On conjugacy of homeomorphisms of the circle possessing periodic points

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Abstract

We give a necessary and sufficient condition for topological conjugacy of homeomorphisms of the circle having periodic points. As an application we get the following theorem on the representation of homeomorphisms. The homeomorphism $F : S^1 \to S^1$ has a periodic point of period $n$ iff there exist a positive integer $q < n$ relatively prime to $n$ and a homeomorphism $\Phi : S^1 \to S^1$ such that the lift of $\Phi^{-1} \circ F \circ \Phi$ restricted to $[0, 1]$ has the form

\[
h(x) = \begin{cases} 
g(x) + \frac{1}{n}, & x \in [0, \frac{1}{n}], \\
x + \frac{q}{n}, & x \in \left[\frac{1}{n}, 1\right], \end{cases}
\]

where $g : [0, \frac{1}{n}] \mapsto [0, \frac{1}{n}]$ is an increasing homeomorphism.

Keywords: Rotation number; Lift; Periodic point; Conjugacy

1.

We begin by recalling the basic definitions and introducing some notations. Let us denote by $S^1$ the unite circle on the complex plane. Define on $S^1$ the following triple ordering relation. For any $u, w, z \in S^1$ there exist unique $t_1, t_2 \in [0, 1)$ such that $ue^{2\pi it_1} = z, we^{2\pi it_2} = u$. Define

\[u < w < z\] if and only if \[0 < t_1 < t_2\] (see [1,2]).
For any distinct elements \( u, v \in S^1 \) the sets \( (u, z) := \{ w \in S^1: u < w < z \}, (u, u) := S^1 \setminus \{ u \}, (u, z) := (u, z) \cup \{ u \} \) and \( [u, z] := (u, v) \cup \{ u \} \cup \{ z \} \) are said to be the arcs.

Let \( A \subset S^1 \) be a nonempty set. We say that a function \( F: A \to S^1 \) preserves (reverses) orientation if for any \( u, w, z \in A \) such that \( u < w < z \) we have \( F(u) < F(w) < F(z) \) (\( F(z) < F(w) < F(u) \)). Note that if \( \text{card} \ A \leq 2 \), then every bijection preserves and reserves orientation.

The continuous mappings \( F, G: S^1 \to S^1 \) are said to be conjugate if there exists a homeomorphism \( \Phi: S^1 \to S^1 \) satisfying the equation \( \Phi \circ F = G \circ \Phi \).

If, furthermore, \( \Phi \) preserves (reverses) orientation, then \( F \) and \( G \) are said to be positively (negatively) conjugate.

For every orientation-preserving homeomorphism \( F: S^1 \to S^1 \) there exists an increasing homeomorphism \( f: \mathbb{R} \to \mathbb{R} \), which is unique up to a translation by an integer, such that
\[
F(e^{2\pi ix}) = e^{2\pi if(x)}, \quad x \in \mathbb{R},
\]
and
\[
f(x + 1) = f(x) + 1, \quad x \in \mathbb{R}.
\]
The function \( f \) is called a lift of \( F \) (see, e.g., [5]).

For every orientation-preserving homeomorphism \( F: S^1 \to S^1 \) the number \( \alpha(F) \in [0, 1) \) defined by
\[
\alpha(F) = \lim_{n \to \infty} \frac{f^n(x)}{n} \pmod{1}, \quad x \in \mathbb{R},
\]
is called the rotation number of \( F \). This number always exists and does not depend on \( x \) and the choice of the lift \( f \).

It is well know (see, for instance, [5]) that a homeomorphism \( F: S^1 \to S^1 \) has a periodic point if and only if its rotation number \( \alpha(F) \) is rational. Then all periodic points have the same prime period (see [7,8]). More precisely, if \( \alpha(F) = q/n \), where \( q, n \in \mathbb{N}, n \geq 2 \) and \( \text{gcd}(q, n) = 1 \), then the set \( \text{Per}_F \) of all periodic points of \( F \) equals \( \text{Per}(F, n) \), where
\[
\text{Per}(F, n) := \{ z \in S^1: F^n(z) = z, F^k(z) \neq z, 1 \leq k < n \}.
\]
However, if \( \alpha(F) = 0 \), then \( \text{Per}_F = \{ z \in S^1: F(z) = z \} = \text{Per}(F, 1) \). Moreover, if a homeomorphism \( F \) reverses orientation, then \( \text{Per}_F = \text{Per}(F, 1) \cup \text{Per}(F, 2) \) and \( \text{card} \ \text{Per}(F, 1) = 2 \).

2.

In this section we will present some properties of homeomorphisms possessing periodic points. From now on we assume that \( F \) is an orientation-preserving homeomorphism with a rational rotation number. We begin by recalling the following result (see [8]).

**Proposition 1.** Let \( z \in \text{Per}_F \) have a prime period of order \( n \geq 2 \). If
\[
\{ z, F(z), \ldots, F^{n-1}(z) \} = \{ b_0, b_1, \ldots, b_{n-1} \},
\]
where \( b_0 = z \) and
\[
\frac{b_i}{b_0} < \frac{b_{i+1}}{b_0} < 2\pi, \quad i = 0, \ldots, n-2,
\]
then
\[ F(b_k) = b_{(k+q)(\text{mod } n)}, \]
where \( q = n\alpha(F) \).

We shall use the following well-known result of number theory.

**Lemma 2.** If \( \gcd(q, n) = 1, 1 \leq q < n \), then there exists a unique \( p \in \{1, \ldots, n - 1\} \) such that
\[ pq \pmod{n} = 1, \tag{1} \]
moreover, \( \gcd(p, n) = 1 \).

The number \( p \) defined above will be called the **conjugate number to** \( q \).

Note that every homeomorphism \( F : S^1 \to S^1 \) having periodic points of order greater than one determines uniquely the integer \( p \) which is conjugate to the integer \( q \), where \( \alpha(F) = g/n \) and \( \gcd(q, n) = 1 \). This integer \( p \) will be called the **characteristic integer of a homeomorphism** \( F \).

