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A New Variational Method for Finding Einstein Metrics on Compact Kähler Manifolds, I

M. S. BERGER* AND P. CHERRIER†

Department of Mathematics and Statistics, Graduate Research Center, University of Massachusetts, Amherst, Massachusetts 01003, and Département de Mathématiques, Université de Paris VI, 4 Place Jussieu, 75230 Paris Cedex 05, France

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Let X be a compact Kähler manifold of complex dimension m. One knows that the determination of Einstein–Kähler metrics on X can be reduced to the solution of complex Monge–Ampère equations, depending on the sign of the first Chern class of X. Here, one studies a unified variational principle in the Sobolev space $W_{2,m}(X)$, whose Euler–Lagrange equations are the p.d.e. in question. The resulting variational problem results in a limit case for the associated Sobolev inequalities.

INTRODUCTION

In this article, we take up a new variational principle for the determination of Einstein–Kähler metrics on a compact Kähler manifold X_{2m} of arbitrary complex dimension m. We restrict the manifold by making necessary assumptions on its first Chern class $C_1(X)$, which must have prescribed sign, and show the calculus of variations point of view leads to a unified approach to the problem we study. In particular, starting from a given Kähler metric g on X, we consider Kähler deformations of g of the form $g_\phi = g + \partial\bar{\partial}\phi$; they preserve the Kähler structure of X and, when $C_1(x) \neq 0$, if g is chosen such that its first fundamental form represents $C_1(X)$ up to sign, they also remain in $C_1(X)$. We seek a Kähler deformation g_ϕ (with real valued globally defined function ϕ) which is Einstein–Kähler, i.e., the Ricci tensor is proportional to the metric tensor.

Such deformations have been studied by numerous authors. E. Calabi [3] first initiated the study of this problem in 1954. He proposed various

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† Present address: 26 avenue du Château, 92340 Bourg la Reine, France. Research partially supported by an NSF grant from the Institute for Advanced Study.

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methods for studying it, one of which involved calculus of variations based on minimizing the L_2 norm of deformed Ricci tensors in the space of Kähler metrics defined by the above deformations. The method we present below is substantially different, and consists in writing down the partial differential equations satisfied by the function ϕ associated to the desired Einstein–Kähler metric g_ϕ and then computing, by well-established functional analytic results, the convenient variational principle whose Euler–Lagrange equations are indeed the p.d.e. in question. In addition, we are able to formulate the problem in the Sobolev space $W_{2,m}(X)$ which is adapted to the associated variational principle. In [2, 3, 14], T. Aubin and S. T. Yau solved successfully the p.d.e. when $C_1(X) \leqslant 0$ by the method of continuity, using elaborate a priori estimates. In a recent paper [4], Aubin studied the case $C_1(X) > 0$ also via continuity method and, under the geometric hypothesis $\int_X \overset{m}{\Lambda}\, C_1 < (m+1)^{2m}(2m)^{-m}$ (which generalizes nicely the condition $\chi(X_2) < 2$ concerning Euler–Poincaré characteristic obtained by Berger [5] when $m = 1$), reduced the problem to an integral inequality (69) not yet proved. This inequality is an extension of Berger [5], who treated the one-dimensional case in a uniform manner by an isoperimetric variational principle using the Gauss–Bonnet theorem. In this paper, we try to extend this unified pattern to the higher-dimensional case and to make first steps toward solving the variational problem in significant cases.

The use of calculus of variations is extremely natural from both geometric and physical points of view. Einstein [11] and Hilbert attempted novel variational principles to describe Einstein's field equations in general relativity. However, the elaboration of these techniques was impeded by the lack of progress in the mathematical understanding of the calculus of variations. In differential geometry, the use of variational methods in multi-dimensional problems centers around the work of Yamabe [12], who investigated deformations of the scalar curvature under conformal transformations of the metric. The p.d.e. one is led to is naturally posed in $W_{1,2}(X)$ and the non-linearity which occurs is limit and corresponds to the continuous but non-compact Sobolev inclusion $W_{1,2}(X) \subset L_p(X)$. In [5], Berger studied the case of one complex dimension and Gaussian curvature. Amazingly, the same limiting behaviour which appears in [5] will become apparent in the problem of Einstein–Kähler metrics, except we consider the geometric problem in terms of the Sobolev space $W_{2,m}(X)$, where the compact Kähler manifold X has real dimension $2m$.

Our arguments in Section III show that higher-order ($\geqslant 3$) derivatives of the deformation ϕ can be controlled in terms of the lower-order derivatives of ϕ (in an appropriate L_p sense). In particular the variational problems for the Einstein–Kähler deformations reduce to a study of L_m norms of the second deriatives of ϕ over X. This is desirable, for example, in the case of algebraic surfaces ($m = 2$), since global geometric invariants can often be expressed in terms of $L_2(X)$ norms of an appropriate deformation ϕ and its first and second derivatives.

I. The Deformation Equations and the Calculus of Variations Principles Adapted to Their Study

Before defining the deformation equations leading to an Einstein–Kähler metric, it is important to present the differential geometric context of this problem. In addition, it is necessary to discuss complex differential geometry, and its utilisation in reducing a complicated non-linear system of partial differential equations to the search for the determination of one smooth function satisfying a single non-linear partial differential equation. Indeed, it will turn out that, by solving a well-defined Monge–Ampère equation globally over the compact manifold in question, an Einstein–Kähler metric can be determined provided the solution is smooth.

This section is divided in three parts. Section I.1 gives some notations and definitions concerning first Chern class of Hermitian manifolds and Kähler deformations of metrics. In Section I.2, the problem of existence of Einstein–Kähler metrics is rephrased analytically; translated in terms of analysis, it leads to complex Monge–Ampère equations (25) and (26). Section I.3 describes a variational principle and shows that the smooth solutions of (33) are the smooth critical points of the functional $\mathfrak{J}$ (34).

1. *First Chern Class, Kähler Deformations of Metrics, and Complex Monge–Ampère Operator.*

(i) Let X_{2m} be a connected complex manifold of complex dimension m and let $z^\lambda = x^\lambda + iy^\lambda)_{\lambda=1,\dots,m}$ be local complex coordinates. We write $\overline{z^\lambda} = z^{\bar\lambda}$ and set

$$e_a = \partial_a = \frac{\partial}{\partial z^a}, \qquad \partial_{ab} = \frac{\partial^2}{\partial z^a\,\partial z^b},$$

where greek indices $\lambda, \mu \cdots$ run from 1 to m while Latin ones run through $1, \dots, m, \bar 1, \dots, \bar m$.

d is the operator of exterior differentiation, and d' and d'' its $(1, 0)$ and $(0, 1)$ parts defined on a function ϕ by

$$d'\phi = \partial_\lambda \phi\, dz^\lambda, \qquad d''\phi = \partial_{\bar\lambda}\phi\, dz^{\bar\lambda}.$$

So $d = d' + d''$ and, if $d^c = d' - d''$, $dd^c = -2d'd''$; in particular, for any $\phi \in C^2(X)$,

$$dd^c\phi = -2\partial_{\lambda\bar\mu}\phi\, dz^\lambda \wedge dz^{\bar\mu}. \tag{1}$$

A real (1, 1) form χ can locally be written

$$\chi = i a_{\lambda\bar{\mu}}\, dz^{\lambda} \wedge dz^{\bar{\mu}},$$

where the matrix $(a_{\lambda\bar{\mu}})_{\lambda,\mu}$ is everywhere Hermitian: $a_{\mu\bar{\lambda}} = \overline{a_{\lambda\bar{\mu}}}$.

(ii) Let g be a Hermitian metric on X, i.e., a real symmetric covariant 2-tensor field $g = g_{ab}\, dz^a \otimes dz^b$ such that, for any indices,

$$g_{\bar{a}\bar{b}} = \overline{g_{ab}}, \qquad g_{\lambda\mu} = g_{\bar{\lambda}\bar{\mu}} = 0$$

and the matrix $(g_{\lambda\bar{\mu}})_{\lambda,\mu}$ is everywhere Hermitian positive definite. Thus, by restriction to the real tangent bundle, g defines a Riemannian metric on X. Every paracompact complex manifold admits a Hermitian metric.

The tensor $g^{-1} = g^{ab} e_a \otimes e_b$, defined by the inverse matrix (g^{ab}), is also real symmetric with

$$g^{\lambda\mu} = g^{\bar{\lambda}\bar{\mu}} = 0 \qquad \text{and} \qquad g^{\lambda\bar{\mu}} g_{\mu\bar{\nu}} = \delta^{\lambda}_{\mu}.$$

g and g^{-1} allow us to lower and raise indices in arbitrary tensors.

Setting

$$|g| = \det(g_{\lambda\bar{\mu}}),$$

the oriented volume element is the globally defined $2m$-form

$$\begin{aligned} dV &= (i/2)^m |g|\, dz^1 \wedge dz^{\bar{1}} \wedge \cdots \wedge dz^m \wedge dz^{\bar{m}} \\ &= |g|\, dx^1 \wedge dy^1 \wedge \cdots \wedge dx^m \wedge dy^m. \end{aligned} \tag{2}$$

The first fundamental form of the Hermitian manifold (X, g) is the real (1, 1) form

$$\omega = (i/2\pi)\, g_{\lambda\bar{\mu}}\, dz^{\lambda} \wedge dz^{\bar{\mu}}; \tag{3}$$

hence

$$dV = \frac{\pi^m}{m!}\, \omega \wedge \cdots \wedge \omega = \frac{\pi^m}{m!}\, \omega^m. \tag{4}$$

g is said to be Kähler if ω is closed. Equivalently, in a local chart adapted to the complex structure,

$$\partial_{\lambda} g_{\mu\bar{\nu}} = \partial_{\mu} g_{\lambda\bar{\nu}} \qquad \text{for any} \quad \lambda, \mu, \nu.$$

(iii) On a Hermitian manifold (X, g), the Chern's connection ∇ (extended to the complexified tensor algebra) is the unique real metric connection ($\nabla g = 0$) such that, if

$$\nabla_{e_a} e_b = \Gamma^c_{ab} e_c \qquad (\text{and so } \nabla_{e_a} dz^b = -\Gamma^b_{ac}\, dz^c)$$

in an adapted frame, all Christoffel's symbols of mixed type vanish and only $\Gamma^{\nu}_{\lambda\mu}=\overline{\Gamma^{\bar\nu}_{\bar\lambda\bar\mu}}$ may be $\neq 0$; then, necessarily,

$$\Gamma^{\nu}_{\lambda\mu}=g^{\nu\bar\rho}\,\partial_\lambda\, g_{\mu\bar\rho}.$$

When g is Kähler, ∇ coincides with the Levi–Civita connection of the underlying Riemannian metric.

