Journal of Computational and Applied Mathematics 223 (2009) 567–577

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Skew-symmetric methods for nonsymmetric linear systems with multiple right-hand sides ☆

Chuanqing Gu *, Hongjun Qian

Department of Mathematics, Shanghai University, Shanghai 200444, China

ARTICLE INFO

Article history:
Received 11 April 2007
Received in revised form 27 July 2007

Keywords:
Global Arnoldi algorithm
Matrix Krylov subspace
Nonsymmetric linear systems
Skew-symmetric methods
Multiple right-hand sides

ABSTRACT

By transforming nonsymmetric linear systems to the extended skew-symmetric ones, we present the skew-symmetric methods for solving nonsymmetric linear systems with multiple right-hand sides. These methods are based on the block and global Arnoldi algorithm which is formed by implementing orthogonal projections of the initial matrix residual onto a matrix Krylov subspace. The algorithms avoid the tediously long Arnoldi process and highly reduce expensive storage. Numerical experiments show that these algorithms are effective and give better practical performances than global GMRES for solving nonsymmetric linear systems with multiple right-hand sides.

1. Introduction

Consider solving nonsymmetric linear systems with the same coefficient matrix and different right-hand sides

$$Ax^{(i)} = b^{(i)} \quad i = 1, \dots, s. \tag{1.1}$$

When all the $b^{(i)}$ are available simultaneously, Eq. (1.1) can be written as the matrix equation

$$AX = B, \tag{1.2}$$

where A is an $N \times N$ real nonsymmetric matrix, B and X are $N \times s$ rectangular matrices whose columns are $b^{(1)}, b^{(2)}, \dots, b^{(s)}$ and $x^{(1)}, x^{(2)}, \dots, x^{(s)}$, respectively. In practice, s is of moderate size $s \ll N$.

For nonsymmetric linear systems, some block Krylov subspace methods have been developed during the last years. The well-known works include the global FOM and global GMRES algorithms [2], the block biconjugate gradient (Bl-BCG) [4], the block generalized minimal residual (BGMRES) algorithm [6], the block quasi-minimum residual (Bl-QMR) algorithm [1,3], etc. Recently, the left conjugate direction (LCD) method [7] was presented for solving the matrix equation (1.2). The method reduces to the usual CG-type method when A is symmetric positive definite.

Let $\mathbb{E} = M_{N,s}$ denote the vector space on the filed $\mathbf{R}$, of rectangular matrices of dimension $N \times s$. For $X, Y \in \mathbb{E}$, the inner product is defined by $(X, Y)_F = \mathrm{tr}(X^T Y)$, where $\mathrm{tr}(Z)$ denotes the trace of the square matrix Z and Z^T denotes the transpose of the matrix Z. The associated norm is the well-known Frobenius norm denoted by $\|.\|_F$. Let V be an $N \times s$ rectangular matrix and the matrix Krylov subspace $K_m(A, V) = \mathrm{span}\{V, AV, \dots, A^{m-1}V\}$. A system of vectors of $\mathbb{E}$ is said to be F-orthogonal if it is orthogonal with respect to the scalar product $(., .)_F$.

☆ The work was supported by the National Natural Science Foundation of China (10271074) and by the Special Funds for Major Specialities of Shanghai Education Committee.

* Corresponding author.
E-mail address: cqgu@staff.shu.edu.cn (C. Gu).

0377-0427/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2008.01.001

For $V_i \in \mathbb{E}$, $i = 1, 2, \ldots, k$, let $\mathcal{V}_k = [V_1, V_2, \ldots, V_k]$, and let H_k denote an $k \times k$ matrix, $H_{.,j}$ denote the jth column of the matrix H_k. For $\alpha = (\alpha_1, \ldots, \alpha_k)^{\mathrm{T}} \in \mathbf{R}^k$, we use the notation $*$ for the following product [2]

$$\mathcal{V}_k * \alpha = \sum_{i=1}^{k} \alpha_i V_i,$$

and

$$\mathcal{V}_k * H_k = [\mathcal{V}_k * H_{.,1}, \mathcal{V}_k * H_{.,2}, \ldots, \mathcal{V}_k * H_{.,k}].$$

It is easy to see that the following relations are satisfied

$$V * (\alpha + \beta) = (V * \alpha) + (V * \beta) \quad \text{and} \quad (V * H_k) * \alpha = V * (H_k \alpha),$$

where α and β are two vectors of R^k.

This paper is organized as follows. Section 2 presents a global Arnoldi algorithm for solving skew-symmetric systems with multiple right-hand sides. Section 3 constructs a new global Arnoldi algorithm by transforming nonsymmetric linear systems to the extended skew-symmetric ones. Section 4 improves the skew-symmetric method obtained via QR factorization for nonsymmetric linear systems. Section 5 discusses the convergence of the global Arnoldi process which is much simpler than global Arnoldi process in [2]. Section 6 gives some numerical experiments.

2. Global and block Arnoldi process for skew-symmetric systems

Consider the skew-symmetric system

$$AX = B, \tag{2.1}$$

where A is an $N \times N$ real skew-symmetric matrix, i.e. $A^{\mathrm{T}} = -A$, B and X are $N \times s$ rectangular matrices. Choose an initial matrix X_0 and define the residual

$$R_0 = B - A * X_0.$$

Then, for $Q_1 = R_0/\|R_0\|_F$, an orthonormal basis $Q_1, Q_2, \ldots, Q_m$ can be obtained by the global Arnoldi process

$$A * Q_k = -h_{k,k-1} * Q_{k-1} + h_{k+1,k} * Q_{k+1} \quad \text{for } k = 1, 2, \ldots, m, \tag{2.2}$$

where Q_0 is a zero matrix. The algorithm is described as follows:

Algorithm 1. Global Arnoldi algorithm for skew-symmetric matrix equation in (2.1)

1. Choose an $N \times s$ matrix Q_1 such that $\|Q_1\|_F = 1$, $h_{1,0} = 0$.
2. For $k = 1, 2, \ldots, m$
   $W = A * Q_k + h_{k,k-1} * Q_{k-1}$,
   $h_{k+1,k} = \|W\|_F$,
   if $h_{k+1,k} = 0$ stop, else $Q_{k+1} = w/h_{k+1,k}$.