We shall write \( p = \text{char} \ F \).

If \( n = 1 \), then \( \text{char} \ F := 0 \).

**Lemma 3.** If \( \text{Per}(F, n) \neq \emptyset \) for an \( n \geq 2 \), then \( \text{Per}(F^p, n) \neq \emptyset \), where \( p = \text{char} \ F \) and for every \( z \in \text{Per}(F) \),
\[ \text{Arg} \frac{F^{ip}(z)}{z} < \text{Arg} \frac{F^{(i+1)p}(z)}{z}, \quad i = 0, \ldots, i - 2. \]
Moreover,
\[ F^p(b_k) = b_{(k+1)\pmod{n}}, \quad k = 0, \ldots, n - 1, \]
where \( \{b_0, \ldots, b_{n-1}\} \) are defined as in Proposition 1.

**Proof.** By Proposition 1, \( F(b_k) = b_{(k+q)\pmod{n}}, \) where \( q = n\alpha(F) \). Suppose
\[ F^i(b_k) = b_{(k+iq)\pmod{n}} \]
for \( 1 \leq i < p \). Then again by Proposition 1 we have
\[ F^{i+1}(b_k) = F(b_{(k+iq)\pmod{n}}) = b_{[(k+iq)\pmod{n}]+q\pmod{n}} = b_{(k+(i+1)q)\pmod{n}}. \]
Hence by (1) and by induction it follows that for \( i = p - 1 \) we have
\[ F^p(b_k) = b_{(k+pq)\pmod{n}} = b_{(k+1)\pmod{n}}. \]
Putting in the last formula \( k = 0 \) and again using induction we obtain
\[ F^{pj}(b_0) = b_{j\pmod{n}}, \quad j = 0, \ldots, n - 1. \]
Thus by the definition of the sequence \( \{b_j\} \) we get the remaining results. \( \square \)

Let \( I_1, I_2, I_3 \subset S^1 \). We shall write \( I_1 \prec I_2 \prec I_3 \) if \( u \prec v \prec w \) for every \( u \in I_1, v \in I_2, w \in I_3 \).

**Corollary 4.** If \( z \in \text{Per}(F, n), n \geq 2 \) and \( p = \text{char} \ F \), then the arcs \( I_k(z) := [F^{kp}(z), F^{(k+1)p}(z)] \),
\( k = 0, \ldots, n - 1 \), have the following properties:
Proof. The proof is a direct consequence of Lemma 3 and the fact that the homeomorphism $F$ preserves orientation, since $b_k = F^{k}(z)$, $k = 0, \ldots, n-1$ for $z = b_0$. \Box

Suppose that $\emptyset \neq \text{Per} F \neq S^1$. Since $\text{Per} F$ is a closed subset of $S^1$ it follows that $S^1 \setminus \text{Per} F$ is a sum of at most countable pairwise disjoint open arcs. Denote the family of these arcs by $B_F$. Put

\[ M := \{ c(I): I \in B_F \}, \]

where $c(I)$ denotes the middle point of the arc $I \subset S^1$. For any $x \in M$ put $I_x := c^{-1}([x])$.

We have

Lemma 5. If $\text{Per}(F, n) \neq \emptyset$ for $n \geq 2$, then for every $x \in M$ there exists a unique $y \in M$ such that

\[ F(I_x) = I_y, \quad F^n[I_x] = I_x \quad \text{and} \quad F^k[I_x] \neq I_x \quad \text{for} \quad 1 \leq k < n. \]

Proof. Clear. \Box

Let $\text{Per}(F, n) \neq \emptyset$ for $n \geq 2$. For $x \in M$ put

\[ O_F(x) := \{ I_x, F[I_x], \ldots, F^{n-1}[I_x] \} \quad (\text{arc orbit}). \]

Let $I_{x,k}, x \in M, k = 0, \ldots, n-1$, be the arcs defined by

\[ I_{x,0} = I_x, \quad O_F(x) = \{ I_{x,0}, I_{x,1}, \ldots, I_{x,n-1} \} \quad (2) \]

such that

\[ \text{Arg} \frac{v_i}{v_0} < \text{Arg} \frac{v_{i+1}}{v_0}, \quad \text{for} \quad v_i \in I_{x,i}, \quad i = 0, \ldots, n-1. \quad (3) \]

Note that $\{I_{x,0}, I_{x,1}, \ldots, I_{x,n-1}\}$ is a sequence of consecutive arcs. We can summarize the above results in the following:

Lemma 6. Suppose that $\text{Per}(F, n) \neq \emptyset$, $n \geq 2$, $\alpha(F) = q/n$, $\gcd(q, n) = 1$ and $p = \text{char} F$. Then for every $x \in M$ and $k = 0, \ldots, n-1$,

\[ F[I_{x,k}] = I_{x,(k+q)(\mod n)}, \quad (4) \]

\[ F^p[I_{x,k}] = I_{x,(k+1)(\mod n)}, \quad (5) \]

\[ F^{kp}[I_{x,0}] = I_{x,k}. \quad (6) \]

Moreover, if $I_{x,0} \prec I_{y,0} \prec I_{x,1}$ for some $x, y \in M$, then

\[ I_{x,k} \prec I_{y,k} \prec I_{x,(k+1)(\mod n)} \quad \text{for} \quad k = 0, \ldots, n-1. \]
Proof. Fix \( x \in \mathcal{M} \). Note that \( I_{x,0} = (z_1, z_2) \) for some \( z_1, z_2 \in \text{Per}(F, n) \). By Lemma 3 we have \( F^k(b_0) = b_{k(\text{mod } n)} \). Putting respectively \( b_0 = z_1 \) and \( b_0 = z_2 \) we get
\[
I_{x,k(\text{mod } n)} = (F^{kp}(z_1), F^{kp}(z_2)) = F^{kp}(I_{x,0}), \quad k \in \mathbb{N}.
\]
(7)