The Laplacian and the square length of the gradient of a function ϕ are defined by

$$\Delta\phi=-\nabla^{\lambda}{}_{\lambda}\phi=-g^{\lambda\bar\mu}\,\partial_{\lambda\bar\mu}\phi$$

and

$$|\nabla\phi|^2=\nabla^\lambda\phi\nabla_\lambda\phi=g^{\lambda\bar\mu}\,\partial_\lambda\phi\,\partial_{\bar\mu}\phi.$$

The components of the curvature tensor of Chern's connection

$$R^{d}{}_{cab}\equiv\langle(\nabla_{e_a}\nabla_{e_b}-\nabla_{e_b}\nabla_{e_a})\,e_c,\,dz^d\rangle$$

are shown to be $\neq 0$ only if a, b are of different type and c, d have same type. Also, lowering index d and defining

$$\begin{aligned}R_{dcab}&\equiv g((\nabla_{e_a}\nabla_{e_b}-\nabla_{e_b}\nabla_{e_a})\,e_c,\,e_d)\\&=g_{de}R^{e}{}_{cab},\end{aligned}$$

only mixed components $R_{\alpha\bar\beta\gamma\bar\delta}$ are $\neq 0$. One has

$$\begin{aligned}R^{\delta}{}_{\gamma\alpha\bar\beta}&=-\overline{R^{\bar\delta}{}_{\bar\gamma\beta\bar\alpha}}=-\partial_{\bar\beta}\Gamma^{\delta}_{\alpha\gamma}\\&=-g^{\delta\bar\varepsilon}\,\partial_{\alpha\bar\beta}\,g_{\gamma\bar\varepsilon}+g^{\delta\bar\mu}g^{\nu\bar\varepsilon}\,\partial_\alpha\,g_{\gamma\bar\varepsilon}\,\partial_{\bar\beta}\,g_{\nu\bar\mu}.\end{aligned}\tag{5}$$

Consequently, for a vector field $\theta=\theta^c e_c$ (resp. a 1-form $\sigma=\sigma_c\,dz^c$), commutation of indices in iterated covariant derivative yields, in the Kähler case,

$$\nabla_{ab}\theta^c-\nabla_{ba}\theta^c=R^{c}{}_{dab}\,\theta^d\tag{6}$$

(resp. $\nabla_{ab}\sigma_c-\nabla_{ba}\sigma_c=-R^{d}{}_{cab}\,\sigma_d$).

In particular, when g is Kähler, for any tensor field t,

$$\nabla_{\lambda\mu}t=\nabla_{\mu\lambda}t,\qquad\nabla^{\lambda\mu}t=\nabla^{\mu\lambda}t;$$

taking $t=\nabla^\nu\phi$, where $\phi\in C^\infty(X)$, one gets

$$\nabla_{\lambda\mu}{}^{\nu}\phi=\nabla_{\mu\lambda}{}^{\nu}\phi,\qquad\nabla^{\lambda\mu}{}_{\nu}\phi=\nabla^{\mu\lambda}{}_{\nu}\phi.\tag{7}$$

And now let us define the Ricci tensor as the real symmetric covariant 2-tensor $R_{ab}\, dz^a \otimes dz^b$ such that

$$R_{ab} = \overline{R_{\bar{a}\bar{b}}}, \qquad R_{\alpha\beta} = 0$$

and the mixed component $R_{\alpha\beta}$ is given by the contraction

$$R_{\alpha\beta} = R^{\gamma}{}_{\gamma\alpha\beta} = -\partial_{\alpha\beta} \operatorname{Log} |g|, \tag{8}$$

the last equality resulting from (5).

The Ricci form of the Hermitian metric g,

$$\begin{aligned} \psi &= (i/2\pi)\, R_{\lambda\bar{\mu}}\, dz^{\lambda} \wedge dz^{\bar{\mu}} \\ &= (i/4\pi)\, dd^c \operatorname{Log}|g| \qquad \text{(by (8) and (1))}, \end{aligned} \tag{9}$$

is a closed real (1, 1) form. One proves that its cohomology class $[\psi] \in H^2(X, \mathbb{R})$ is independent of the metric g. It is the first Chern class of the complex manifold X, denoted by $C_1(X)$. When X is compact, the topological invariant

$$C_1^m = \int_X \overset{m}{\Lambda}\, \psi \tag{10}$$

is, apart from a scaling factor, always an integer. In the case of positive Chern class, $C_1(X) > 0$, as defined in paragraph (iv), we shall restrict X via an inequality for C_1^m (see (Proposition 4).

(iv) From now on, we shall suppose X is a compact Kähler manifold (i.e., admits Kähler metrics). The following lemma describes the elements of a given cohomology class of (1, 1) forms. (See "Complex Manifolds" by Kodaira and Morrow, Holt, Rinehart & Winston, New York, 1971, for a proof (cf. [8], [9], [13]).)

LEMMA 1. *Let $\chi = ia_{\lambda\bar{\mu}}\, dz^{\lambda} \wedge dz^{\bar{\mu}}$ be a closed real* (1, 1) *form on a compact Kähler manifold X. Then χ is cohomologous to zero if and only if there exists $f \in C^{\infty}(X)$ such that*

$$\chi = id'd''f, \qquad \text{i.e.,} \quad a_{\lambda\bar{\mu}} = \partial_{\lambda\bar{\mu}} f.$$

A cohomology class $\mathfrak{C}$ of real (1, 1) forms is said to be positive definite (resp. negative definite or null) if $\mathfrak{C}$ contains a representative $\chi = ia_{\lambda\bar{\mu}}\, dz^{\lambda} \wedge dz^{\bar{\mu}}$ such that the Hermitian matrix $(a_{\lambda\bar{\mu}})_{\lambda,\mu}$ is everywhere postitive definite (resp. negative definite or null). Using Lemma 1, one shows that these three properties are mutually exclusive. Thus it is meaningful to say that $C_1(X)$ has prescribed sign (>0, <0, or null).

Let g and g' be two Kähler metrics. The quantities corresponding to g' are indicated with a prime ($'$). g' is called a Kähler deformation of g if the first fundamental forms (see (3)) ω and ω' are cohomologous; equivalently, according to Lemma 1, if there exists $\phi \in C^\infty(X)$ such that

$$g'_{\lambda\bar{\mu}} = g_{\lambda\bar{\mu}} + \partial_{\lambda\bar{\mu}}\phi \tag{11}$$

or

$$\omega' = \omega + \frac{i}{2\pi} d'd''\phi = \omega - \frac{i}{4\pi} dd^c\phi. \tag{11'}$$

We shall write

$$g' = g_\phi = g + \partial\bar{\partial}\phi,$$

and for g' to define a metric on X, ϕ must belong to the convex set $\mathfrak{A}$ of admissible functions defined by

$$\mathfrak{A} = \{\phi \in C^2(X);\text{ the matrix } (g_{\lambda\bar{\mu}} + \partial_{\lambda\bar{\mu}}\phi)_{\lambda,\mu} \text{ is everywhere positive definite.}\} \tag{12}$$

If $\phi \in \mathfrak{A}$, we can choose local coordinates adapted to a given point $P \in X$, i.e., such that, in P, the matrix $(g_{\lambda\bar{\mu}})_{\lambda,\mu}$ and $(\partial_{\lambda\bar{\mu}}\phi)_{\lambda,\mu}$ are respectively the identity and diagonal. Then, for every direction λ, $(1 + \partial_{\lambda\bar{\lambda}}\phi)(P) > 0$ and the trace $g^{\lambda\bar{\mu}}g'_{\lambda\bar{\mu}} = m + \Delta\phi$ is positive.

Under condition (11), the volume elements relative to g and g' are related by

$$dV' = M(\phi)\, dV,$$

where M is the complex Monge–Ampère operator

$$M(\phi) = \frac{|g'|}{|g|} = \det(\delta^\lambda_\mu + \nabla^\lambda_\mu\phi), \tag{13}$$

since, in the Kähler metric g, $\nabla^\lambda_\mu\phi = g^{\lambda\bar{\nu}}\partial_{\mu\bar{\nu}}\phi$.

Let us also compare Ricci forms (9) ψ and ψ'. By (8) and (13),

$$R'_{\lambda\bar{\mu}} - R_{\lambda\bar{\mu}} = -\partial_{\lambda\bar{\mu}} \operatorname{Log}\frac{|g'|}{|g|} = -\partial_{\lambda\bar{\mu}} \operatorname{Log} M(\phi), \tag{14}$$

so

$$\psi' - \psi = (i/4\pi)\, dd^c \operatorname{Log} M(\phi).$$

We conclude[1] this section by displaying the development of the determinant (13) defining $M(\phi)$ as a sum of homogeneous terms in ϕ of degrees 0, 1, ..., m. We can write

$$M(\phi)=1+\sum_{k=1}^{m}\sum_{I,J}\operatorname{sgn}(I,J)\,\nabla_{j_1}^{i_1}\phi\cdots\nabla_{j_k}^{i_k}\phi,$$

with $I\in\{(i_1,...,i_k);\ 1\leqslant i_1<\cdots<i_k\leqslant m\}$ and J a permutation of I of signature $\operatorname{sgn}(I,J)$. Setting

$$\varepsilon_{\lambda_1\cdots\lambda_k}^{\mu_1\cdots\mu_k}=(k!)^{-1}\,\delta_{\lambda_1\cdots\lambda_k}^{\mu_1\cdots\mu_k}, \tag{15}$$

where $\delta_{\lambda_1\cdots\lambda_k}^{\mu_1\cdots\mu_k}$ is the Kronecker tensor (components $\neq 0$ and equal to $+1$ or -1 if and only if $(\mu_1,...,\mu_k)$ is a sequence of k distinct integers between 1 and m which is an even or odd permutation of $(\lambda_1,...,\lambda_k)$), we obtain

$$M(\phi)=1+\sum_{k=1}^{m}\varepsilon_{\lambda_1\cdots\lambda_k}^{\mu_1\cdots\mu_k}\nabla_{\mu_1}^{\lambda_1}\phi\cdots\nabla_{\mu_k}^{\lambda_k}\phi \tag{16}$$

Compare Eq. (16) with (16′) of the Appendix.

