(G)$ which covers all elements of $V(G) \cup E(G)$. The total covering number $\alpha_2(G)$ of a graph G is the minimum cardinality of a total cover in G. In [1], it is proven that $\alpha_2(G) \leqslant \lceil n/2 \rceil$ for a connected graph G of order n. Here we consider the extremal case and give some properties of connected graphs which have a total covering number $\lceil n/2 \rceil$. We prove that such a graph with even order has a 1-factor and such a graph with odd order is factor-critical.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The elements of $V(G) \cup E(G)$ are called elements of G. A vertex v of G is said to cover itself, all edges incident to v and all vertices adjacent to v. Similarly, an edge e of G covers itself, the two end vertices, and all edges adjacent to e. A set C of the elements of G is called a *total cover* if each element of G is covered by some element of C. The *total covering number* of G denoted by $\alpha_2(G)$, is the $\min\{|C| \mid C$ is a total cover of $G\}$. A total cover C of G is called minimum if $|C| = \alpha_2(G)$.

We denote the edge independence number of G by $\beta_1(G)$ and the vertex covering number of G which is the fewest number of vertices needed to cover all edges by $\alpha(G)$. Then, it follows from the definition that $\alpha_2(G) \leqslant \alpha(G)$ for any graph G without isolated vertices. In this paper, we will prove that $\alpha_2(G) \leqslant \beta_1(G) + 1$ for any connected graph and discuss some properties of a connected graph G with $\alpha_2(G) = \lceil n/2 \rceil$, where $n = |V(G)|$. The notations not defined here can be found in [2].

2. Main results

First, we state some definitions and the Gallai–Edmonds' Structure Theorem [3].

Definition 1. A graph G is *factor-critical* if $G - v$ has a 1-factor for any $v \in V(G)$.

Definition 2. A *near-perfect matching* of G is a perfect matching of $G - v$ for some $v \in V(G)$.

Definition 3. Given a graph G, we define
$D(G) = \{v \in G \mid v$ is not covered by at least one maximum matching of $G\}$,
$A(G) = \{v \in V(G) - D(G) \mid v$ is adjacent to at least one vertex in $D(G)\}$, and
$C(G) = V(G) - A(G) - D(G)$.

Gallai–Edmonds' Structure Theorem [3, p. 94]. *For any graph G, the following structure properties hold:*

(1) *The components of the subgraph induced by $D(G)$ are factor-critical.*

(2) *The subgraph induced by $C(G)$ has a perfect matching.*

(3) *If M is any maximum matching of G, then it contains a near-perfect matching of each component of $\langle D(G) \rangle$, a perfect matching of $\langle C(G) \rangle$, and matchies all vertices of $A(G)$ with vertices in distinct components in $\langle D(G) \rangle$.*

(4) $\beta_1(G) = \frac{1}{2}[|V(G)| - k(\langle D(G) \rangle) + |A(G)|]$, *where $k(G) =$ the number of components in G.*

With the above theorem, we now develop several results.

Lemma 1. *For a connected graph G, if $A(G) \neq \emptyset$, then $\alpha_2(G) \leq \beta_1(G)$.*

Proof. Suppose that G is a connected graph with $A(G) \neq \emptyset$. Since G is connected, each component of $\langle D(G) \rangle$ has a vertex which is adjacent to some vertex in $A(G)$. *Let* $D_1, D_2, \ldots, D_k$ be components of $\langle D(G) \rangle$ and $v_i \in V(D_i)$ such that v_i is adjacent to some vertex in $A(G)$ for $1 \leq i \leq k$. By Gallai–Edmonds' Structure Theorem, each $D_i - v_i$ has a perfect matching E_0 and $\langle C(G) \rangle$ has a perfect matching E_0. Now, it is clear that $Q = E_0 \cup E_1 \cup \cdots \cup E_k \cup (A(G))$ is a total cover of G. Applying Gallai–Edmonds' Structure Theorem again, we have $|Q| = \beta_1(G)$. Thus $\alpha_2(G) \leq \beta_1(G)$. $\square$

Theorem 1. *If G is a connected graph, then $\alpha_2(G) \leq \beta_1(G) + 1$.*

Proof. Let G be a connected graph. If $A(G) \neq \emptyset$, then it follows from Lemma 1 that $\alpha_2(G) \leqslant \beta_1(G)$. So we assume $A(G) = \emptyset$. Since G is connected, it follows from the definition of $D(G)$, $A(G)$, and $C(G)$ that either $D(G) = V(G)$ or $C(G) = V(G)$. By Gallai–Edmonds Structure Theorem, if $V(G) = D(G)$, then G is a factor-critical, and so $\alpha_2(G) \leqslant \beta_1(G) + 1$; if $V(G) = C(G)$, then G has a perfect matching, and so $\alpha_2(G) \leqslant \beta_1(G)$. Therefore $\alpha_2(G) \leqslant \beta_1(G) + 1$. □

As a corollary to Theorem 1, we obtain again the upper bound of $\alpha_2(G)$ derived in [1].

Corollary 1. *If G is a connected graph of order n, then $\alpha_2(G) \leqslant \lceil n/2 \rceil$.*

Proof. If n is odd, then $\beta_1(G) \leqslant (n-1)/2$. It follows from Theorem 1 that

$$\alpha_2(G) \leqslant \beta_1(G) + 1 \leqslant \frac{n-1}{2} + 1 = \left\lceil \frac{n}{2} \right\rceil.$$

Assume that n is even. If $\beta_1(G) = n/2$, then G has a perfect matching, and so $\alpha_2(G) \leqslant \beta_1(G) = n/2$. For $\beta_1(G) \leqslant n/2 - 1$, by Theorem 1, $\alpha_2(G) \leqslant \beta_1(G) + 1 \leqslant n/2$. Thus the corollary is true. □

Corollary 1 shows that $\alpha_2(G) \leqslant \lceil n/2 \rceil$ for any connected graph of order n. We now develop properties of the connected graphs for which the total covering numbers attain the upper bound.