Thus we get (6). Hence
\[
F^p[I_{x,k}] = F^p[F^{kp}[I_{x,0}]] = F^{(k+1)p}[I_{x,0}] = I_{x,(k+1)(\text{mod } n)}
\]
for \( k = 0, \ldots, n - 1 \). By the first equality in (7) and Lemma 2 we obtain
\[
F[I_{x,k}] = (F(F^{kp}(z_1)), F(F^{kp}(z_2))) = (F^{kp+1}(z_1), F^{kp+1}(z_2))
\]
\[
= (F^{kp+qp}(z_1), F^{kp+qp}(z_2)) = (F^{(k+1)p}(z_1), F^{(k+1)p}(z_2))
\]
\[
= I_{x,(k+1)(\text{mod } n)}.
\]

Since \( F^p \) preserves orientation we have
\[
F^{kp}[I_{x,0}] \prec F^{kp}[I_{y,0}] \prec F^{kp}[I_{x,1}].
\]

Applying (6) and (5) we get the result. \( \square \)

Let \( A, B \subsetneq S^1 \) be closed sets and \( \text{card } A = \text{card } B \geq 2 \). We have the decompositions
\[
S^1 \setminus A = \bigcup_{x \in \mathcal{M}_A} I_x, \quad S^1 \setminus B = \bigcup_{x \in \mathcal{M}_B} J_x,
\]
(8)

where \( I_x, x \in \mathcal{M}_A \), are open pairwise disjoint arcs and similarly \( J_x, x \in \mathcal{M}_B \).

Put
\[
I_x = (a_x, b_x), \quad J_x = (c_x, d_x),
\]

where \( x \) is respectively a middle point of \( I_x \) and \( J_x \).

Lemma 7. Let \( A, B \subsetneq S^1 \) be closed sets, \( \text{card } A = \text{card } B \geq 2 \), \( \Gamma : A \to B \) be an orientation-preserving (reversing) bijection, then there exists a unique orientation-preserving (reversing) bijection \( f : \mathcal{M}_A \to \mathcal{M}_B \) such that
\[
(\Gamma(a_x), \Gamma(b_x)) = (c_{f(x)}, d_{f(x)}) \quad (\Gamma(b_x), \Gamma(a_x)) = (c_{f(x)}, d_{f(x)}), \quad x \in \mathcal{M}_A.
\]
(9)

Proof. Let us note that, if \( \Gamma \) preserves (reverses) orientation, then
\[
(\Gamma(a_x), \Gamma(b_x)) \cap B = \emptyset \quad ((\Gamma(b_x), \Gamma(a_x)) \cap B = \emptyset) \quad \text{for } x \in \mathcal{M}_A.
\]

Hence it is clear that there exists a unique bijection \( f \) such that (9) holds. It is also easy to check that \( f \) respectively preserves or reverses orientation. \( \square \)

Definition 8. A function \( f : \mathcal{M}_A \to \mathcal{M}_A \) satisfying (9) is said to be the bijection generated by \( \Gamma \). We shall write \( f = : \text{gen } \Gamma \).

We shall use the following proposition on the extension.
Proposition 9. Let $A, B \subseteq S^1$ be nonempty closed sets, $\Gamma : A \to B$ be a bijection preserving (reversing) orientation and $\Phi_x : I_x \to J_{f(x)}$, $x \in M_A$, be homeomorphisms preserving (reversing) orientation, where $f = \text{gen } \Gamma$. Then

$$\Phi(z) = \begin{cases} \Phi_x(z), & z \in I_x, \\ \Gamma(z), & z \in A, \end{cases}$$

is an orientation-preserving (reversing) homeomorphism.

**Proof.** Let us note that, if $\Gamma$ preserves (reverses) orientation and $I_{x_1} < I_{x_2} < I_{x_3}$, then $J_{f(x_1)} < J_{f(x_2)} < J_{f(x_3)}$ ($J_{f(x_3)} < J_{f(x_2)} < J_{f(x_1)}$). Now, it is not hard to verify that $\Phi$ preserves (reverses) orientation (see the method in paper ([3])). Since every orientation-preserving (reversing) bijection of the circle is a homeomorphism (see [2]) we get the result. □

3.

Now we consider the problem of the conjugacy of homeomorphisms with periodic points that is with a rational rotation number. We give a necessary and sufficient condition for the conjugacy and we determine all homeomorphisms $\Phi : S^1 \to S^1$ such that

$$\Phi(F(z)) = G(\Phi(z)), \quad z \in S^1,$$

where $F$ and $G$ are given homeomorphisms. The problem of conjugacy of homeomorphisms with an irrational rotation number has been investigated in paper [4].

We know that if homeomorphisms $F$ and $G$ are positively conjugate, then they have the same rotation number $\alpha(F)$ (see [5]) and consequently they have the same prime period and the same characteristic integer. However, if $F$ and $G$ are negatively conjugate, then $\alpha(F) = 1 - \alpha(G)$ (see [4]) and it is readily to verify that $\text{char } F = n - \text{char } G$, where $n$ is a prime period of $F$.

Let us start from a few simple properties of conjugating functions.

**Lemma 10.** If a bijection $\Phi : S^1 \to S^1$ satisfies (10), then $\Phi[\text{Per } F] = \text{Per } G$.

**Proof.** If $x \in \text{Per } F$, then $F^n(x) = x$ for $n \in \mathbb{N}$. Hence

$$\Phi(x) = \Phi(F^n(x)) = G^n(\Phi(x)).$$

Thus $\Phi(x) \in \text{Per } G$.