Since the Kronecker tensor has null covariant derivative and is antisymmetric in μ indices, by commutativity property (7) of covariant derivatives on a Kähler manifold, for $l=1,...,k-1$, we have

$$\varepsilon_{\lambda_1\cdots\lambda_k}^{\mu_1\cdots\mu_k}\nabla_{\mu_1}^{\lambda_1}\phi\cdots\nabla_{\mu_k\mu_l}^{\lambda_l}\phi\cdots\nabla^{\lambda_k}\phi=0$$

and $1-M(\phi)$ appears as a divergence (a key property from our viewpoint),

$$1-M(\phi)=\nabla_\mu X^\mu(\phi), \tag{17}$$

where the vector field $X^\mu(\phi)$ is defined by

$$X^\mu(\phi)=\nabla^\mu\phi+\sum_{k=2}^{m}\varepsilon_{\lambda_1\cdots\lambda_k}^{\mu_1\cdots\mu_{k-1}\mu}\nabla_{\mu_1}^{\lambda_1}\phi\cdots\nabla_{\mu_{k-1}}^{\lambda_{k-1}}\phi\,\nabla^{\lambda_k}\phi. \tag{18}$$

So, in a Kähler deformation, volume is preserved: $\int_X dV=\int_X dV'$. This is clear from the definition of $M(\phi)$ and (17) by integration.

For later use, notice that if $u_1,...,u_k$ belong to $C^3(X)$, the preceding argument yields that

$$\varepsilon_{\lambda_1\cdots\lambda_k}^{\mu_1\cdots\mu_k}\nabla_{\mu_1}^{\lambda_1}u_1\cdots\nabla_{\mu_k\mu_l}{}^{\lambda_l}u_l\cdots\nabla_{\mu_{k-1}}^{\lambda_{k-1}}u_{k-1}\,\nabla^{\lambda_k}u_k=0; \tag{19}$$

also, antisymmetry of $\varepsilon_{\lambda_1\cdots\lambda_k}^{\mu_1\cdots\mu_k}$ in λ indices and equality $\nabla_{\mu_l}^{\lambda_k\lambda_l}u_l=\nabla_{\mu_l}^{\lambda_l\lambda_k}u_l$ imply

$$\varepsilon_{\lambda_1\cdots\lambda_k}^{\mu_1\cdots\mu_k}\nabla_{\mu_1}^{\lambda_1}u_1\cdots\nabla_{\mu_l}^{\lambda_k\lambda_l}u_l\cdots\nabla_{\mu_{k-1}}^{\lambda_{k-1}}u_{k-1}\,\nabla_{\mu_k}u_k=0. \tag{20}$$

[1] See the Appendix for an alternative differential form formulation.

2. *The Equations Leading to Einstein–Kähler Metrics.*

A Kähler metric on a Kähler manifold X is said to be Einstein if there exists a real number k such that

$$\psi' = k\omega', \qquad \text{i.e.,} \quad R'_{\lambda\bar{\mu}} = g'_{\lambda\bar{\mu}}. \tag{21}$$

A priori (21) is a system for the unknown metric g'. When X is compact, let us prove that it can be reduced to a single equation.

A necessary condition for solving (21) is for $C_1(X)$ to be of prescribed sign. First let us suppose $C_1(X) \neq 0$. The coefficient k will have the sign of $C_1(X)$ and, since a homothety on the metric preserves Ricci tensor, we can search g' such that

$$\psi' = \varepsilon\omega' \qquad \text{or} \qquad R'_{\lambda\bar{\mu}} = \varepsilon g'_{\lambda\bar{\mu}}, \tag{22}$$

with ε equal to 1 or -1 according whether $C_1(X)$ is >0 or <0.

Choose a real (1–1) form $\chi = (i/2\pi)\ \varepsilon g_{\lambda\bar{\mu}}\, dz^\lambda \wedge dz^{\bar{\mu}}$ belonging to $C_1(X)$ such that the matrix $(g_{\lambda\bar{\mu}})_{\lambda,\mu}$ is everywhere positive definite and thus defines a Kähler metric g on X (since χ is closed) with first fundamental form $\omega = \varepsilon\chi$. Hence $\varepsilon\omega$, as well as ψ, belongs to $C_1(X)$ and, by Lemma 1, there exists $f \in C^\infty(X)$ such that

$$R_{\lambda\bar{\mu}} = \varepsilon g_{\lambda\bar{\mu}} + \partial_{\lambda\bar{\mu}} f. \tag{23}$$

On the other hand, (22) shows that $\varepsilon\omega' \in C_1(X)$; thus ω and ω' are cohomologous and g' must be a Kähler deformation of g,

$$g'_{\lambda\bar{\mu}} = g_{\lambda\bar{\mu}} + \partial_{\lambda\bar{\mu}}\phi, \tag{24}$$

where the unknown function ϕ satisfies an equation to be exhibited. Relations (22), (23), (24) imply

$$R'_{\lambda\bar{\mu}} - R_{\lambda\bar{\mu}} = \varepsilon(g'_{\lambda\bar{\mu}} - g_{\lambda\bar{\mu}}) - \partial_{\lambda\bar{\mu}} f = \partial_{\lambda\bar{\mu}}(\varepsilon\phi - f);$$

since by (14)

$$R'_{\lambda\bar{\mu}} - R_{\lambda\bar{\mu}} = -\partial_{\lambda\bar{\mu}} \operatorname{Log} M(\phi),$$

we have

$$\partial_{\lambda\bar{\mu}}[\operatorname{Log} M(\phi) + \varepsilon\phi - f] = 0.$$

Consequently, the function $\operatorname{Log} M(\phi) + \varepsilon\phi - f$, whose Laplacian is zero, must be constant and, because f is defined up to additive constants, ϕ verifies

$$M(\phi) = \exp(-\varepsilon\phi + f). \tag{25}$$

Now, if $C_1(X)=0$, start with an arbitrary initial Kähler metric g. Then ψ, cohomologous to zero, can be written $(i/2\pi)\, d'd''f$, that is, $R_{\lambda\bar{\mu}}=\partial_{\lambda\bar{\mu}}f$, a substitute to (23). If we search a Kähler deformation $g'=g+\partial\bar{\partial}\phi$ of g with null Ricci tensor, the preceding argument leads to the equation

$$M(\phi)=e^{f}. \tag{26}$$

This is a special case of Calabi conjecture which asserts that, on any compact Kähler manifold (X, g), every element

$$\chi=(i/2\pi)\, a_{\lambda\bar{\mu}}\, dz^{\lambda}\wedge dz^{\bar{\mu}}\in C_1(X)$$

is the Ricci form of some Kähler metric g'. In fact, since χ and ψ belong to $C_1(X)$,

$$R_{\lambda\bar{\mu}}-R'_{\lambda\bar{\mu}}=R_{\lambda\bar{\mu}}-a_{\lambda\bar{\mu}}=\partial_{\lambda\bar{\mu}}f$$

for some function $f\in C^{\infty}(X)$, and if we search g' as a Kähler deformation of g, equality (14) yields Eq. (26).

3. *Variational Principles for the Deformation Equations*

In complex dimension $m=1$, $1-M(\phi)=\Delta\phi$, and the smooth solutions of $\Delta\phi+f=0$ are the smooth critical points of the functional $\int_X(\frac{1}{2}|\nabla\phi|^2+f\phi)\, dV$. It is natural to ask if this extends in higher dimensions and if the deformation equations can be written as the Euler–Lagrange equations of some functionals. Following Berger [6, p. 93, Sect. 2.5, concerning gradient operators], one is led to introduce

$$J(\phi)=\int_X\int_0^1 \phi[1-M(s\phi)]\, ds\, dV \qquad \text{for} \quad \phi\in C^2(X). \tag{27}$$