Theorem 2. *Let G be a connected graph of odd order n. If $\alpha_2(G) = \lceil n/2 \rceil$, then G is factor-critical and $M_v \cup \{v\}$ is a minimum total cover of G for any $v \in V(G)$, where M_v is a perfect matching of $G - v$.*

Proof. Let G be a connected graph of odd order n with $\alpha_2(G) = \lceil n/2 \rceil = (n+1)/2$. Then it follows from Theorem 1 that

$$\beta_1(G) \geqslant \alpha_2(G) - 1 = \left\lceil \frac{n}{2} \right\rceil - 1 = \frac{n-1}{2}.$$

Since n is odd, $\beta_1(G) \leqslant (n-1)/2$, and so $\beta_1(G) = (n-1)/2$. We now conclude that $A(G) = \emptyset$ for otherwise Lemma 1 implies that $\alpha_2(G) \leqslant \beta_1(G) = (n-1)/2$. Since G is connected, it follows that either $V(G) = D(G)$ or $V(G) = C(G)$. Since n is odd, the Gallai–Edmonds' Structure Theorem implies that $V(G) = D(G)$ and G is factor-critical. It is now clear that $M_v \cup \{v\}$ is a minimum total cover of G for any $v \in V(G)$. This completes the proof of Theorem 2. □

Theorem 3. *Let G be a connected graph of even order n. If $\alpha_2(G) = n/2$, then G has a perfect matching.*

Proof. Let G be a connected graph of even order n with $\alpha_2(G)=n/2$. For $A(G)=\emptyset$, as in the discussion in the proof of Theorem 2, we easily see that $V(G)=C(G)$ and G has a perfect matching. If $A(G)\neq\emptyset$, then it follows from Lemma 1 that $\beta_1(G)\geqslant\alpha_2(G)=n/2$. Thus $\beta_1(G)=n/2$ and G has a perfect matching. □

Using the fact that for complete graphs $\alpha_2(K_n)=\lceil n/2\rceil$, we prove the following two results.

Theorem 4. *Let G be a connected graph of even order n with $\alpha_2(G)=n/2$. Then G has the property that every minimum total cover of G contains at most one vertex if and only if G is a complete graph.*

Proof. Let G be a connected graph of even order n with $\alpha_2(G)=n/2$. Suppose that every minimum total cover of G contains at most one vertex. Assume, to the contrary, that G is not complete. By Theorem 3, we let

$$V(G)=\{v_1, v_2, \ldots, v_{n/2}, u_1, u_2, \ldots, u_{n/2}\}$$

such that $\{u_iv_i \mid 1\leqslant i\leqslant n/2\}$ is a perfect matching of G. Since G is not complete, without loss of generality, we assume that $v_1v_2\notin E(G)$. Then

$$C=\{u_1, u_2\}\cup\left\{u_iv_i \,\middle|\, 3\leqslant i\leqslant\frac{n}{2}\right\}$$

is a minimum total cover of G which contains two vertices, contradicting the assumption. Thus G is complete. On the other hand, suppose that G is complete. Let C be any minimum total cover of G. Let

$$C_1=C\cap V(G),\qquad C_2=C\cap E(G),\qquad t_1=|C_1|,\quad\text{and}\quad t_2=|C_2|.$$

Then $t_1+t_2=n/2$ and $\beta_1(G-C_1)=t_2$. Since $G-C_1$ is a complete graph on $n-t_1$ vertices,

$$\beta_1(G-C_1)=\left\lceil\frac{n-t_1-1}{2}\right\rceil.$$

Thus

$$\frac{n}{2}=t_1+t_2=t_1+\left\lceil\frac{n-t_1-1}{2}\right\rceil=\left\lceil\frac{n+t_1-1}{2}\right\rceil.$$

This implies that $t_1\leqslant 1$, completing the proof of Theorem 4. □

Theorem 5. *Let G be a connected graph of odd order n with $\alpha_2(G)=\lceil n/2\rceil$. Then G has the property that every minimum total cover of G contains at most two vertices if and only if G is complete.*

Proof. Let G be a connected graph of odd order n with $\alpha_2(G) = \lceil n/2 \rceil = (n+1)/2$. Suppose that every minimum total cover of G contains at most two vertices. If G is not complete, then there are two non-adjacent vertices v_1 and v_2 in G. By Theorem 2, we may assume that

$$V(G) = \{u_1, u_2, \ldots, u_{(n-1)/2}, v_1, v_2, \ldots, v_{(n-1)/2}, w\}$$

such that $\{u_i v_i \mid 1 \leqslant i \leqslant (n-1)/2\}$ is a matching of G. Then

$$C = \{w, u_1, u_2\} \cup \left\{u_i v_i \,\middle|\, 3 \leqslant i \leqslant \frac{n-1}{2}\right\}$$

is a minimum total cover of G which contains three vertices, contradicting the assumption. Therefore G is complete. On the other hand, suppose G is complete. Similar to the proof of Theorem 4, we see that every minimum total cover of G contains at most two vertices. □

Acknowledgments

The authors wish to thank the referees for pointing out the result in [3] and other helpful suggestions.

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