Conversely, if $x \in \text{Per } G$, then $G^n(x) = x$ for $n \in \mathbb{N}$ and $x = \Phi(y)$ for $y \in S^1$. Hence $\Phi(F^n(y)) = G^n(\Phi(y)) = G^n(x) = x = \Phi(y)$. Thus $F^n(y) = y$ that is $y \in \text{Per } F$. □

Put in (8) $A = \text{Per } F$ and $B = \text{Per } G$. Then we have the following

**Lemma 11.** If a homeomorphism $\Phi : S^1 \to S^1$ satisfies (10), $\text{Per}(F, n) \neq \emptyset$ and $f = \text{gen } \Phi|_{\text{Per } F}$, then

$$\Phi[F^k[I_x]] = G^k[J_{f(x)}], \quad x \in M, \; k \in \mathbb{N}.$$  \hspace{1cm} (11)

If $\Phi$ preserves orientation, then

$$\Phi[I_{x,k}] = J_{f(x),k}, \quad k = 0, \ldots, n - 1,$$  \hspace{1cm} (12)

and if $\Phi$ reverses orientation, then
\[ \Phi[I_{x,k}] = J_{f(x),n-k}, \quad k = 0, \ldots, n - 1. \]  

**Proof.** According to Lemma 10 we have
\[
\bigcup_{x \in \mathcal{M}} \Phi[I_x] = \Phi\left[ \bigcup_{x \in \mathcal{M}} I_x \right] = \bigcup_{y \in \mathcal{M}'} J_y.
\]
Thus for every \( x \in \mathcal{M} \) there exists \( y \in \mathcal{M}' \) such that \( J_y = \Phi[I_x] \). Further by (9) we get
\[ \Phi[I_x] = J_{f(x)}. \]  

(14)

Hence by (6) and (10),
\[ \Phi[I_{x,k}] = G^{kp}[J_{f(x)}] = J_{f(x),n-k}, \]
where \( p := \text{char } F \).

If \( \Phi \) preserves orientation, then \( F \) and \( G \) have the same characteristic number \( p \). Hence, according to (6) we get
\[ G^{kp}[J_{f(x)}] = J_{f(x),n-k}, \]
and (12) is proved.

If \( \Phi \) reverses orientation, then \( p := \text{char } G = n - p \). Again by (6) with respect to the function \( G \) we have
\[ G^{kp}[J_{f(x)}] = G^{(n-k)p}[J_{f(x)}] = J_{f(x),n-k}, \]
since \( G^n[J_{f(x)}] = J_{f(x)} \). This gives (13). \( \square \)

**Remark 12.** Suppose that homeomorphisms \( F, G \) and a bijection \( \Phi \) satisfy (10), \( \text{Per}(F,n) \neq \emptyset \) and \( \Gamma := \Phi|_{\text{Per} F} \), then
\[ F[\text{Per } F] = \text{Per } F, \quad G[\text{Per } G] = \text{Per } G, \quad \Gamma[\text{Per } F] = \text{Per } G \]
and
\[ \Gamma \circ F(z) = G \circ \Gamma(z), \quad z \in \text{Per } F, \]
\[ F^n(z) = z, \quad z \in \text{Per } F, \]
\[ G^n(z) = z, \quad z \in \text{Per } G. \]  

(15)

**Proof.** Clear. \( \square \)

**Remark 13.** If \( a \in \text{Per } F \neq \emptyset, p = \text{char } F \) and an orientation-preserving bijection \( \Gamma \) satisfies (15), then
\[ \Gamma[\text{Per } F \cap [a, F^p(a)]] = \text{Per } G \cap [b, G^p(b)] \]
for \( b = \Gamma(a) \).

If \( \Phi \) reverses orientation, then \( p := \text{char } G = n - p \) and
\[ \Gamma[\text{Per } F \cap [a, F^p(a)]] = \text{Per } G \cap [b, G^{p}(b)] \]
for \( b = G^p(\Gamma(a)) \).
**Proof.** Clear. □

The following lemma shows that the simple above property plays a crucial role in the conjugacy problem.

**Lemma 14.** Let $\text{Per} F \neq \emptyset$, $p := \text{char} F$, $a \in \text{Per} F$, $b \in \text{Per} G$, $\alpha(F) = \alpha(G)$ and

$$\Gamma_0 : \text{Per} F \cap [a, F^p(a)] \rightarrow \text{Per} G \cap [b, G^p(b)]$$

be an orientation-preserving bijection, then there exists a unique mapping $\Gamma : \text{Per} F \rightarrow \text{Per} G$ satisfying (15) such that $\Gamma_0 = \Gamma | [a, F^p(a)]$. $\Gamma$ is an orientation-preserving homeomorphism.

If $\alpha(F) = 1 - \alpha(G)$ and

$$\Gamma_0 : \text{Per} F \cap [a, F^p(a)] \rightarrow \text{Per} G \cap [b, G^{n-p}(b)]$$

be an orientation-reversing bijection, then there exists a unique $\Gamma : \text{Per} F \rightarrow \text{Per} G$ satisfying (15) such that $\Gamma_0 = \Gamma | [a, F^p(a)]$. $\Gamma$ is an orientation-reversing homeomorphism.

**Proof.** If $\text{Per} F = \text{Per}(F, 1)$, then result is trivial, since then $p = 0$ and $[a, a] = S^1$. Further, assume that $\text{Per}(F, n) \neq \emptyset$ for $n \geq 2$ and $\alpha(F) = \alpha(G)$. Put

$$A_i := \text{Per} F \cap [F^i p(a), F^{(i+1)p}(a)]$$

and define $\Gamma_i : A_i \rightarrow S^1$ by the formula

$$\Gamma_i(z) := G^{ip} \circ \Gamma_0 \circ F^{-ip}(z), \quad z \in A_i, \ i = 0, \ldots, n - 1. \quad (16)$$

Set $\Gamma := \bigcup_{i=0}^{n-1} \Gamma_i$, that is

$$\Gamma(z) := \Gamma_i(z), \quad z \in A_i. \quad (17)$$

It is easy to see that $\Gamma$ preserves orientation and $\Gamma[\text{Per} F] = \text{Per} G$. Hence $\Gamma$ is a homeomorphism (see [8]).