It will turn out that all the operators in (25) and (26) are gradient operators. In Lemma 2 below, we prove the Gateaux derivative of $J(\varphi)$ is $1-M(\varphi)$. We also define (for comparison purposes) a generalization of the quadratic form associated with $1-M(\varphi)$, as follows:

$$I(\phi)=\int_X \phi[1-M(\phi)]\, dV. \tag{28}$$

The development (16) of $M(\phi)$ and the definitions (28), (27) imply that

$$I=\sum_{k=1}^{m} I_k \qquad \text{and} \qquad J=\sum_{k=1}^{m}\frac{I_k}{k+1}, \tag{29}$$

when I_k is homogeneous in φ of degree $k+1$ and is the polarization of the following $(k+1)$-linear form on $C^2(X)$:

$$H_k(u_1, ..., u_{k+1}) = -\int_X \varepsilon^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} \nabla^{\lambda_1}_{\mu_1} u_1 \cdots \nabla^{\lambda_k}_{\mu_k} u_k u_{k+1}\, dV; \tag{30}$$

so

$$I_k(\phi) = H_k(\phi, ..., \phi). \tag{31}$$

As a consequence of antisymmetry of Kronecker tensor and commutativity properties (7) of covariant derivatives on Kähler manifolds, one proves that H_k *is symmetric*[2] (a crucial property in our variational formulation). It is clear that $H_k(u_1, ..., u_{k+1})$ is preserved by permutation of u_i and u_j $(i<j\leqslant k)$ since $\varepsilon^{\mu_1 \cdots \mu_i \cdots \mu_j \cdots \mu_k}_{\lambda_1 \cdots \lambda_i \cdots \lambda_j \cdots \lambda_k} = \varepsilon^{\mu_1 \cdots \mu_j \cdots \mu_i \cdots \mu_k}_{\lambda_1 \cdots \lambda_j \cdots \lambda_i \cdots \lambda_k}$. Thus the symmetry of H_k will be shown if we see

$$H_k(u_1, ..., u_k, u_{k+1}) = H_k(u_1, ..., u_{k+1}, u_k).$$

Integrating (30) by parts twice yields

$$\begin{aligned} H_k(u_1, ..., u_{k+1}) &= \int_X \varepsilon^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} \nabla^{\lambda_1}_{\mu_1} u_1 \cdots \nabla^{\lambda_{k-1}}_{\mu_{k-1}} u_{k-1}\, \nabla^{\lambda_k} u_k\, \nabla_{\mu_k} u_{k+1}\, dV \\ &\qquad \text{(using } \nabla \varepsilon^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} = 0 \text{ and (19))} \\ &= -\int_X \varepsilon^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} \nabla^{\lambda_1}_{\mu_1} u_1 \cdots \nabla^{\lambda_{k-1}}_{\mu_{k-1}} u_{k-1}\, \nabla^{\lambda_k}_{\mu_k} u_{k+1} u_k\, dV \quad \text{(by (20))} \\ &= H_k(u_1, ..., u_{k+1}, u_k). \end{aligned} \tag{32}$$

To carry out these integrations by parts, we need in fact the functions $u_k \in C^3(X)$. However, we obtain the result when they are only C^2 by a density argument.

Let us rewrite the deformation equations under the form

$$\begin{aligned} \Gamma_{\pm} &= [1 - M(\phi)] - 1 + \exp(-\varepsilon\phi + f) = 0, \qquad \varepsilon = \pm 1 \\ \Gamma_0(\phi) &= [1 - M(\phi)] + e^f - 1 = 0 \end{aligned} \tag{33}$$

and let us consider the functionals

$$\begin{aligned} \mathfrak{J}_{\pm}(\phi) &= J(\phi) - \int_X \phi\, dV - \varepsilon \int_X e^{-\varepsilon\phi + f} dV \\ \mathfrak{J}_0(\phi) &= J(\phi) + \int_X (e^f - 1)\, \phi\, dV. \end{aligned} \tag{34}$$

[2] For an alternative proof see Eq. (82) of the Appendix.

LEMMA 2. *The C^2 critical points of the functional $\mathfrak{J}_+$, $\mathfrak{J}_-$, or $\mathfrak{J}_0$ are the C^2 solutions of the corresponding deformation equation.*

Proof. We must only verify that if u and v belong to $C^2(X)$ and $j(t) = J(u+tv)$, then

$$j'(0) = \int_X [1 - M(u)]\, v\, dV.$$

By symmetry of the $(k+1)$-linear form H_k,

$$\frac{d}{dt} I_k(u+tv)|_{t=0} = \text{coefficient of } t \text{ in } H_k(u+tv, ..., u+tv)$$

$$= (k+1)\, H_k(u, ..., u, v),$$

hence

$$j'(0) = \sum_{k=1}^{m} H_k(u, ..., u, v) \qquad \text{(by (29) and (31))}$$

$$= \int_X -\left[\sum_{k=1}^{m} \varepsilon^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} \nabla^{\lambda_1}_{\mu_1} u \cdots \nabla^{\lambda_k}_{\mu_k} u\right] v\, dV \qquad \text{(by (30))}$$

$$= \int_X [1 - M(u)]\, v\, dV \qquad \text{(by (16))}.$$

A weak solution of a deformation equation (33) is a critical point of the corresponding functional $\mathfrak{J}_+$, $\mathfrak{J}_-$, or $\mathfrak{J}_0$ in the Sobolev space $W_{2,m}(X)$ (see Proposition 1 below).

We now turn to a discussion of such critical points.

II. SOBOLEV SPACE SETTING FOR THE VARIATIONAL APPROACH TO DEFORMATION EQUATIONS

Let us denote by $\mathfrak{J}$ any of the functionals $\mathfrak{J}_+$, $\mathfrak{J}_-$, and $\mathfrak{J}_0$ defined in (34). For investigating critical points of $\mathfrak{J}$ by means of functional analysis, it is natural to work on Sobolev spaces. The appropriate one turns out to be the Sobolev space $W_{2,m}(X)$, defined as the completion of $C^2(X)$ with respect to any of the equivalent norms

$$\|u\|_{W_{2,m}} = \|\nabla^2 u\|_m + \|\nabla u\|_m + \|u\|_m$$

(where $|\nabla^2 u|^2 = \frac{1}{2}\nabla^{ab}u\,\nabla_{ab}u = \nabla^{\alpha\beta}u\,\nabla_{\alpha\beta}u + \nabla^{\alpha\bar\beta}u\,\nabla_{\alpha\bar\beta}u$ and $\|\ \|_p$ is the $L_p(X)$ norm for $p \geqslant 1$)

$$u \to \|\nabla^2 u\|_m + \|u\|_m \qquad \text{or} \qquad u \to \|\Delta u\|_m + \left|\int_X u\,dV\right|.$$

Since the real dimension of X is $2m$, we are in the limit case of the Sobolev inclusion theorem and, for any $u \in W_{2,m}(X)$, e^u is integrable with convenient inequalities and compactness properties of the mapping

$$u \in W_{2,m}(X) \to e^u \in L_q(X),$$

according to Cherrier [10, Theorems 3 and 5]. More precisely, there exist real numbers A and B such that, if $u \in W_{2,m}(X)$,

$$\int_X e^u\,dV \leqslant A \exp\left[B\|\Delta u\|_m^m + \frac{\int_X u\,dV}{\int_X dV}\right] \tag{35}$$

and thus, for some constants C and μ independent of u,

$$\int_X e^u\,dV \leqslant C\exp(\mu\|u\|_{W_{2,m}}^m). \tag{36}$$

On the other hand, if (u_i) is a bounded sequence of $W_{2,m}(X)$, there exist a subsequence (u_{i_k}) and an element $u \in W_{2,m}(X)$ such that, for any $q \geqslant 1$, $\exp(u_{i_k})$ converges strongly to e^u in $L_q(X)$; by reflexivity of $W_{2,m}(X)$, we can also suppose u_{i_k} converges weakly to u in $W_{2,m}(X)$.

Proposition 1 extends J and $\mathfrak{J}$ to $W_{2,m}(X)$ and defines the weak solutions of the deformation equations as the critical points of the extended functional $\mathfrak{J}$, the C^2 ones being the admissible C^α solutions of these equations. In Proposition 2, positivity and convexity properties of J are studied.

PROPOSITION 1. (i) *H_k (see (30) and (32)) can be extended as a continuous $(k+1)$ linear symmetric form on $W_{2,m}(X)$; thus I and J, defined in (28), (27), extend as polynomial C^∞ functionals on $W_{2,m}(X)$.*

(ii) *$\mathfrak{J}$ extends as a C^∞ bounded real valued mapping on $W_{2,m}(X)$. The zeros of the Fréchet derivative*

$$d\mathfrak{J}\colon W_{2,m}(X) \to (W_{2,m}(X))^*$$

are called the weak solutions of the deformation equations. The weak solutions of class C^2 are in fact smooth C^∞ admissible solutions of these equations.

Proof. (i) Let $\mathfrak{D}_{2,m}(X)$ be the space $C^2(X)$ endowed with $W_{2,m}$-norm. First we show that, for any $k=1, ..., m$, H_k is continuous on $(\mathfrak{D}_{2,m}(X))^{k+1}$. If $u_1, ..., u_{k+1}$ belong to $\mathfrak{D}_{2,m}(X)$, let us put

$$E_k(u_1, ..., u_{k+1}) = \varepsilon^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} \nabla^{\lambda_1}_{\mu_1} u_1 \cdots \nabla^{\lambda_{k-1}}_{\mu_{k-1}} u_{k-1} \nabla_{\mu_k} u_k \nabla^{\lambda_k} u_{k+1}; \quad (37)$$

$E_k(u_1, ..., u_{k+1})$ is the integrand of $H_k(u_1, ..., u_{k+1})$ written under form (32), i.e., after performing one integration by parts in the initial definition (30). Notice that[3]

$$|E_k(u_1, ..., u_{k+1})| \leqslant \text{Const } |\nabla^2 u_1| \cdots |\nabla^2 u_{k-1}| \, |\nabla u_k| \, |\nabla u_{k+1}| \quad (38)$$

since, in a g-orthonormal frame ($g_{\lambda\bar{\mu}} = \delta_{\lambda\mu}$), we can write

$$|E_k| = \left| \sum \varepsilon^{\mu_1 \cdots \mu_1}_{\lambda_1 \cdots \lambda_1} \partial_{\lambda_1 \bar{\mu}_1} u_1 \cdots \partial_{\lambda_{k-1} \overline{\mu_{k-1}}} u_{k-1} \, \partial_{\lambda_k} u_k \, \partial_{\overline{\mu_k}} u_{k+1} \right|$$

(summation extended to all subsets $(\lambda_1, ..., \lambda_k)$, $(\mu_1, ..., \mu_k)$ of k integers between 1 and m)

$$\leqslant \sum (k!)^{-1} |\partial_{\lambda_1 \bar{\mu}_1} u_1| \cdots |\partial_{\lambda_{k-1} \overline{\mu_{k-1}}} u_{k-1}| \, |\partial_{\lambda_k} u_k| \, |\partial_{\overline{\mu_k}} u_{k+1}|$$

$$\leqslant (k!)^{-1} (\sqrt{2m})^{k-1} m |\nabla^2 u_1| \cdots |\nabla^2 u_{k-1}| \, |\nabla u_k| \, |\nabla u_{k+1}|,$$

by Schwarz inequality, taking into account that in the chosen frame, if $u \in C^2(X)$,

$$|\nabla u|^2 = \sum_{\alpha=1}^{m} |\partial_\alpha u|^2 \quad \text{and} \quad |\nabla^2 u|^2 = \sum_{\alpha,\beta=1}^{m} (|\nabla_{\alpha\beta} u|^2 + |\partial_{\alpha\beta} u|^2) \geqslant \sum_{\alpha,\beta} |\partial_{\alpha\beta} u|^2.$$

Defining r_k by

$$\frac{k-1}{m} + \frac{2}{r_k} = 1,$$

r_k increases from 2 to $2m$ when k goes from 1 to m and, by Hölder's inequality, when $v_1, ..., v_{k+1}$ belong to $C^2(X)$,

$$\|v_1 \cdots v_{k+1}\|_1 \leqslant \left(\prod_{l=1}^{k-1} \|v_l\|_m \right) \|v_k\|_{r_k} \|v_{k+1}\|_{r_k}$$

$$\leqslant C(\text{vol}(X, g), m) \left(\prod_{l=1}^{k-1} \|v_l\|_m \right) \|v_k\|_{2m} \, \|v_{k+1}\|_{2m}. \quad (39)$$

[3] See the Appendix for an alternate proof of (38) via differential forms (e.g., Eq. (38′)).