Proposition 1. *If A is a skew-symmetric matrix, then*

$$h_{j,k} = (A * Q_k, Q_j)_F = 0 \quad \text{for } j = 1, 2, \ldots, k-2, k$$

and

$$h_{k.k+1} = -h_{k+1,k}.$$

Proof. In terms of (2.2), for $j = 1, 2, \ldots,$ it follows that

$$\begin{aligned} h_{j,k} &= (A * Q_k, Q_j)_F = (Q_k, A^{\mathrm{T}} * Q_j)_F = -(Q_k, A * Q_j)_F \\ &= -(Q_k, -h_{j,j-1} * Q_{j-1} + h_{j+1,j} * Q_{j+1})_F = 0, \end{aligned}$$

$$\begin{aligned} h_{k,k+1} &= (A * Q_{k+1}, Q_k)_F = -(Q_{k+1}, A * Q_k)_F \\ &= -(Q_{k+1}, -h_{k,k-1} * Q_{k-1} + h_{k+1,k} * Q_{k+1})_F \\ &= -h_{k+1,k}. \quad \square \end{aligned}$$

Proposition 2. *Let $\theta_m = [Q_1, Q_2, \ldots, Q_m]$, where the $N \times s$ matrices Q_i, $i = 1, , 2, \ldots, m$, are produced by Algorithm 1. Then*

$$\|\theta_m * \alpha\|_F = \|\alpha\|_2, \tag{2.3}$$

where α is a vector of R^m.

The proof for (2.3) is similar to the one given in [2]. Let $Z_{m+1} = (Q_1, Q_2, \ldots, Q_{m+1})$, $Z_m = (Q_1, Q_2, \ldots, Q_m)$, *and let*

$$T_m = \begin{bmatrix} 0 & -h_{21} & & & \\ h_{21} & 0 & -h_{32} & & \\ & \ddots & \ddots & \ddots & \\ & & h_{m-1,m-2} & 0 & -h_{m,m-1} \\ & & & h_{m,m-1} & 0 \end{bmatrix}, \qquad T_{m+1} = \begin{bmatrix} T_m \\ 0, \ldots, 0, h_{m+1,m} \end{bmatrix}.$$

Theorem 1. *Let* Z_{2m}, Z_{2m+1}, T_{2m} *and* T_{2m+1} *be given above. Then using the product* $*$, *the following relations hold*

$$A * Z_m = Z_m * T_m + h_{m+1,m}[0_{N\times s}, 0_{N\times s}, \ldots, Q_{m+1}],$$

and

$$A * Z_m = Z_{m+1} * T_{m+1}. \tag{2.4}$$

Let the approximate solution of Eq. (2.1) be

$$X_m = X_0 + Z_m * f \tag{2.5}$$

with $f = (f_1, f_2, \ldots, f_m)^{\mathrm{T}}$. Then, the mth residual R_m of Eq. (2.1) can be expressed as

$$R_m = B - A * X_m = R_0 - A * Z_m * f = R_0 - Z_{m+1} * T_{m+1} * f,$$

which leads to

$$\begin{aligned} \|R_m\|_F &= \min_{f \in R^{m\times 1}} \| \|R_0\|_F\, e_1 - T_{m+1} * f \|_2 \\ &= \| \|R_0\|_F\, e_1 - T_{m+1} * f^* \|_2. \end{aligned} \tag{2.6}$$

Introduce a real error tolerance satisfying $|\varepsilon| = \mathrm{tol}$ and rewrite (2.6) as the linear equation

$$\begin{bmatrix} T_m \\ 0, 0, \ldots, 0, h_{m+1,m} \end{bmatrix} * \begin{bmatrix} f1 \\ f2 \\ \vdots \\ f_{m-2} \\ f_{m-1} \\ f_m \end{bmatrix} = \begin{bmatrix} \|R_0\|_F \\ 0 \\ \vdots \\ 0 \\ 0 \\ \varepsilon \end{bmatrix}, \tag{2.7}$$

where m is a even number, and

$$\begin{bmatrix} T_m \\ 0, 0, \ldots, 0, h_{m+1,m} \end{bmatrix} * \begin{bmatrix} f1 \\ f2 \\ \vdots \\ f_{m-2} \\ f_{m-1} \\ f_m \end{bmatrix} = \begin{bmatrix} \|R_0\|_F \\ 0 \\ \vdots \\ 0 \\ \varepsilon \\ 0 \end{bmatrix},$$

where m is an odd number. In the following we only discuss the case of even number.

Solve Eq. (2.7) to get

$$\begin{cases} -h_{21}f_2 = \|R_0\|_F \\ h_{21}f_1 - h_{32}f_3 = 0 \\ h_{32}f_2 - h_{43}f_4 = 0 \\ \cdots \\ h_{m-1,m-2}f_{m-2} - h_{m-4,m-3}f_{m-4} = 0 \\ h_{m,m-1}f_{m-1} = 0 \\ h_{m+1,m}f_m = \varepsilon. \end{cases} \tag{2.8}$$

From (2.8) we obtain the minimization solution $g_m^* = (f_2{}^*, f_4{}^*, \ldots, f_m^*)^{\mathrm{T}}$ which can be explicitly expressed as

$$\begin{cases} f_{2i-1}^* = 0 \quad \text{for } i = 1, 2, \ldots, m/2 \\ f_2^* = -\|R_0\|_F / h_{21} \\ f_{2i}^* = f_{2i-2}^* h_{2i-1,2i-2} / h_{2i,2i-1}, \quad \text{for } i = 2, 3, \ldots, (m-2)/2 \\ f_m^* = \varepsilon / h_{m+1,m}. \end{cases} \tag{2.9}$$