Since $F^p[A_i] = A_{i+1}, i = 0, \ldots, n - 2$, we have for $z \in A_i$,

$$\Gamma \circ F^p(z) = \Gamma_{i+1} \circ F^p(z) = G^{(i+1)p} \circ \Gamma_0 \circ F^{-ip-p} \circ F^p(z)$$

Similarly, from $F^p[A_{n-1}] = A_0$ it follows that

$$\Gamma \circ F^p(z) = \Gamma_0 \circ F^p(z) \quad \text{for} \ z \in A_{n-1}. \quad (18)$$

Moreover,

$$G^p \circ \Gamma(z) = G^p \circ \Gamma_{n-1}(z) = G^p \circ G^{(n-1)p} \circ \Gamma_0 \circ F^{-(n-1)p}(z) = G^p \circ \Gamma_0 \circ F^{-np} \circ F^p(z) = \Gamma_0 \circ F^p(z)$$

for $z \in A_{n-1}$, since $G^{np}(u) = u$ for $u \in \text{Per} G$ and $F^{np}(u) = u$ for $u \in \text{Per} F$.

Let $q$ be such that $pq(\mod n) = 1$, then $F^{pq}(z) = F(z)$ for $z \in \text{Per} F$, $G^{pq}(z) = G(z)$ for $z \in \text{Per} G$ and

$$\Gamma(F^{pq}(z)) = G^{pq}(\Gamma(z)).$$
Hence we get (15).

The uniqueness is clear, since (15) implies that $\Gamma(F^{ip}(z)) = G^{ip}(\Gamma(z))$ for $z \in \text{Per} F$. Thus $\Gamma$ is given by (16) and (17).

In the case $\alpha(F) = 1 - \alpha(G)$ the proof is analogous. □

To introduce the next definition we quote the following result (see [8]).

**Proposition 15.** If $\text{Per}(F, n) \neq \emptyset$ and $z_0 \in I_x$ for $x \in \mathcal{M}$, then $F^n(z_0) \in I_x$ and either for every $z \in I_x$, $(z, F^n(z)) \subset I_x$ or for every $z \in I_x$, $(F^n(z), z) \subset I_x$.

Proposition 15 lets us to define the following mapping $\xi_F : \mathcal{M} \to \{ -1, 1 \}$:

$$\xi_F(x) := \begin{cases} -1, & \text{if } (F^n(z), z) \subset I_x \text{ for } z \in I_x, \\ 1, & \text{if } (z, F^n(z)) \subset I_x \text{ for } z \in I_x. \end{cases}$$

**Lemma 16.** Suppose that homeomorphisms $F, G, \Phi$ satisfy (10) and $\text{Per} F \neq \emptyset$. If there exists an orientation-preserving (reversing) homeomorphism $\Phi$ satisfying (10), then

$$\xi_F(x) = \xi_G(f(x)) \quad \text{and} \quad \xi_F(x) = -\xi_G(f(x)),$$

$x \in \mathcal{M}_F$,

where $f := \text{gen } \Phi|_{\text{Per} F}$.

**Proof.** We have $\Phi[I_x] = J_{f(x)}$, $x \in \mathcal{M}$. If $\xi_F(x) = -1$, then $(F^n(z), z) \subset I_x$, for $z \in I_x$. If $\Phi$ preserves orientation

$$J_{f(x)} = \Phi[I_x] \supset \Phi[(F^n(z), z)] = (\Phi(F^n(z)), \Phi(z)) = (G^n(\Phi(z)), \Phi(z)).$$

Thus $(G^n(u), u) \subset J_{f(x)}$, for $u \in J_{f(x)}$ and consequently $\xi_G(f(x)) = -1$. Similarly we verify in the case $\xi_F(x) = 1$ and an orientation-reversing $\Phi$. □

Now we give a necessary and sufficient condition for the conjugacy of homeomorphisms with periodic points. In the proof of this theorem we give a general construction of all homeomorphisms conjugating given functions, that is satisfying functional equation (10). First we introduce the following notation.

Let $a \in \text{Per} F$ and $p = \text{char } F$. Then put

$$\mathcal{M}_a := \mathcal{M} \cap [a, F^p(a)].$$

**Theorem 17.** Let $F$ and $G$ be homeomorphisms such that $\text{Per} F \neq \emptyset$ and $\text{Per} G \neq \emptyset$.

The mappings $F$ and $G$ are positively conjugate if and only if $\alpha(F) = \alpha(G)$ and there exist $a \in \text{Per} F$, $b \in \text{Per} G$ and an orientation-preserving bijection

$$\Gamma_0 : \text{Per } F \cap [a, F^p(a)] \to \text{Per } G \cap [b, G^p(b)],$$

such that

$$\xi_F(x) = \xi_G(\text{gen } \Gamma_0(x)), \quad x \in \mathcal{M}_a, \quad (18)$$

where $p = \text{char } F$.

The mappings $F$ and $G$ are negatively conjugate if and only if $\alpha(F) = 1 - \alpha(G)$ and there exist $a \in \text{Per} F$, $b \in \text{Per} G$ and an orientation-preserving bijection

$$\Gamma_0 : \text{Per } F \cap [a, F^p(a)] \to \text{Per } G \cap [b, G^p(b)],$$
such that
\[ \xi_F(x) = -\xi_G(\text{gen } \Gamma_0(x)), \quad x \in \mathcal{M}_a, \]
where \( \overline{p} = \text{char } G \).

**Proof.** The necessary condition follows directly from Lemmas 10 and 16 and Remark 13. The mapping \( \Gamma_0 \) we define as a restriction of the homeomorphism \( \Phi \) satisfying (10) to the set \( \{a, F^p(a)\} \). If \( \Phi \) preserves orientation then (18) holds but if \( \Phi \) reverses orientation then (19) holds.

Conversely, we shall show that there exists a homeomorphic solution of (10) and it depends on an arbitrary homeomorphism defined on an arc.

Assume that \( \alpha(F) = \alpha(G) \). By Lemma 14 there exists a unique orientation-preserving homeomorphism \( \Gamma : \text{Per}_F \rightarrow \text{Per}_G \) satisfying (15) such that \( \Gamma_0 = \Gamma|_{[a,F^p(a)]} \).