On the other hand, by the Sobolev inclusion theorem, $W_{2,m}(X) \subset W_{1,2m}(X)$ and thus

$$\|\nabla u\|_{2m} \leqslant C(X)\|u\|_{W_{2,m}} \quad \text{for} \quad u \in W_{2,m}(X). \tag{40}$$

Combining inequalities (38), (39), and (40), we get

$$|H_k(u_1, ..., u_{k+1})| \leqslant \int_X |E_k(u_1, ..., u_{k+1})| \, dV \leqslant \text{Const} \prod_{l=1}^{k+1} \|u_l\|_{W_{2,m}},$$

proving by the way the continuity of H_k on $(\mathfrak{D}_{2,m}(X))^{k+1}$. By density, H_k extends as a continuous $(k+1)$-linear symmetric form on $W_{2,m}(X)$, always noted H_k. In fact, the proof shows that H_k can be prolonged to the space

$$(W_{2,m}(X))^{k-1} \times (W_{1,2m}(X))^2. \tag{41}$$

Now relations (28), (30) imply that we can also extend I and J as continuous polynomial (and so C^∞) functionals on $W_{2,m}(X)$, the following equalities remaining true for any $\phi \in W_{2,m}(X)$:

$$I(\phi) = \sum_{k=1}^{m} H_k(\phi, ..., \phi), \qquad J(\phi) = \sum_{k=1}^{m} \frac{H_k(\phi, ..., \phi)}{k+1}.$$

Lemma 2 shows that the Fréchet derivative of J is given by

$$\langle dJ(u), \xi \rangle = \sum_{k=1}^{m} H_k(u, ..., u, \xi), \qquad u \text{ and } \xi \in W_{2,m}(X). \tag{42}$$

Here $\langle \, , \, \rangle$ is the natural pairing between a Banach space and its dual. dJ: $W_{2,m}(X) \to (W_{2,m}(X))^*$ is a smooth C^∞ mapping.

(ii) $\mathfrak{J}$ (34) extends as a C^∞ bounded mapping from $W_{2,m}(X)$ to $\mathbb{R}$. Let us prove it for $\mathfrak{J}_\pm$. If $u \in W_{2,m}(X)$, u and e^u are integrable; so we can set

$$\mathfrak{J}_\pm(u) = J(u) - \int_X u \, dV - \varepsilon \int_X \exp(-\varepsilon u + f) \, dV.$$

This functional is the required extension since we are going to see it is continuous on $W_{2,m}(X)$. Boundedness follows from the C^∞ polynomial nature of J and inequality (36). As regards smoothness, defining

$$E(u) = \int_X e^{u+f} \, dV,$$

one verifies by induction on l that

$$(D^lE(u))(\xi_1, ..., \xi_l) = \int_X e^{u+f}\xi_1 \cdots \xi_l \, dV, \qquad \text{for} \quad u, \xi_1, ..., \xi_l \in W_{2,m}(X). \tag{43}$$

Since e^u and the ξ_k's belong to $L_q(X)$ for any $q \geq 1$, the integral of (43) is well defined according to Hölder's inequality. We must check that the application D^lE sends differentiably $W_{2,m}(X)$ in $\mathfrak{L}^l(W_{2,m}(X); \mathbb{R})$, the Banach space of l-continuous forms on $W_{2,m}(X)$, with $D(D^lE) = D^{l+1}E$. If $h \in W_{2,m}(X)$, we have

$$|e^h - 1 - h| \leq \frac{h^2}{2} e^{|h|}$$

and thus

$$\left|(D^lE(u+h) - D^lE(u))(\xi_1, ..., \xi_l) - \int_X e^{u+f}h\xi_1 \cdots \xi_l \, dV\right|$$
$$\leq \frac{1}{2}\int_X e^f e^u h^2 e^{|h|} |\xi_1 \cdots \xi_l| \, dV$$
$$\leq \frac{1}{2} e^{\max f} \|e^u\|_q \|e^{|h|}\|_q \|h\|_q^2 \prod_{k=1}^{l} \|\xi_k\|_q,$$

with $q = (l+4)^{-1}$ by Hölder. Taking into account the continuous embedding $W_{2,m}(X) \subset L_q(X)$ and the fact that, for any bounded subset $\mathfrak{B} \subset W_{2,m}(X)$,

$$\sup_{\mathfrak{B}} \|e^{|h|}\|_q \leq \sup_{\mathfrak{B}} (\|e^h\|_q + \|e^{-h}\|_q) < \infty \qquad \text{(by (36))},$$

for some constant A independent of h, we can write

$$\|D^lE(u+h) - D^lE(u) - (D^{l+1}E(u))(h, ., ..., .)\|_{\mathfrak{L}^l(W_{2,m}(X);\mathbb{R})}$$
$$\leq A\|e^{|h|}\|_q \|h\|_{W_{2,m}}^2 = o(\|h\|_{W_{2,m}}),$$

which justifies (43).

The critical points $u \in W_{2,m}(X)$ of $\mathfrak{J}$, i.e., the weak solutions of deformation equations, satisfy $d\mathfrak{J}(u) = 0$, that is, for $\delta = 0, 1, -1$ and $\xi \in W_{2,m}(X)$,

$$\langle d\mathfrak{J}_\delta(u), \xi \rangle = \langle dJ(u), \xi \rangle + \int_X (e^{-\delta u + f} - 1)\, \xi \, dV = 0. \tag{44}$$

As we saw H_k can be extended by continuity to the product space (41), notice that, according to (42) and inclusion $W_{1,2m}(X) \subset L_q(X)$, the right-hand side of (44) is meaningful when $\xi \in W_{1,2m}(X)$, Eq. (44) remaining satisfied for such ξ at a critical point u.

The admissibility (12) and C^∞ smoothness of C^2 weak solutions follow from Aubin [1, p. 144].

Let $\bar{\mathfrak{A}}$ be the $W_{2,m}$-closure of the set $\mathfrak{A}$ of admissible functions (12). It is a closed convex subset of $W_{2,m}(X)$ for both strong and weak topologies. As regards convexity and sign of J on $\bar{\mathfrak{A}}$, we can state the

PROPOSITION 2. *On* $\bar{\mathfrak{A}}$,

(i) *the functional J is convex and, in fact, the second variation of J (47′) is the Dirichlet integral in the deformed metric;*

(ii) *I and J are $\geqslant 0$, the following inequality being verified:*

$$0 \leqslant J \leqslant I \leqslant (m+1)\, J. \tag{45}$$

Note. Inequality (45) appears in Aubin [4, p. 146], where it is shown in an inductive procedure. Here, the proof is different and shorter.

Proof. (i) Let $u, v, w \in W_{2,m}(X)$. Equation (42) yields

$$\begin{aligned}(d(dJ)(u))(v) &= \frac{d}{dt}\, dJ(u+tv)|_{t=0}\\ &= \frac{d}{dt} \sum_{k=1}^{m} H_k(u+tv, ..., u+tv, .)|_{t=0}\\ &= \sum_{k=1}^{m} kH_k(u, ..., u, v, .), \qquad \text{by symmetry of } H_k.\end{aligned}$$

Thus the second Fréchet derivative of J in u is given by[4]

$$\begin{aligned}(d^2J(u))(v, w) &= \sum_{k=1}^{m} kH_k(u, ..., u, v, w)\\ &= \int_X \sum_{k=1}^{m} \frac{\delta^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k}}{(k-1)!} \nabla^{\lambda_1}_{\mu_1} u \cdots \nabla^{\lambda_{k-1}}_{\mu_{k-1}} u\, \nabla_{\mu_k} v\, \nabla^{\lambda_k} w\, dV, \end{aligned} \tag{46}$$

using the remark concluding part (i) of the proof of Proposition 1 and (15).

[4] See the Appendix (46′) for an alternate representation.