From (2.8) and (2.9), we have

$$\|\tilde{R}_m\|_F = \|\|R_0\|_F e_1 - \tilde{T}_{2m+1} * f^*\|_2 = |h_{m+1,m} f_m^*| = |\varepsilon|.$$

By (2.5) the iterative solution X_m and X_{m-1} in (2.9) can be written as, respectively,

$$X_m = X_0 + \sum_{i=1}^{m/2} f_{2i}^* Q_{2i} \tag{2.10}$$

and

$$X_{m-1} = X_0 + \sum_{i=1}^{(m-2)/2} f_{2i}^* Q_{2i}. \tag{2.11}$$

Comparing the iterative relation (2.11) with (2.10) leads to

$$X_m = X_{m-1} + f_m^* Q_m.$$

Thus, we present a new global Arnoldi algorithm for the iteration solution X_m in (2.1) for skew-symmetric systems as follows:

Algorithm 2. Global Arnoldi algorithm for skew-symmetric systems

1. Choose X_0, and compute $R_0 = B - AX_0$, $Q_1 = R_0/\|R_0\|_F$.
2. For $i = 1, 2, \ldots$ construct $Q_1, Q_2, \ldots$ by Algorithm 1.
3. Compute $f_2^* = -\|R_0\|_F/h_{21}$, for $i = 2, 3, \ldots$ and compute

   $$f_{2i}^* = f_{2i-2}^* h_{2i-1,2i-2}/h_{2i,2i-1}.$$
4. $X_m = X_{m-1} + Q_m f_m^*$.
5. Given a tol > 0, if $|\varepsilon| = $ tol, stop; otherwise continue.

3. Skew-symmetric method with global Arnoldi algorithm solving for nonsymmetric linear systems

We go back to the matrix equation (1.2) and construct a new matrix system

$$\tilde{A}\tilde{X} = \tilde{B}, \tag{3.1}$$

where

$$\tilde{A} = \begin{bmatrix} 0 & A \\ -A^{\mathrm{T}} & 0 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}, \qquad \tilde{X} = \begin{bmatrix} Y \\ X \end{bmatrix}. \tag{3.2}$$

Notice that the matrix $\tilde{A}$ is a skew-symmetric matrix. Moreover, Eq. (3.1) becomes the two equations: $AX = B$ and $A^{\mathrm{T}}Y = -C$ with the given matrix C. In this way, solving the matrix equation (1.2) is equivalent to solving the matrix, Eq. (3.1). Since $\tilde{A}$ is a skew-symmetric matrix, the global Arnoldi process can be used to solve the nonsymmetric linear equations which is much simpler than the global Arnoldi process in [2]. Choose an initial $N \times s$ matrix

$$\hat{X}_0 = \begin{bmatrix} Y_0 \\ X_0 \end{bmatrix}$$

such that $A^{\mathrm{T}}Y_0 = -C$. For $R_0 = B - AX_0$, the residual of the Eq. (3.1) is denoted by

$$\tilde{R_0} = \tilde{B} - \tilde{A}\tilde{X_0} = \begin{bmatrix} R_0 \\ 0 \end{bmatrix}.$$

Suppose that $R_0 \neq 0$, otherwise X_0 is a solution of (3.1). As in Section 2, the global Arnoldi process is presented as follows:

(i) Set $\tilde{Q}_1 = \frac{\tilde{R}_0}{\|\tilde{R}_0\|_F} = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix}$ with $Q_1 = \frac{R_0}{\|R_0\|_F}$.

(ii) Set $\beta_0 = 0$, and

$$\beta_j \tilde{Q}_{j+1} = \tilde{A}\tilde{Q}_j + \beta_{j-1}\tilde{Q}_{j-1} \quad \text{for } j = 1, 2, \ldots, 2m \tag{3.3}$$

with

$$\beta_j = \|\tilde{A}\tilde{Q}_j + \beta_{j-1}\tilde{Q}_{j-1}\|_F.$$

The following relations can be gained by induction

$$\beta_{2k-1} = \| -A^{\mathrm{T}}Q_{2k-1} + \beta_{2k-2}Q_{2k-2}\|_F, \quad \text{if } \beta_{2k-1} \neq 0, \text{ then}$$
$$\tilde{Q}_{2k} = \begin{bmatrix} 0 \\ Q_{2k} \end{bmatrix}, \qquad Q_{2k} = (-A^{\mathrm{T}}Q_{2k-1} + \beta_{2k-2}Q_{2k-2})/\beta_{2k-1};$$
$$\beta_{2k} = \|AQ_{2k} + \beta_{2k-1}Q_{2k-1}\|_F, \quad \text{if } \beta_{2k} \neq 0, \quad \text{then}$$
$$\tilde{Q}_{2k+1} = \begin{bmatrix} Q_{2k+1} \\ 0 \end{bmatrix}, \qquad Q_{2k+1} = (AQ_{2k} + \beta_{2k-1}Q_{2k-1})/\beta_{2k} \quad \text{for } k = 1, 2, \ldots, m.$$

From the above discussions the global Arnoldi algorithm is summarized as follows:

Algorithm 3. Global Arnoldi algorithm for nonsymmetric matrix in (1.2)

1. Choose an $N \times s$ matrix Q_1 such that$\|Q_1\|_F = 1$.
2. For $k = 1, 2, \ldots, m$
   $W = -A^{\mathrm{T}}Q_{2k-1} + \beta_{2k-2}Q_{2k-2}$,
   $\beta_{2k-1} = \|W\|_F$,
   if $\beta_{2k-1} = 0$ stop, else $Q_{2k} = w/\beta_{2k-1}$;
   $W = AQ_{2k} + \beta_{2k-1}Q_{2k-1}$,
   $\beta_{2k} = \|W\|_F$,
   if $\beta_{2k} = 0$ stop, else $Q_{2k+1} = W/\beta_{2k}$.