Put
\[ f := \text{gen } F. \]
We have
\[ \text{Per } F = \text{Per } (F, n) \quad \text{and} \quad \text{Per } G = \text{Per } (G, n), \]
where \( \alpha(F) = q/n \) and \( \gcd(q,n) = 1 \).

Let us assume that (18) holds and put in (8) \( A = \text{Per } F \) and \( B = \text{Per } G \). For every \( x \in \mathcal{M} \cap [a, F^p(a)] = \mathcal{M}_a \), where \( p = \text{char } F \) define an orientation-preserving homeomorphism \( \Phi_{x,0} : I_x \rightarrow J_f(x) \) such that
\[ \Phi_{x,0}(F^n(z)) = G^n(\Phi_{x,0}(z)), \quad z \in I_x,0 := I_x. \]
(20)
We can construct such homeomorphisms by the classical method of successive extensions. In fact, assume \( \xi_F(x) = 1 \). Since \( F^n[I_x] = I_x, G^n[J_f(x)] = J_f(x) \) as well as \( F^n(z) \neq z \) for \( z \in \text{Int } I_x \) and \( G^n(z) \neq z \) for \( z \in \text{Int } J_f(x) \) we infer that for every \( z_x \in I_x \) and every homeomorphism \( \Psi_x : [z_x,F^n(z_x)] \rightarrow J_f(x) \) such that
\[ \Psi_x[F^n(z_x)] = G^n(\Psi_x(z_x)) \]
(21)
there exists a unique homeomorphism \( \Phi_{x,0} : I_x \rightarrow J_f(x) \) satisfying (20) and
\[ \Psi_x = \Phi_{x,0} |_{[z_x,F^n(z_x)]} \]
(see [6, p. 70]). It follows by (18) and (21) that \( \Phi_{x,0} \) preserves orientation. Similarly, we have in the case \( \xi(x) = -1 \). Note that if \( \alpha(F) = 1 - \alpha(G) \) and (19) holds then \( \Phi_{x,0} \) reverses orientation.

The case \( n = 1 \) is trivial and we get the result directly by Proposition 9. Suppose now \( n \geq 2 \). By Lemma 6 we have
\[ F^m[I_{x,k}] = I_{x,(k+mq)(\text{mod } n)}, \quad G^m[J_{x,k}] = J_{x,(k+mq)(\text{mod } n)}, \]
(22)
where \( I_{x,k} \) for \( k = 0, 1, \ldots, n-1 \) are defined by (2) and (3).

Define the mappings
\[ \Phi_{x,k} : I_{x,kq}(\text{mod } n) \rightarrow J_{f(x),kq}(\text{mod } n) \]
by the following formula
\[ \Phi_{x,k}(z) := G^{k-n} \circ \Phi_{x,0} \circ F^{n-k}(z), \quad z \in I_{x,kq}(\text{mod } n), \]
(23)
and put
\[
\Phi(z) := \begin{cases} 
\Phi_{x,k}(z), & z \in I_{x,kq({\text{mod}} \ n)}, \ x \in M_a, \\
\Gamma(z), & z \in \text{Per } F.
\end{cases}
\] (24)

We show that \(\Phi\) is a homeomorphism. For this purpose, we shall show the following equality
\[
\bigcup_{x \in M_a} \bigcup_{k=0}^{n-1} I_{x,kq({\text{mod}} \ n)} = S^1 \setminus \text{Per } F.
\] (25)

Suppose \(y \in M\). By Corollary 4(i) there exists \(k \in \{0, 1, \ldots, n-1\}\) such that \(I_y \subset I_k(a) := \left[ F^{kp}(a), F^{(k+1)p}(a) \right]\). On the other hand, by Lemma 5 there exists \(x \in M\) such that \(F^{(n-k)p}[I_y] = I_x\). Further, in view of Corollary 4(iv) we have
\[
I_x \subset F^{(n-k)p}[I_k(a)] = F^{(n-k)p}[F^{kp}[I_0(a)]] = F^{np}[I_0(a)] = I_0(a).
\]
Hence we infer that \(x \in M_a\) and \(I_y = F^{kp}[I_x]\). By Lemma 6, \(F^{kp}[I_x] = I_{x,k({\text{mod}} \ n)}\).

Summarizing we obtain the equality
\[
\bigcup_{y \in M} I_y = \bigcup_{x \in M_a} \bigcup_{k=0}^{n-1} I_{x,k}.
\]

Since
\[
\bigcup_{y \in M} I_y = S^1 \setminus \text{Per } F
\]
and the mapping \(k \mapsto kq({\text{mod}} \ n)\) is a bijection of the set \(\{0, 1, \ldots, n-1\}\) onto itself we get (25).

Next, we conclude from (22) and (23) that for \(x \in M_a\) and \(k = 0, \ldots, n-1\),
\[
\Phi_{x,k}[I_{x,kq({\text{mod}} \ n)}] = G^{k-n} \circ \Phi_{x,0} \circ F^{n-k}[I_{x,kq({\text{mod}} \ n)}]
= G^{k-n} \left[ \Phi_{x,0}[I_{x,[kq({\text{mod}} \ n)+(n-k)q)({\text{mod}} \ n)}] \right]
= G^{k-n} \left[ \Phi_{x,0}[I_{x,0}] \right] = G^{k-n} [J_{f(x),0}]
= J_{f(x),(k-n)q({\text{mod}} \ n)} = J_{f(x),kq({\text{mod}} \ n)}.
\]
Hence it follows by Proposition 9 and (25) that \(\Phi\) is an orientation-preserving homeomorphism.

If (19) holds then the same reasoning shows that \(\Phi\) is an orientation-reversing homeomorphism.