Pick u in $\mathfrak{A}$. To prove convexity of J in u, we shall show that the symmetric bilinear form $d^2J(u)$ is non-negative, i.e.,

$$(d^2J(u))(v, v) = \frac{d^2}{dt^2} J(u+tv)|_{t=0} \geqslant 0 \qquad \text{for any } v \in W_{2,m}(X). \qquad (47)$$

By smoothness of J and density, we can suppose $u \in \mathfrak{A}$ and $v \in C^2(X)$. Let A be the integrand of the integral (46) defining $(d^2J(u))(v, v)$. In $P \in X$, choose a frame adapted to u (i.e., g-orthonormal and diagonalizing the matrix $(\partial_{\lambda\bar{\mu}} u(P))_{\lambda,\mu}$) and set

$$a_\lambda = \partial_{\lambda\bar{\lambda}} u(P), \qquad b_\lambda = 1 + a_\lambda.$$

For $\mu = 1, ..., m$ and $k = 0, 1, ..., m-1$, we denote by $E_{k,\mu}$ (resp. $F_{k,\mu}$) the kth elementary symmetric function of the $m-1$ variables a_λ (resp. b_λ), $\lambda \neq \mu$. Thus $E_{0,\mu} = 1$ and, if $k \geqslant 1$,

$$E_{k,\mu} = \sum_{\substack{1 \leqslant \lambda_1 < \cdots \lambda_k \leqslant m \\ \lambda_l \neq \mu}} a_{\lambda_1} \cdots a_{\lambda_k}.$$

Now $A(P)$ is equal to

$$\begin{aligned} A(P) &= \sum_{k=1}^{m} \sum_{\substack{\lambda_1 \neq \cdots \neq \lambda_k \\ \lambda_l \in \{1,...,m\}}} ((k-1)!)^{-1} a_{\lambda_1} \cdots a_{\lambda_{k-1}} \partial_{\lambda_k} v \, \partial_{\bar{\lambda}_k} v \\ &= \sum_{\mu=1}^{m} \left[1 + \sum_{k=1}^{m} \sum_{\substack{1 \leqslant \lambda_1 < \cdots < \lambda_k \leqslant m \\ \lambda_l \neq \mu}} a_{\lambda_1} \cdots a_{\lambda_k} \right] |\partial_\mu v|^2 \\ &= \sum_{\mu=1}^{m} \left(\sum_{k=0}^{m-1} E_{k,\mu} \right) |\partial_\mu v|^2 = \sum_{\mu=1}^{m} \left(\prod_{k \neq \mu} (1 + a_k) \right) |\partial_\mu v|^2 \\ &= \sum_{\mu=1}^{m} F_{m-1,\mu} |\partial_\mu v|^2, \end{aligned}$$

yielding (47) since by hypothesis of admissibility on u, all the coefficients $F_{m-1,\mu}$ are >0. Notice that if $u \in \mathfrak{A}$, $A(P)$ can be written

$$\begin{aligned} A(P) &= \left[\sum_{\mu=1}^{m} (1 + a_\mu)^{-1} |\partial_\mu v|^2 \right] \left[\prod_{k=1}^{m} (1 + a_k) \right] \\ &= g'^{\lambda\bar{\mu}} \partial_\lambda v \, \partial_{\bar{\mu}} v \, M(u) = |\nabla' v|^2 M(u) \end{aligned}$$

and consequently

$$\frac{d^2}{dt^2} J(u+tv)|_{t=0} = \int_X A(P) \, dV(P) = \int_X |\nabla' v|^2 \, dV', \qquad (47')$$

the quantities being related to the Kähler deformation g_u (11′) of g associated to u; so, when $v \neq$ Const., J is strictly convex in the v-direction. (See the end of the Appendix for a proof via differential forms.)

Following the pattern of Proposition 1, let us now define the C^∞ polynomial functionals

$$\tilde{H}_k(u_1, ..., u_{k+1}) = \int_X \varepsilon^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} (\delta^{\lambda_1}_{\mu_1} + \nabla^{\lambda_1}_{\mu_1} u_1) \cdots (\delta^{\lambda_{k-1}}_{\mu_{k-1}} + \nabla^{\lambda_{k-1}}_{\mu_{k-1}} u_{k-1}) \times \nabla_{\mu_k} u_k \, \nabla^{\lambda_k} u_{k+1} \, dV \tag{48}$$

$$\tilde{I}_k(u) = \tilde{H}_k(u, ..., u),$$

with $k = 1, ..., m$, $u_1, ..., u_{k-1} \in W_{2,m}(X)$ and $u_k, u_{k+1} \in W_{1,2m}(X)$. The previous computation shows that, for $u \in \tilde{\mathfrak{A}}$ and $v, w, w_1 \in W_{2,m}(X)$,

$$\tilde{H}_k(u, ..., u, v, v) \geqslant 0, \qquad (d^2 J(v))(w, w_1) = m\tilde{H}_m(v, ..., v, w, w_1). \tag{49}$$

Remark $\tilde{\mathfrak{A}}$ is the largest convex subset of $W_{2,m}(X)$ containing $\mathfrak{A}$ on which J is convex. Indeed, choose $u \in C^2(X)$ such that the metric $g_u = g + \partial\bar{\partial} u$ is $\geqslant 0$ but not > 0, i.e., admits somewhere one zero eigenvalue. Arbitrarily close to u in C^2 topology, we can find u' with the property $(d^2 J(u'))(v, v) < 0$, by picking v with support sufficiently small contained in the open set of X on which the tensor $g_{u'}$ has at least one negative eigenvalue.

(ii) Concerning the sign of $I(u)$ and $J(u)$, by continuity, we can take $u \in C^2(X)$. Taking into account the definitions (29) of I, J and (31) of I_k and H_k, if we set $I(u) = \int_X B \, dV$, $J(u) = \int_X C \, dV$, we see that, in the previously used frame adapted to u, the integrands B and C for I and J are then respectively given by

$$B = \sum_{\mu=1}^{m} \left(\sum_{k=0}^{m-1} \frac{E_{k,\mu}}{k+1} \right) |\partial_\mu u|^2, \qquad C = \sum_{\mu=1}^{m} \left(\sum_{k=0}^{m-1} \frac{E_{k,\mu}}{(k+1)(k+2)} \right) |\partial_\mu u|^2.$$

(Recall that the elementary functions $E_{k,\mu}$ and $F_{k,\mu}$ are defined in part (i) of the proof.) The inequalities (45) follow from

$$0 \leqslant C \leqslant B \leqslant (m+1)\, C \quad \text{when} \quad u \in \mathfrak{A}. \tag{45′}$$

Proof of (45′). We are going to express

$$\beta_\mu = \sum_{k=0}^{m-1} \frac{E_{k,\mu}}{k+1} \quad \text{and} \quad \gamma_\mu = \sum_{k=0}^{m-1} \frac{E_{k,\mu}}{(k+1)(k+2)} \tag{50}$$

as linear combinations of the $(F_{l,\mu})_l$ with positive coefficients. Suppressing the index μ, we set

$$P(x)=\sum_{k=0}^{m-1}(-1)^k E_k x^{m-1-k}, \qquad Q(x)=\sum_{k=0}^{m-1}(-1)^k F_k x^{m-1-k}.$$

Then, by Taylor's formula,

$$P(x)=Q(1+x)=\sum_{k=0}^{m-1} Q^{(m-1-k)}(1)\frac{x^{m-1-k}}{(m-1-k)!}$$

and

$$E_k=(-1)^k\frac{Q^{(m-1-k)}(1)}{(m-1-k)!}=\frac{(-1)^h}{(m-1-k)!}\sum_{l=0}^{k}(-1)^l$$
$$\times(m-1-l)\cdots(k-l+1)\,F_l,$$

yielding

$$E_k=\sum_{l=0}^{k}(-1)^{k+l}C_{m-1-l}^{m-1-k}F_l. \tag{51}$$

Combining (50) and (51), we obtain

$$\beta=\sum_{l=0}^{m-1}s_lF_l, \qquad \gamma=\sum_{l=0}^{m-1}t_lF_l, \tag{52}$$

where

$$s_l=\sum_{k=l}^{m-1}(-1)^{k+l}\frac{C_{m-1-l}^{m-1-k}}{k+1}, \qquad t_l=\sum_{k=l}^{m-1}(-1)^{k+l}\frac{C_{m-1-l}^{m-1-k}}{(k+1)(k+2)}$$

and thus

$$t_l=s_l-\sum_{k=l}^{m-1}(-1)^{k+l}\frac{C_{m-1-l}^{m-1-k}}{k+2}. \tag{53}$$

Now, using

$$\int_0^1 x^k(1-x)^l\,dx=\frac{k!\,l!}{(k+l+1)!},$$

we can write

$$s_l = \sum_{k=0}^{m-1-l} (-1)^{m-1-l-k} \frac{C^k_{m-1-l}}{m-k} = \int_0^1 (1-x)^{m-1-l} x^l \, dx$$

$$= \frac{(m-1-l)!\, l!}{m!} = (mC^l_{m-1})^{-1}, \tag{54}$$

since $(1-x)^{m-1-l} x^l = \sum_{k=0}^{m-1-l} (-1)^{m-1-l-k} C^k_{m-1-l} x^{m-1-k}$. In the same way,

$$\sum_{k=l}^{m-1} (-1)^{k+l} \frac{C^{m-1-k}_{m-1-l}}{k+2}$$

$$= \sum_{k=0}^{m-1-l} (-1)^{m-1-l-k} \frac{C^k_{m-1-l}}{m+1-k}$$

$$= \int_0^1 (1-x)^{m-1-l} x^{l+1} dx = \frac{(m-1-l)!\,(l+1)!}{(m+1)!} = \frac{l+1}{m+1} s_l;$$

thus, by (53), (54),

$$t_l = \frac{m-l}{m+1} s_l = \frac{(m-l)!\, l!}{(m+1)!} = [(m+1)\, C^l_m]^{-1} \tag{55}$$

and

$$0 \leqslant t_l \leqslant s_l \leqslant (m+1)\, t_l,$$

which leads to (45′), by (52) and positivity of the F_l's.