Let $\tilde{Z}_{2m+1} = (\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_{2m+1})$, $\tilde{Z}_{2m} = (\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_{2m})$, and let

$$\tilde{T}_{2m} = \begin{bmatrix} 0 & -\beta_1 & & & \\ \beta_1 & 0 & -\beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{2m-2} & 0 & -\beta_{2m-1} \\ & & & \beta_{2m-1} & 0 \end{bmatrix}, \qquad \tilde{T}_{2m+1} = \begin{bmatrix} T_{2m} \\ 0, 0, \ldots, 0, \beta_{2m} \end{bmatrix}.$$

Theorem 2. *Let $\tilde{Z}_{2m}, \tilde{Z}_{2m+1}, \tilde{T}_{2m}$ and $\tilde{T}_{2m+1}$ be as given above. Then using the product $*$, the following relations hold*

$$\tilde{A}\tilde{Z}_{2m} = \tilde{Z}_{2m} * \tilde{T}_{2m} + \beta_{2m}[0_{N\times s}, 0_{N\times s}, \ldots, \tilde{Q}_{2m+1}],$$

and

$$\tilde{A}\tilde{Z}_{2m} = \tilde{Z}_{2m+1} * \tilde{T}_{2m+1}. \tag{3.4}$$

Let the approximate solution of Eq. (3.1) be

$$\tilde{X}_m = \begin{bmatrix} Y_m \\ X_m \end{bmatrix} = \tilde{X}_0 + \tilde{Z}_{2m} * f \tag{3.5}$$

with $f = (f_1, f_2, \ldots, f_{2m})^{\mathrm{T}}$. Then, the mth residual R_m of Eq. (3.1) can be expressed as

$$\begin{aligned} \tilde{R}_m &= \tilde{B} - \tilde{A} * \tilde{X}_m = \begin{bmatrix} B \\ C \end{bmatrix} - \tilde{A} * \tilde{X}_m \\ &= \tilde{R}_0 - \tilde{A} * \tilde{Z}_{2m} * f \\ &= \tilde{R}_0 - \tilde{Z}_{2m+1} * \tilde{T}_{2m+1} * f \\ &= \|R_0\|_F \tilde{Q}_1 - \tilde{Z}_{2m+1} * \tilde{T}_{2m+1} * f. \end{aligned} \tag{3.6}$$

From (3.4) we get

$$\begin{aligned} \|\tilde{R}_m\|_F &= \min_{f \in R^{2m\times 1}} \| \|R_0\|_F e_1 - \tilde{T}_{2m+1} * f\|_2 \\ &= \| \|R_0\|_F e_1 - \tilde{T}_{2m+1} * f^*\|_2 \end{aligned} \tag{3.7}$$

with the minimization solution f^*.

In the same way, introduce a real error tolerance satisfying $|\varepsilon| = \text{tol}$ and rewrite (3.7) as the linear equation

$$\begin{bmatrix} 0 & -\beta_1 & & & & \\ \beta_1 & 0 & -\beta_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \beta_{2m-2} & 0 & -\beta_{2m-1} \\ & & & \beta_{2m-1} & 0 \\ & & & & \beta_{2m} \end{bmatrix} * \begin{bmatrix} f1 \\ f2 \\ \vdots \\ f_{2m-2} \\ f_{2m-1} \\ f_{2m} \end{bmatrix} = \begin{bmatrix} \|R_0\|_F \\ 0 \\ \vdots \\ 0 \\ 0 \\ \varepsilon \end{bmatrix}. \tag{3.8}$$

Solve Eq. (3.8) to get

$$\begin{cases} -\beta_1 f_2 = \|R_0\|_F \\ \beta_1 f_1 - \beta_2 f_3 = 0 \\ \beta_2 f_2 - \beta_3 f_4 = 0 \\ \cdots \\ \beta_{2m-2} f_{2m-2} - \beta_{2m-3} f_{2m-4} = 0 \\ \beta_{2m-1} f_{2m-1} = 0 \\ \beta_{2m} f_{2m} = \varepsilon. \end{cases} \tag{3.9}$$

On the basis of (3.9) we obtain the minimization solution $f_m^* = (f_2{}^*, f_4{}^*, \ldots, f_{2m}^*)^{\mathrm{T}}$ of (3.7) which can be explicitly expressed as

$$\begin{cases} f_{2i-1}^* = 0 \quad \text{for } i = 1, 2, \ldots, m \\ f_2^* = -\|R_0\|_F / \beta_1 \\ f_{2i}^* = f_{2i-2}^* \beta_{2i-2} / \beta_{2i-1} \quad \text{for } i = 2, 3, \ldots, m-1 \\ f_{2m}^* = \varepsilon / \beta_{2m}. \end{cases} \tag{3.10}$$

From (3.7) and (3.10), we have

$$\|\tilde{R}_m\|_F = \| \|R_0\|_F\, e_1 - \tilde{T}_{2m+1} * f^* \|_2 = |\beta_{2m} f_{2m}^*| = |\varepsilon| = \text{tol}.$$

Then, the approximation solution in (1.2) can be expressed as

$$X_m = X_{m-1} + f_{2m}^* Q_{2m}.$$

Now a new global skew-symmetric method for the iteration solution X_m in (1.2) can be given as follows:

Algorithm 4. Global Arnoldi algorithm for nonsymmetric linear systems in (1.2)

1. Choose X_0, and compute $R_0 = B - AX_0$, $Q_1 = R_0 / \|R_0\|_F$.
2. For $i = 1, 2, \ldots$ construct $Q_1, Q_2, \ldots$ by Algorithm 3.
3. Compute $f_2^* = -\|R_0\|_F / \beta_1$, for $i = 2, 3, \ldots$ and compute
$$f_{2i}^* = f_{2i-2}^* \beta_{2i-2} / \beta_{2i-1}.$$
4. $X_m = X_{m-1} + Q_{2m} f_{2m}^*$.
5. Given a tol > 0, if $|\varepsilon| \le$ tol, stop; otherwise continue.