Finally we check that \(\Phi\) given by (24) satisfies (10). Fix \(x \in M_a\) and \(0 \leq k \leq n-1\). Suppose \(z \in I_{x,kq({\text{mod}} \ n)}\). Then by (22),
\[
F(z) \in F[I_{x,kq({\text{mod}} \ n)}] = I_{x,(k+1)q({\text{mod}} \ n)}.
\]
This relation together with (23) and (24) imply that for \(0 \leq k < n-1\),
\[
\Phi(F(z)) = \Phi_{x,k+1}(F(z)) = G^{k+1-n} \circ \Phi_{x,0} \circ F^{n-k-1}(F(z))
= G^{k+1-n} \circ \Phi_{x,0} \circ F^{n-k}(z) = G \circ G^{k-n} \circ \Phi_{x,0} \circ F^{n-k}(z)
= G \circ \Phi_{x,k}(z) = G(\Phi(z)).
\]
If \( k = n - 1 \), then \( F(z) \in I_{x,0} \) and again by (23) and (24) we have

\[
G(\Phi(z)) = G \circ \Phi_{x,n-1}(z) = G \circ G^{n-1-n} \circ \Phi_{x,0} \circ F^{n+1-n}(z) = \Phi_{x,0} \circ F(z)
\]

= \( \Phi(F(z)) \).

By Lemma 14 the function \( F \) satisfies (15), so by (25) and (24) \( \Phi \) fulfills (10) on \( S^1 \). It is easily to see from Lemmas 11 and 6 and (22) that every orientation-preserving homeomorphic solution \( \Phi \) of (10) is given by (24) and (23).

The same proof remains valid for the case where \( \alpha(F) = 1 - \alpha(G) \) and (19) holds. The only difference is that the mappings \( \Phi_{x,k} : I_{x,kq(mod\ n)} \to J_{x,(k+m(n-q))(mod\ n)} \) and in formula (22) should be \( G^m[J_{x,k}] = J_{x,(k+m(n-q))(mod\ n)} \).

4.

In this section we determine the form of all homeomorphisms of the circle possessing periodic points. We establish also the strict relation between homeomorphisms of the circle with periodic points and homeomorphisms of real intervals. For this purpose we prove two lemmas.

First, let us introduce the following notation \( e_k := e^{\frac{2\pi ik}{n}}, k = 0, \ldots, n - 1 \).

**Lemma 18.** Let \( \alpha(F) = q/n, \gcd(q,n) = 1 \). Then for every \( a \in \text{Per} F \) there exist orientation-preserving homeomorphisms \( G : S^1 \to S^1 \) and \( \Phi : S^1 \to S^1 \) such that \( \Phi(a) = e_0 \),

\[
G(e_k) = e_{k+1}, \quad k = 0, 1, \ldots, n - 2, \quad G(e_{n-1}) = e_0
\]

and

\[
\Phi \circ F = G \circ \Phi.
\]

**Proof.** Set \( p = \text{char } F \). Let

\[
\Phi_k : [F^{pk}(a), F^{p(k+1)}(a)] \to [e_{kp(mod\ n)}, e_{(k+1)p(mod\ n)}]
\]

for \( k = 0, \ldots, n - 1 \) be arbitrary orientation-preserving homeomorphisms. Define \( \Phi := \bigcup_{i=0}^{n-1} \Phi_i \), that is,

\[
\Phi(z) := \Phi_k(z), \quad z \in [F^{pk}(a), F^{p(k+1)}(a)], \quad k = 0, \ldots, n - 1.
\]

Proposition 9 shows that \( \Phi \) is an orientation-preserving homeomorphism (here \( A := \{a, F(a), \ldots, F^{n-1}(a)\}, B := \{e_0, e_1, \ldots, e_{n-1}\} \) and \( \Gamma(F^i(a)) := e_i \)). We define the homeomorphism \( G \) by the following formula:

\[
G := \Phi \circ F \circ \Phi^{-1}.
\]

Let us note that

\[
\Phi(F^k(a)) = e_k, \quad k = 0, \ldots, n - 1.
\]

and for \( k = 0, 1, \ldots, n - 2 \),

\[
G(e_k) = \Phi \circ F \circ \Phi^{-1}(e_k) = \Phi \circ F \circ F^k(a) = \Phi(F^{k+1}(a)) = e_{k+1}
\]
and
\[ G(e_{n-1}) = \Phi \circ F \circ \Phi^{-1}(e_{n-1}) = \Phi \circ F \circ F^{n-1}(a) = \Phi(F^n(a)) = \Phi(a) = e_0. \]

**Remark 19.** Formulas (26) and (27) determine all orientation-preserving homeomorphisms \( \Phi \) such that \( \Phi(F^k(a)) = e_k, k = 0, \ldots, n-1 \).

**Proof.** Clear. \( \Box \)

**Lemma 20.** Let \( \alpha(G) = \alpha(T), n \geq 2 \) be a prime period of \( G, T^n = \text{id}, p = \text{char} G \), then for every \( b \in \text{Per}\, G \) such that \( G^k(b) = T^k(b), k = 0, \ldots, n-1 \),
\[ H(z) := \begin{cases} T \circ G^n(z), & z \in [b, T^p(b)], \\ T(z), & z \in S^1 \setminus [b, T^p(b)], \end{cases} \]
(29)
is a homeomorphism conjugated to \( G \).

**Proof.** Let us note that \( G^m(b) = T^m(b) \) for every integer \( m \). Put
\[ L_k := \frac{G^{kp}(b), G^{(k+1)p}(b)}{[T^{kp}(b), T^{(k+1)p}(b)]}. \]
It follows by Proposition 1 and Lemma 3 that
\[ T[L_k] = L_{(k+q)(\text{mod} \, n)}, \]
(30)
where \( q = \alpha(G)n \). Moreover, \( G^n[L_k] = L_k, k = 0, \ldots, n-1 \), so
\[ T \circ G^n[L_k] = L_q \]
and
\[ T[S^1 \setminus L_0] = S^1 \setminus L_q. \]
Hence by Proposition 9 we infer that \( H \) defined by (29) is a homeomorphism which preserves orientation.