Remark. (1) With the aid of functionals $\tilde{I}_k$ defined in (48), I and J can be expressed on $W_{2,m}(X)$ under the form

$$I = \sum_{k=1}^{m} k s_{k-1} \tilde{I}_k, \qquad J = \sum_{k=1}^{m} k t_{k-1} \tilde{I}_k,$$

the assertion justified by comparing, in an adapted frame, the integrands of both sides of these equations, using (52). For example, if B and $\tilde{B}_k$ are the integrands of I and $\tilde{I}_k$, we have

$$B = \sum_{\mu=1}^{m} \beta_\mu |\partial_\mu u|^2 = \sum_{\mu=1}^{m} \sum_{k=0}^{m-1} s_k F_{k,\mu} |\partial_\mu u|^2$$

$$= \sum_{k=0}^{m-1} (k+1)\, s_k \left[\sum_{\mu=1}^{m} \frac{F_{k,\mu}}{k+1} |\partial_\mu u|^2 \right] = \sum_{k=1}^{m} k s_{k-1} \tilde{B}_k.$$

Hence, by (54) and (55),

$$I = \sum_{k=1}^{m} (C_m^k)^{-1} \tilde{I}_k, \qquad J = \sum_{k=1}^{m} (C_{m+1}^k)^{-1} \tilde{I}_k. \tag{56}$$

Since $\tilde{I}_k$ is $\geqslant 0$ on $\mathfrak{A}$ (49) and $\tilde{I}_1(u) = \int_X |\nabla u|^2 \, dV$, we obtain

$$I(u) \geqslant \frac{1}{m} \|\nabla u\|_2^2, \qquad J(u) \geqslant \frac{1}{m+1} \|\nabla u\|_2^2 \qquad \text{when} \quad u \in \mathfrak{A}. \tag{57}$$

On the other hand, since $\sum_{k=1}^m H_k = \sum_{k=1}^m (C_m^k)^{-1} \tilde{H}_k$ and $dJ(u) = \sum_{k=1}^m H_k(u, ..., u, .)$ by (42), we infer for any $u \in W_{2,m}(X)$ and $\xi \in W_{1,2m}(X)$, that

$$\langle dJ(u), \xi \rangle = \sum_{k=1}^{m} (C_m^k)^{-1} \tilde{H}_k(u, ..., u, \xi) \tag{58}$$

(recalling that $dJ(u)$ can be extended by continuity to $W_{1,2m}(X)$). Now let h be an increasing C^1 real valued function on $\mathbb{R}$, with $h' \in L_\infty(\mathbb{R})$, more generally, we may suppose h increasing Lipschitzian. In (58), take $u \in \mathfrak{A}$ and $\xi = h(u)$. Then

$$\langle dJ(u), h(u) \rangle \geqslant \frac{1}{m} \tilde{H}_1(u, h(u)) = \frac{1}{m} \int_X h'(u) |\nabla u|^2 \, dV, \tag{59}$$

since, for $k = 2, ..., m$,

$$\tilde{H}_k(u, ..., u, h(u))$$
$$= \int_X \varepsilon^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} (\delta^{\lambda_1}_{\mu_1} + \nabla^{\lambda_1}_{\mu_1} u) \cdots (\delta^{\lambda_{k-1}}_{\mu_{k-1}} + \nabla^{\lambda_{k-1}}_{\mu_{k-1}} u) \, |\nabla u|^2 h'(u) \, dV \geqslant 0$$

(the integrand is $\geqslant 0$ because $u \in \mathfrak{A}$ and $h'(u) \geqslant 0$). Equation (59) is analogous to Aubin's inequality (18) in [1, p. 149].

In the next section, we will use (57) for getting lower bounds of $\mathfrak{J}$ and (59) is the cornerstone in the proof of boundedness of weak solutions of deformation equations (Proposition 4 below).

(2) Unlike what happens for convexity, minorations of type (57), (59) are valid on subsets of $W_{2,m}(X)$ larger than $\mathfrak{A}$, e.g., on

$$\overline{\mathfrak{A}_c} = \overline{c\mathfrak{A}} \qquad \text{when} \quad 1 \leqslant c < c(m). \tag{60}$$

($\mathfrak{A}_c$ is the set of $u \in C^2(X)$ such that the tensor $g_u = cg + \partial\bar{\partial} u$ is everywhere positive definite). Of course, m^{-1} and $(m+1)^{-1}$ must be replaced by lower positive constants depending on c and m. As in (48) one introduces the functionals

$$\tilde{H}_{c,k}(u_1, ..., u_{k+1}) = \int_X \varepsilon^{\mu_1 \cdots \mu_k}_{\lambda_1 \cdots \lambda_k} \prod_{l=1}^{k-1} (c\delta^{\lambda_l}_{\mu_l} + \nabla^{\lambda_l}_{\mu_l} u_l) \nabla_{\mu_k} u_k \nabla^{\lambda_k} u_{k+1}\, dV$$

$$= c^{k+1} \tilde{H}_k \left(\frac{u_1}{c}, ..., \frac{u_{k+1}}{c}\right), \qquad k \geqslant 2,$$

$$\tilde{I}_{c,k}(u) = \tilde{H}_{c,k}(u, ..., u),$$

$\tilde{I}_{c,k}$ being $\geqslant 0$ on $\overline{\mathfrak{A}_c}$. One expands I and J as linear combinations of $\tilde{I}_1$ and $\tilde{I}_{c,k}$, $2 \leqslant k \leqslant m$. The coefficients can be explicitly computed, but are less simple than in (56), (58), and are >0 when c is sufficiently close to 1. For instance, in case $m = 2$, one immediately obtains

$$I(u) = \frac{2-c}{2} \|\nabla u\|_2^2 + \tilde{I}_{c,2}(u), \qquad J(u) = \frac{3-c}{6} \|\nabla u\|_2^2 + \frac{1}{3} \tilde{I}_{c,2}(u).$$

III. On Existence and Regularity of Weak Solutions of Deformation Equations

For any function $u \in L_1(X)$, we note $\bar{u} = V^{-1} \int_X u\, dV$ the average of u over the manifold (X, g) of volume $V = \int_X dV$. Recall that the topology of $W_{1,q}(X)$ can be defined by the norm $u \to \|\nabla u\|_q + |\bar{u}|$.

Proposition 3. *Let (X_{2m}, g) be a compact Kähler manifold with first Chern class $C_1(X)$ of prescribed sign.*

(i) *Case $C_1(X) < 0$. On $\mathfrak{A}$, the functional*

$$\mathfrak{J}_-(u) = J(u) - \int_X u\, dV + \int_X e^{u+f}\, dV$$

is bounded from below, convex, and thus weakly lower semicontinuous.

(ii) *Case $C_1(X) = 0$. On $\mathfrak{A}$, the functional*

$$\mathfrak{J}_0(u) = J(u) + \int_X (e^f - 1)\, u\, dV$$

is convex, weakly lower semicontinuous, and bounded from below if

$$\int_X e^f\, dV = V. \tag{61}$$

Setting

$$H_l = \left\{ u \in W_{2,m}(X); \int_X u \, dV = l \right\}, \qquad l \in \mathbb{R},$$

under condition (61), H_l *is a natural constraint for* $J_0(u)$ *over* $W_{2,m}(X)$ (*in the sense defined below*).

(iii) *Case* $C_1(X) > 0$. *The functional*

$$\mathfrak{J}_+(u) = J(u) - \int_X u \, dV - \int_X e^{-u+f} \, dV$$

over $W_{2,m}(X)$ *has as a natural constraint* (*see the definition below*) *the weakly closed* C^∞ *hypersurface*

$$\Sigma = \left\{ u \in W_{2,m}(X); G(u) = \int_X e^{-u+f} \, dV = V \right\}. \tag{62}$$

$\mathfrak{J}_+$ *is weakly lower semicontinuous on* $\mathfrak{A}$. *Assuming the first fundamental form* ω *belongs to* $C_1(X)$ *and Aubin's inequality* (69), $\mathfrak{J}_+$ *is bounded from below on* $\mathfrak{A} \cap \Sigma$ *if we restrict the manifold X by the geometric assumption*

$$\int_X \overset{m}{\Lambda} C_1 < (m+1)^{2m} (2m)^{-m}. \tag{63}$$

(*When* $m = 1$, *note that this requires* $0 < \chi(x) < 2$ *as in* [5, *p.* 331].)

(iv) *For any minimizing sequence* (u_k) *relative to* $\mathfrak{J}_-|_{\mathfrak{A}}$, $\mathfrak{J}_0|_{\mathfrak{A} \cap H_l}$, *or* $\mathfrak{J}_+|_{\mathfrak{A} \cap \Sigma}$, *the sequences* $(J(u_k))$ *and* $(\|u_k\|_{W_{1,2}})$ *are bounded.*

Proof. (i) $C_1(X) < 0$. By Proposition 2, J is convex on $\mathfrak{A}$ and so is $\mathfrak{J}_-$ because the mapping $u \to -\int_X u \, dV + \int_X e^{u+f} \, dV$ is clearly convex on $W_{2,m}(X)$. Hence $\mathfrak{J}_-$ is weakly lower semicontinuous on $\mathfrak{A}$ (Berger [6, p. 301]), i.e., if a sequence (v_k) of $\mathfrak{A}$ converges weakly in $W_{2,m}(X)$ to v ($v \in \mathfrak{A}$ since $\mathfrak{A}$ is weakly closed), then $\mathfrak{J}_-(v) \leqslant \liminf_{k \to \infty} \mathfrak{J}_-(v_k)$.

On the other hand, for any $u \in \mathfrak{A}$, using $e^{u+f} \geqslant 1 + u + f$, $J(u) \geqslant 0$ (Proposition 2) and setting $C_1 = V + \int_X f \, dV$, we have

$$\mathfrak{J}_-(u) \geqslant J(u) + C_1 \geqslant C_1. \tag{64}$$

Thus $\mu = \operatorname{Inf}_{u \in \mathfrak{A}} \mathfrak{J}_-(u) > -\infty$. Now consider a minimizing sequence $\{u_k\} \in \mathfrak{A}$. Thus if $u_k \in \mathfrak{A}$ and $\lim_{k \to \infty} \mathfrak{J}_-(u_k) = \mu$, $\mathfrak{J}_-(u_k) \leqslant C_2$. So, according to (64), (57),

$$J(u_k) \leqslant C_2 - C_1 \qquad \text{and} \qquad \|\nabla u_k\|_2 \leqslant [(m+1)(C_2 - C_1)]^{1/2}. \tag{65}$$

Then

$$\int_X u_k \, dV = -\mathfrak{J}_-(u_k) + \left(J(u_k) + \int_X e^{u_k+f} \, dV\right) \geq -C_2.$$

Thanks to the inequality $e^{f+t} + C_3 \geq 2t$ ($t \in \mathbb{R}$) for some positive constant C_3 depending on $\min_X f$, we can also write

$$\mathfrak{J}_-(u_k) \geq \int_X (e^{u_k+f} - u_k) \, dV \geq \int_X u_k \, dV - C_3$$

and

$$\int_X u_k \, dV \leq C_2 + C_3.$$

Therefore, $|\int_X u_k \, dV| \leq C_4$ and, by (65), $\sup_k \|u_k\|_{W_{1,2}} < \infty$. Thus the minimizing sequence $\{u_k\}$ has a weakly convergent subsequence in $W_{1,2}(X)$ with limit $\bar{u}$.