4. Skew-symmetric method with *QR* factorization solving for nonsymmetric linear systems

We start to improve the skew-symmetric method obtained via *QR* factorization for nonsymmetric linear system (1.2). As stated in Section 3, the same matrix equation (3.1) will be formed. Choose an initial $N \times s$ matrix

$$\hat{X}_0 = \begin{bmatrix} Y_0 \\ X_0 \end{bmatrix}$$

such that $A^{\mathrm{T}} Y_0 = -C$. The residual of Eq. (3.1) is denoted by

$$\tilde{R}_0 = \tilde{B} - \tilde{A}\tilde{X}_0 = \begin{bmatrix} B - AX_0 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 R \\ 0 \end{bmatrix},$$

where $Q_1 R$ is a *QR* factorization of $B - AX_0$, and Q_1 is an $N \times s$ orthonormal matrix and R is $s \times s$ upper-triangular. The block Arnoldi process is defined by

(i) Set $\tilde{Q}_1 = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix}$.

(ii) Set $R_0 = [0]_{s\times s}$, $\tilde{Q}_0 = [0]_{N\times s}$ and

$$\tilde{Q}_{j+1}R_j = \tilde{A}\tilde{Q}_j + \tilde{Q}_{j-1}R_{j-1}^{\mathrm{T}}, \quad j = 1, 2, \ldots, 2m \ (\tilde{Q}R \text{ factorization}). \tag{4.1}$$

The following relations can be shown by induction

$$\tilde{Q}_{2k} = \begin{bmatrix} 0 \\ Q_{2k} \end{bmatrix}, \qquad Q_{2k}R_{2k-1} = -A^{\mathrm{T}}Q_{2k-1} + Q_{2k-2}R_{2k-2}^{\mathrm{T}} \quad (QR \text{ factorization}),$$

$$\tilde{Q}_{2k+1} = \begin{bmatrix} Q_{2k+1} \\ 0 \end{bmatrix}, \qquad Q_{2k+1}R_{2k} = AQ_{2k} + Q_{2k-1}R_{2k-1}^{\mathrm{T}} \quad (QR \text{ factorization}), \quad k = 1, 2, \ldots, m.$$

Algorithm 5. Block Arnoldi algorithm for nonsymmetric matrix in (1.2)

1. Choose an $N \times s$ matrix X_0 and compute $Q_1R = B - A * X_0$. Let $R_0 = [0]_{s\times s}$.
2. For $k = 1, 2, \ldots, m$
   $W = -A^{\mathrm{T}}Q_{2k-1} + Q_{2k-2}R_{2k-2}^{\mathrm{T}}$,
   $Q_{2k}R_{2k-1} = W$ (QR factorization);
   $W = AQ_{2k} + Q_{2k-1}R_{2k-1}^{\mathrm{T}}$,
   $Q_{2k+1}R_{2k} = W$ (QR factorization).

For the block Arnoldi algorithm, we have the similar results as in Section 3.

Let $\tilde{Z}_{2m+1} = (\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_{2m+1})$, $\tilde{Z}_{2m} = (\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_{2m})$, and let

$$\tilde{T}_{2m} = \begin{bmatrix} 0 & -R_1^{\mathrm{T}} & & & \\ R_1 & 0 & -R_2^{\mathrm{T}} & & \\ & \ddots & \ddots & \ddots & \\ & & R_{2m-2} & 0 & -R_{2m-1}^{\mathrm{T}} \\ & & & R_{2m-1} & 0 \end{bmatrix}, \qquad \tilde{T}_{2m+1} = \begin{bmatrix} T_{2m} \\ 0, 0, \ldots, 0, R_{2m} \end{bmatrix}.$$

Theorem 3. *Let $\tilde{Z}_{2m}, \tilde{Z}_{2m+1}, \tilde{T}_{2m}$ and $\tilde{T}_{2m+1}$ be as given above. Then using the product $*$, the following relations hold*

$$\tilde{A}\tilde{Z}_{2m} = \tilde{Z}_{2m} * \tilde{T}_{2m} + [0_{N\times s}, 0_{N\times s}, \ldots, \tilde{Q}_{2m+1}]R_{2m},$$

and

$$\tilde{A}\tilde{Z}_{2m} = \tilde{Z}_{2m+1} * \tilde{T}_{2m+1}.$$

According to (3.7) and (3.8), we have

$$\begin{bmatrix} 0 & -R_1^{\mathrm{T}} & & & \\ R_1 & 0 & -R_2^{\mathrm{T}} & & \\ & \ddots & \ddots & \ddots & \\ & & R_{2m-2} & 0 & -R_{2m-1}^{\mathrm{T}} \\ & & & R_{2m-1} & 0 \\ & & & & R_{2m} \end{bmatrix} * \begin{bmatrix} F1 \\ F2 \\ \vdots \\ F_{2m-2} \\ F_{2m-1} \\ F_{2m} \end{bmatrix} = \begin{bmatrix} RI_{s\times s} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \varepsilon I_{s\times s} \end{bmatrix}, \tag{4.2}$$

in which $|\varepsilon| = \mathrm{tol}/s$ is given as real error tolerance and F_i is a series of $s \times s$ matrices.

Use (4.2) to obtain the minimization solution $g_m^* = (F_2{}^*, F_4{}^*, \ldots, F_{2m}^*)^{\mathrm{T}}$ which can be explicitly expressed as

$$\begin{cases} F_{2i-1}^* = 0 \quad \text{for } i = 1, 2, \ldots, m \\ F_2^* = -R_1^{-T}R \\ F_{2i}^* = R_{2i-1}^{-T}R_{2i-2}F_{2i-2}^* \quad \text{for } i = 2, 3, \ldots, m-1 \\ F_{2m}^* = \varepsilon R_{2m}^{-T}. \end{cases} \tag{4.3}$$

Then, we have

$$\|\tilde{R}_m\|_F = \|RE_1 - \tilde{T}_{2m+1} * F^*\|_F = \|R_{2m}F_{2m}^*\|_F = |\varepsilon|s = \mathrm{tol},$$

and

$$X_m = X_{m-1} + Q_{2m}F_{2m}^*.$$

Thus, a new block skew-symmetric method for the iteration solution X_m in (1.2) can be summarized as follows:

Algorithm 6. Block Arnoldi algorithm for nonsymmetric linear systems in (1.2)

1. Choose X_0, and compute Q_1 (QR factorization of $B - A * X_0$).
2. For $i = 1, 2, \ldots$ construct $Q_1, Q_2, \ldots$ by Algorithm 5.
3. Compute $F_2^* = -R_1^{-T}R$ and $F_{2i}^* = R_{2i-1}^{-T}R_{2i-2}F_{2i-2}^*$, for $i = 2, 3, \ldots, m-1$.