We shall show that
\[ H^k(z) = T^k(G^n(z)), \quad z \in L_0, \quad k = 1, \ldots, n. \]
(31)
For this purpose observe that the following equalities
\[ H^i[L_0] = L_{iq(\text{mod} \, n)}, \quad i = 1, \ldots, n, \]
(32)
hold. In fact,
\[ H[L_0] = T[G^n[L_0]] = T[L_0] = L_q \]
and
\[ L_{iq(\text{mod} \, n)} \subset S^1 \setminus L_0 \quad \text{for} \quad i = 1, \ldots, n-1, \]
since \( \gcd(q, n) = 1 \).

Further, from (29) and (30) we get by induction
\[ H^i[L_0] = H[H^{i-1}[L_0]] = H[L_{(i-1)q(\text{mod} \, n)}] = T[L_{(i-1)q(\text{mod} \, n)}] = L_{iq(\text{mod} \, n)} \]
for \( i = 2, \ldots, n. \)
Now, suppose that $z \in L_0$. Then $H(z) \in L_q$. However by (32),
\[ H^i(z) \in S^1 \setminus L_0, \quad i = 1, \ldots, n - 1. \]
Hence, according to (29) we get
\[
H^k(z) = H(H^{k-1}(z)) = T(H^{k-2}(z)) = T^2(H^{k-2}(z)) = \cdots = T^{k-1}(H(z)) = T^{k-1}(T(G^n(z))) = T^k(G^n(z))
\]
for $2 \leq k \leq n$.
Putting in (31) $z = b$ we have
\[ H^k(b) = T^k(b), \quad k = 0, \ldots, n - 1, \]
thus $b$ is a periodic point of $H$ of order $n$ and consequently $\alpha(H) = \bar{q}/n$ for a $1 \leq \bar{q} \leq n$.
Let $\{b_0, \ldots, b_{n-1}\} := \{b, H(b), \ldots, H^{n-1}(b)\} = \{b, T(b), \ldots, T^{n-1}(b)\}$ be defined in the following way: $b_0 = b$ and
\[
\frac{\text{Arg } b_i}{b_0} < \frac{\text{Arg } b_{i+1}}{b_0} < 2\pi, \quad i = 0, \ldots, n - 2.
\]
In view of Proposition 1, $b_{\bar{q}} = H(b_0) = T(b_0) = b_q$. Thus $q = \bar{q}$, so $\alpha(H) = \alpha(T) = \alpha(G)$.
If, on the other hand, in (31) we set $k = p$ and $z = b$, then we obtain
\[ H^p(b) = T^p(G^n(b)) = T^p(b) = G^p(b). \tag{33} \]
Similarly, putting in (31) $k = n$ we get
\[ H^n(z) = T^n(G^n(z)) = G^n(z), \quad z \in L_0. \tag{34} \]
From this and (33) it is easy to see that
\[ \text{Per } H \cap [b, H^p(b)] = \text{Per } G \cap [b, G^p(b)]. \]
Define
\[ \Gamma_0(z) := z \quad \text{for } z \in \text{Per } H \cap [b, H^p(b)]. \]
Note that
\[ \text{gen } \Gamma_0(x) = x, \quad x \in M_b, \]
and we have $I_x \subseteq L_0$ for $x \in M_b = M \cap [b, H^p(b)]$. Hence by (34),
\[ \xi_H(x) = \xi_G(x), \quad x \in M_b. \]
Thus the assumptions of Theorem 17 are fulfilled. This clearly proves that $H$ and $G$ are conjugate.

Now we shall show the following main theorem.

**Theorem 21.** If $F : S^1 \to S^1$ is a homeomorphism, $\alpha(F) = q/n, n \geq 2, \gcd(q, n) = 1$, then $F$ is conjugate to a homeomorphism which lifting restricted to $[0, 1]$ has the following form
\[ h(x) = \begin{cases} g(x) + \frac{q}{n}, & x \in [0, \frac{1}{n}], \\ x + \frac{q}{n}, & x \in [\frac{1}{n}, 1]. \end{cases} \tag{35} \]
where $g : [0, \frac{1}{n}] \to [0, \frac{1}{n}]$ is an increasing homeomorphism.
Proof. Put \( \varepsilon_k = e^{\frac{2\pi i k}{n}} \). For a given \( a \in \text{Per } F \) let \( G \) be a homeomorphism determined in Lemma 18 and \( H \) be given by (29) where \( b = \varepsilon_0 = 1 \) and

\[
T(z) := e^{2\pi i \frac{q}{n} z} = \varepsilon_q z, \quad z \in S^1.
\]

It follows by Lemma 18 that \( F \) and \( G \) are conjugate. On the other hand, Lemma 20 implies that \( G \) and \( H \) are conjugate. Consequently \( F \) and \( H \) are conjugate. Let \( h \) be the lift of \( H \), that is,

\[
H(e^{2\pi i t}) = e^{2\pi i h(t)}, \quad h(t + 1) = h(t) + 1
\]

and \( 0 \leq h(0) < 1 \). Since \( H(1) = T(1) = e^{2\pi i \frac{q}{n}} = \varepsilon_q \) we have \( h(0) = q/n \).

From (29),

\[
H(z) = T(z) = \varepsilon_q z \quad \text{for} \quad z \in [\varepsilon_1, \varepsilon_0].
\]

This gives \( e^{2\pi i \left( \frac{q}{n} + t \right)} = H(e^{2\pi i t}) = e^{2\pi i h(t)} \) for \( t \in [\frac{1}{n}, 1] \), so we have \( h(x) = x + q/n \) for \( x \in [\frac{1}{n}, 1] \).

On the other hand, from (29)

\[
H(z) = T(G^n(z)) \quad \text{for} \quad z \in [\varepsilon_0, \varepsilon_1]
\]

and

\[
T \circ G^n(\varepsilon_0) = T(\varepsilon_0) = \varepsilon_q.
\]

From this we conclude that the lift \( h \) of \( H \) restricted to \([0, 1]\) is of the form (35), where \( g + q/n \) is the lift of \( T \circ G^n \) restricted to \([\varepsilon_0, \varepsilon_1] = [1, e^{2\pi i/n}] \) and such that \( g(0) = 0 \). \( \square \)

References