4. $X_m = X_{m-1} + Q_{2m}F^*_{2m}$.
5. Given an tol > 0, if $|\varepsilon| = \text{tol}/s$, stop; otherwise continue.

We assume that the R_i are nonsingular in Algorithm 6, otherwise, the algorithm will stop.

5. Convergence theorem

In this section, we will give the residual evaluation of global Arnoldi Algorithm 4 and block Arnoldi Algorithm 6. To get the convergence theorem of Algorithm 4, we need the following lemmas.

Lemma 1. *Let $R_m = [r_m^{(1)}, r_m^{(2)}, \ldots, r_m^{(s)}]$, where R_m are the residual of Eq. (1.2). Then*

$$\|R_m\|_F \le \sqrt{s} \max_{i=1,2,\ldots,s} \|r_m^{(i)}\|_2.$$

Proof. It is clear that we obtain

$$\|R_m\|_F = \sqrt{\sum_{i=1}^{s} \|r_m^{(i)}\|_2^2} \le \sqrt{s} \max_{i=1,2,\ldots,s} \|r_m^{(i)}\|_2. \quad \square$$

Lemma 2. *For all Q_i produced by Algorithm 3, then the following relations hold*

$$Q_{2m-1} = \sum_{k=0}^{m-1} d_k^{(m)} (AA^{\mathrm{T}})^k R_0, \tag{5.1}$$

$$Q_{2m} = \sum_{k=0}^{m-1} D_k^{(m)} A^{\mathrm{T}} (AA^{\mathrm{T}})^k R_0, \tag{5.2}$$

where $d_k^{(m)}$ and $D_k^{(m)}$ are constant numbers which can be expressed by each other.

Proof. It is readily derived from Algorithm 3 that

$$Q_1 = \frac{R_0}{\|R_0\|_F} = d_0^{(1)} R_0$$

$d_0^{(1)} = 1/\|R_0\|_F$, and

$$Q_2 = -A^{\mathrm{T}} Q_1/\beta_1 = D_0^{(1)} A^{\mathrm{T}} R_0,$$

where $D_0^{(1)} = -1/\|R_0\|_F \beta_1$. Suppose that (5.1) and (5.2) hold for $i = 1, 2, \ldots, s$. For $i = s+1$, it follows that

$$\begin{aligned} Q_{2s+1} &= (AQ_{2s} + \beta_{2s-1} Q_{2s-1})/\beta_{2s} \\ &= \left[A \sum_{k=0}^{s-1} D_k^{(s)} A^{\mathrm{T}} (AA^{\mathrm{T}})^k R_0 + \beta_{2s-1} \sum_{k=0}^{s-1} d_k^{(s)} (AA^{\mathrm{T}})^k R_0 \right] \Big/ \beta_{2s} \\ &= \left[A \sum_{k=0}^{s-1} D_k^{(s)} A^{\mathrm{T}} (AA^{\mathrm{T}})^k R_0 + \beta_{2s-1} \sum_{k=1}^{s} d_k^{(s)} (AA^{\mathrm{T}})^k R_0 \right] \Big/ \beta_{2s} \\ &= \sum_{k=0}^{s} d_k^{(s+1)} (AA^{\mathrm{T}})^k R_0, \end{aligned}$$

and

$$\begin{aligned} Q_{2s+2} &= (-A^{\mathrm{T}} Q_{2s+1} + \beta_{2s} Q_{2s})/\beta_{2s+1} \\ &= \left[-A^{\mathrm{T}} \sum_{k=0}^{s} D_k^{(s+1)} A^{\mathrm{T}} (AA^{\mathrm{T}})^k R_0 + \beta_{2s} \sum_{k=0}^{s-1} d_k^{(s)} A^{\mathrm{T}} (AA^{\mathrm{T}})^k R_0 \right] \Big/ \beta_{2s+1} \\ &= \sum_{k=0}^{s} D_k^{(s+1)} A^{\mathrm{T}} (AA^{\mathrm{T}})^k R_0, \end{aligned}$$

where

$$d_s^{(s+1)} = 1/\beta_{2s} D_{s-1}^{(s)} \quad \text{and} \quad D_s^{(s+1)} = -1/\beta_{2s+1} d_s^{(s+1)}. \quad \square$$

Theorem 4. *The residual of the solution X_m in Algorithm 4 satisfies*

$$\|R_m\|_F \le \max_{i=1,2,\ldots,s} \frac{\|r_0^{(i)}\|_2\sqrt{s}}{T_m\left(\frac{\delta_n^2+\delta_1^2}{\delta_n^2-\delta_1^2}\right)} \tag{5.3}$$

and

$$\|X^* - X_m\|_F \le \max_{i=1,2,\ldots,s} \frac{\|A^{-1}\|_F\|r_0^{(i)}\|_2\sqrt{s}}{T_m\left(\frac{\delta_n^2+\delta_1^2}{\delta_n^2-\delta_1^2}\right)}, \tag{5.4}$$

where X^* is the exact solution of (1.2).

Proof. It is not difficult to find that

$$\|R_m\|_F = \min_{f_{2i}} \|R_0 - \sum_{i=1}^{m} f_{2i}AQ_{2i}\|_F.$$

From (5.2) we get

$$AQ_{2m} = D_{m-1}^{(m)}(AA^{\mathrm{T}})^m R_0 + D_{m-2}^{(m)}(AA^{\mathrm{T}})^{m-1}R_0 + \cdots + D_0^{(m)}(AA^{\mathrm{T}})R_0.$$

Let Q_m be the set of all polynomials $P_m(\lambda)$ of degree $\le m$ and $P_m(0) = 1$. From Lemma 1 it follows that

$$\begin{aligned}\|R_m\|_F &= \min_{f_{2i}} \|R_0 - \sum_{i=1}^{m} f_{2i}AQ_{2i}\|_F \\ &= \min_{P_m\in Q_m} \|P_m(AA^{\mathrm{T}})R_0\|_F \\ &\le \sqrt{s}\max_{i=1,2,\ldots,s}\min_{P_m\in Q_m} \|P_m(AA^{\mathrm{T}})r_0^{(i)}\|_2.\end{aligned}$$

Let $z_1, z_2, \ldots, z_n$ be the unit orthogonal eigenvectors of AA^{T}, and let $\delta_1^2, \delta_2^2, \ldots, \delta_n^2$ be the corresponding eigenvalues, where δ_i are the singular values of the matrix A. Then

$$\min_{P_m\in Q_m} \|P_m(AA^{\mathrm{T}})r_0^{(i)}\|_2 \le \min_{P_m\in Q_m}\max_{[\delta_1^2,\delta_n^2]} |P_m(\lambda^2)|^2\|r_0^{(i)}\|_2, \tag{5.5}$$

where δ_1 and δ_n are the smallest and largest singular values of A, respectively. The Chebyshev polynomial of degree m can be expressed by

$$T_m(x) = \frac{1}{2}[(x + \sqrt[2]{x^2-1})^m + (x - \sqrt[2]{x^2-1})^m].$$

Let

$$P(X) = T_m\left(\frac{2x}{\delta_n^2-\delta_1^2} - \frac{\delta_n^2+\delta_1^2}{\delta_n^2-\delta_1^2}\right)\Big/ T_m\left(\frac{\delta_n^2+\delta_1^2}{\delta_n^2-\delta_1^2}\right),$$

with $P(x) \in Q_m$. It follows from (5.5) that

$$\begin{aligned}\min_{P_m\in Q_m} \|P_m(AA^{\mathrm{T}})r_0^{(i)}\|_2 &\le \max_{[\delta_1^2,\delta_n^2]} |P(x)|\|r_0^{(i)}\|_2 \\ &= \frac{\|r_0^{(i)}\|_2}{T_m\left(\frac{\delta_n^2+\delta_1^2}{\delta_n^2-\delta_1^2}\right)}.\end{aligned} \tag{5.6}$$

In terms of (5.6) we have

$$\begin{aligned}\|R_m\|_F &\le \sqrt{s}\max_{i=1,2,\ldots,s}\min_{P_m\in Q_m} \|P_m(AA^{\mathrm{T}})r_0^{(i)}\|_2 \\ &\le \max_{i=1,2,\ldots,s} \frac{\|r_0^{(i)}\|_2\sqrt{s}}{T_m\left(\frac{\delta_n^2+\delta_1^2}{\delta_n^2-\delta_1^2}\right)},\end{aligned}$$

which accomplishes the proof of the relation (5.3).

Table 1
The number of iterations and CPU time

Matrix	n	Method	s				
			4	8	16	20	24
sherman 5	3312	Algorithm 6	8079(91.34)	3162(75.86)	1430(81.53)	1374(124)	1282(166)
		block GMRES	328(924)	192(825)	104(563)	85(498)	71(421)
		Algorithm 4	32306(339)	32403(561)	32641(1110)	32918(1750)	33016(2210)
cdde5	961	Algorithm 6	323(1.06)	172(1.26)	92(1.4)	78(1.71)	69(1.71)
		block GMRES	97(11)	60(11.812)	32(8.1)	26(6.95)	23(7.14)
		Algorithm 4	1003(6.98)	1026(12.81)	1027(25.57)	1030(43.53)	1040(82.75)
rdb1048	2048	Algorithm 6	362(3.04)	347(5.7)	248(10.12)	214(14.7)	169(14.57)
		block GMRES	82(13.37)	73(44.62)	43(40.57)	33(29.12)	26(21.9)
		Algorithm 4	1003(6.98)	1026(12.81)	1027(25.57)	1030(43.53)	1040(82.75)
sherman 4	1104	Algorithm 6	268(0.7)	120(0.7)	57(0.84)	49(0.95)	49(1.35)
		block GMRES	64(2.82)	47(5.2)	33(9.45)	29(10.51)	25(10.04)
cavity07	1182	Algorithm 6	7372(102)	2759(57.6)	971(35.62)	678(31.81)	570(33.62)
		block GMRES	228(332)	121(141)	67(105)	56(98.65)	47(84.73)
pde2961	2961	Algorithm 6	873(13.56)	503(16.46)	301(19.56)	250(22.5)	214(32.25)
		block GMRES	149(108)	116(228)	89(507)	83(660)	80(995)
tols2000	2000	Algorithm 6	4274(35.46)	1215(19.73)	367(14.87)	262(13.29)	195(14.37)
		block GMRES	206(237)	106(136)	56(87.9)	45(75.12)	39(123)

Since $X^* - X_m = A^{-1}R_m$ and $R_0 = A(X^* - X_0)$, from the above result, we obtain that

$$\|X^* - X_m\|_F \le \|A^{-1}\|_F \|R_m\|_F \le \max_{i=1,2,\dots,s} \frac{\|A^{-1}\|_F \|r_0^{(i)}\|_2 \sqrt{s}}{T_m\left(\frac{\delta_n^2+\delta_1^2}{\delta_n^2-\delta_1^2}\right)}. \quad \square$$

We use "block" Krylov subspaces $K_m(A, R_0) = \mathrm{span}\{R_0, AR_0, \dots, A^{m-1}R_0\}$, where the block is defined such that

$$B_m = \left\{ \sum_{k=0}^{m-1} A^k R_0 \gamma_k : \gamma_k \epsilon R^{s\times s} \right\},$$

and use "global" Krylov subspaces $K_m(A, R_0) = \mathrm{span}\{R_0, AR_0, \dots, A^{m-1}R_0\}$, where the block is defined such that

$$B_m^{\Delta} = \left\{ \sum_{k=0}^{m-1} \gamma_k A^k R_0 : \gamma_k \epsilon R \right\}.$$

The search space B_m^{Δ} has only m dimension, while B_m has ms^2 dimension [5]. Note that Algorithm 4 requires least scalar work, but the dimension of its search space is s^2 which is smaller than that of Algorithm 6.

6. Numerical examples

All the numerical experiments presented are computed in double precision with some Matlab 6.5 codes. For all the examples the initial guess X_0 is taken to be the zero matrix. The right-hand sides B are chosen such that the exact solution X is a matrix of order $n \times s$ whose all entries are equal to one. We will use the above algorithms to test some numerical example. We pick nonsymmetric matrices from the Matrix Market. For all the experiments, the initial guess was taken to be zero. The tests were stopped as soon as $(\|R_m\|_F/\|R_0\|_F) \ll 10^{-8}$. Fig. 1 shows the convergence of residuals for Algorithm 6 and block GMRES with fidap004 in the case of $s = 8$. From the following figure it is not difficult to show that Algorithm 6 gives a higher practical performance than block GMRES. The following numerical experiments show that the algorithms avoid the tediously long Arnoldi process and highly reduce expensive storage which are produced by the block GMRES (see Table 1).

By carrying out many numerical experiments, we find that "breakdown" does not appear in Algorithm 4 and that Algorithm 6 gives a higher performance than block GMRES. Moreover, we only need four spaces to store less variables in the computation. Therefore the two given algorithms avoid the tediously long Arnoldi process and highly reduce expensive storage. They are really attractive algorithms for solving nonsymmetric linear equation systems.

7. Conclusion

In this paper we present the skew-symmetric system methods for solving nonsymmetric linear systems with multiple right-hand sides. These methods are based on the global Arnoldi Algorithm 4 and the block Arnoldi Algorithm 6. Many numerical experiments verify that Algorithm 6 and Algorithm 4 are attractive for solving nonsymmetric linear systems with multiple right-hand sides.

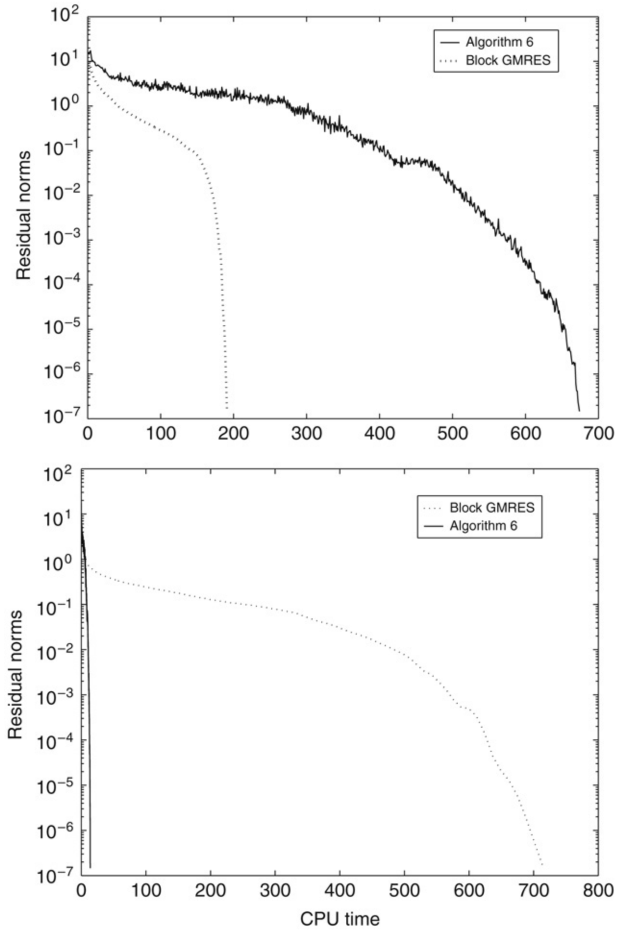

Fig. 1. Algorithm 6 and block GMRES. This figure shows the behaviour of the Frobenius residual norms.

References

[1] R. Freund, M. Malhotra, A block-QMR algorithm for non-Hermitian linear systems with multiple right-hand sides, Linear Algebra and its Applications 254 (1997) 119–157.

[2] K. Jbilou, A. Messaaoudi, H. sadok, Global FOM and GMRES algorithms for matrix equations, Applied Numerical Mathematics 13 (1999) 49–63.

[3] M. Malhotraa, R. Freund, P.M. Pinsky, Iterative solution of multiple radiation and scattering problems in structural acoustics using a block quasi-minimal residual algorithm, Computer Methods in Applied Mechanics and Engineering 146 (1997) 173–196.

[4] C.C. Paige, M.A. Saunders, LSQR: An algorithm for sparse linear equations and sparse least squares, Association for Computing Machinery Transactions on Mathematical Software 8 (1982) 43C71.

[5] Martin H. Gutknecht, Solvers for Systems of equation With Multiple Righthand sides, in: Talk given at the GAMM Workshop on Numerical Linear Algebra in Bielefeld, Sep. 13–14, 2002.

[6] V. Simoncini, E. gallopoulos, Convergence properties of block GMRES and matrix polynomial, Linear Algebra and its Applications 247 (1) (1996) 97–119.

[7] Chuanqing Gu, Zhenxin Yang, Global SCD algorithm for real positive definite linear systems with multiple right-hand sides, Applied Mathematics and Computation 189 (1) (2007) 59–67.