It is natural to conjecture that the infimum μ is reached at $\bar{u}$ since the deformation equation $\Gamma_-(\phi) = 0$ (33) admits a unique solution $\phi \in \mathfrak{A}$ [1], consequently ϕ must be the absolute minimizer of the functional on $\mathfrak{A}$.

(ii) $C_1(X) = 0$. Let ϕ be a C^2 solution of the deformation equation $\Gamma_0(\phi) = 0$ (33). By integration over X, we obtain (61) because the volume is preserved in the Kähler deformation $g + \partial\bar{\partial}\phi$ (17). Equation (61) is a necessary condition that f must satisfy in order that $\Gamma_0(\phi) = 0$ be solvable and under which $\mathfrak{J}_0$ is invariant by addition of constants. We suppose (61) verified.

Notice that the hyperplane H_l is a natural constraint for our variational problem, i.e., any critical point u of $\mathfrak{J}_0|_{H_l}$ is a weak solution. In fact, there exists a Lagrange multiplier λ such that, if $\xi \in W_{2,m}(X)$,

$$\langle d\mathfrak{J}_0(u), \xi \rangle + \lambda \int_X \xi \, dV = \langle dJ(u), \xi \rangle + \int_X (e^f - 1 + \lambda) \, \xi \, dV = 0.$$

Taking $\xi \equiv 1$, from (61), we deduce

$$\int_X e^f \, dV + (\lambda - 1) \, V = \lambda V = 0,$$

thus $\lambda = 0$ and $d\mathfrak{J}_0(u) = 0$.

As previously, $\mathfrak{J}_0$ is clearly convex and thus weakly lower semicontinuous on the convex weakly closed subsets $\mathfrak{A}$ and $\mathfrak{A} \cap H_l$ of $W_{2,m}(X)$.

$\mathfrak{J}_0$ is bounded from below on $\mathfrak{A}$. Let $u \in \mathfrak{A}$ and $u' = u - \bar{u}$. As $\overline{u'} = 0$,

$$\mathfrak{J}_0(u) = \mathfrak{J}_0(u') = J(u') + \int_X e^f u' \, dV \tag{66}$$

and, by Poincaré's inequality,

$$\|u'\|_1 \leqslant \alpha \|\nabla u\|_2. \tag{67}$$

Also $u' \in \mathfrak{A}$. Hence, taking into account (57) and setting $\beta = \exp(\max_X f)$, we can write

$$\begin{aligned} \mathfrak{J}_0(u) &\geqslant \frac{\|\nabla u\|_2^2}{m+1} - \beta \|u'\|_1 \\ &\geqslant \frac{\|\nabla u\|_2^2}{m+1} - \alpha\beta \, \|\nabla u\|_2 \geqslant \gamma, \end{aligned} \tag{68}$$

where $\gamma = \min_{t>0} (t^2/(m+1) - \alpha\beta t)$. So $\nu = \mathrm{Inf}_{\mathfrak{A}} \mathfrak{J}_0 = \mathrm{Inf}_{\mathfrak{A} \cap H_l} \mathfrak{J}_0 > -\infty$.

Let (u_k) be a minimizing sequence belonging to $\mathfrak{A} \cap H_l$. Then $\mathfrak{J}_0(u_k) \leqslant C_1$ and inequality (68) implies that $\|\nabla u_k\|_2 \leqslant C_2$. Since $\bar{u}_k = l$ by definition, $\sup_k \|u_k\|_{W_{1,2}} < \infty$. Also, by (45), (66), and (67),

$$\begin{aligned} 0 \leqslant J(u_k) = \mathfrak{J}_0(u_k) - \int_X e^f u'_k \, dV &\leqslant C_1 + e^\beta \|u'_k\|_1 \\ &\leqslant C_1 + \alpha e^\beta C_2. \end{aligned}$$

Now, a subsequence (u_{k_j}) converges weakly in $W_{1,2}(X)$, toward a limit u. It is reasonable to conjecture that infimum ν is reached in some point, a candidate being u. Indeed, the equation $\Gamma_0(\phi) = 0$ admits a unique solution ϕ in $\mathfrak{A} \cap H_l$, and so ϕ must be the absolute minimum of the functional in $\mathfrak{A} \cap H_l$.

(iii) $C_1(X) > 0$. First, let $\phi \in C^2(X)$ be a solution of $\Gamma_+(\phi) = 0$ (33). Integrating over X, we find

$$G(\phi) = \int_X e^{-\phi + f} \, dV = V,$$

which means that ϕ necessarily belongs to the set Σ defined in (62).

Σ is a strictly convex C^∞ hypersurface of $W_{2,m}(X)$. The proof of Proposition 1, (ii) shows that the function G is smooth and that, for every $u \in W_{2,m}(X)$, $dG(u) \neq 0$ and the hessian $d^2G(u)$ is positive definite. Σ is also weakly closed. This follows from the compactness of the mapping $u \in W_{2,m}(X) \to e^{-u} \in L_q(X)$ recalled at the beginning of Section II.

In fact, G is a weakly continuous function on $W_{2,m}(X)$ and therefore $\mathfrak{J}_+$ is weakly lower semicontinuous on $\mathfrak{A}$ and $\mathfrak{A} \cap \Sigma$ (by convexity of J on $\mathfrak{A}$). Indeed, let (u_n) be a sequence of $W_{2,m}(X)$ weakly converging to u; we have only to prove that $G(u)$ is the unique cluster value of the sequence $(G(u_n))$. Let $(v_k) = (u_{n_k})$ be a subsequence of (u_n) such that $\lim_{k \to \infty} G(v_k) = l$. Since v_k tends weakly to u, $\sup_k (\|v_k\|_{W_{2,m}}) < \infty$. By compactness of the embeddings $W_{2,m}(X) \subset L_q(X)$, Kondrakov's theorem, a subsequence $(w_j) = (v_{k_j})$ converges to u in any $L_q(X)$ and we may suppose $\lim_{j \to \infty} w_j(x) = u(x)$ for almost all $x \in X$. Now inequality (36) implies the sequence (e^{-w_j}) is bounded in the Lebesgue spaces $L_q(X)$ and, because e^{-w_j} converges to e^{-u} almost everywhere, e^{-w_j} tends also to e^{-u} weakly in $L_q(X)$, by a well-known integration result. Consequently $l = G(u)$ since

$$\int_X e^{-u+f}\, dV = \lim_{j \to \infty} \int_X e^{-w_j} e^f\, dV = \lim_{j \to \infty} G(w_j) = l.$$

Σ is a natural constraint for the variational problem under study, which is thus an isoperimetric one (see Berger [6, p. 324]). This means that the critical points u of the restriction of $\mathfrak{J}_+$ to Σ are the critical points of $\mathfrak{J}_+$ on $W_{2,m}(X)$ itself; hence weak solutions appear to be of saddle point type. In fact, writing at such a point u the proportionality of $d\mathfrak{J}_+(u)$ and $dG(u)$, we get a Lagrange multiplier λ such that, for any $\xi \in W_{2,m}(X)$,

$$\langle d\mathfrak{J}_+(u) + \lambda\, dG(u), \xi \rangle = \langle dJ(u), \xi \rangle - \int_X \xi\, dV + (\lambda + 1) \int_X e^{-u+f} \xi\, dV = 0.$$

Picking $\xi \equiv 1$ yields

$$(\lambda + 1) \int_X e^{-u+f}\, dV - V = 0;$$

hence, since $u \in \Sigma$, $\lambda \int_X e^{-u+f}\, dV = 0$ and $\lambda = 0$, so $d\mathfrak{J}_+(u) = 0$.

Remark that constraint Σ is the exact analogue of the Gauss–Bonnet theorem in complex dimension $m = 1$ (Berger [5]).

Note that $\mathfrak{J}_+$ is not bounded from below on $\mathfrak{A}$, because $\lim_{k \to \infty} \mathfrak{J}_+(k) = -\infty$, for appropriate constants k. According to the previous paragraph, we are led to minimize $\mathfrak{J}_+$ over $\mathfrak{A} \cap \Sigma$. To this end we shall need Aubin's fundamental hypothesis [4, p. 148], an extension of (35). For any compact Kähler manifold (X_{2m}, g), there exist two constants C and ξ such that, if $p \geqslant 1$, every function $u \in \mathfrak{A}$ satisfies

$$\int_X e^{-pu}\, dV \leqslant C \exp\left[\xi p^{m+1} I(u) - p\bar{u}\right]. \tag{69}$$

When $C_1(X)>0$ and $\omega \in C_1(X)$, the best constant ξ_m (i.e., the infimum of real numbers ξ for which there exists $C(\xi)$ such that (69) is verified) is obtained by studying the case of a ball in $\mathbb{C}^m$, so that

$$\xi_m = (2/\pi)^m m^{m+1} (m+1)^{-2m-1} (m-1)!. \tag{70}$$

Assume $\omega \in C_1(X)$ and take $u \in \mathfrak{A} \cap \Sigma$. Since $\int_X e^{-u+f}\, dV = V$,

$$\min_X f + \operatorname{Log}\left(\int_X e^{-u}\, dV\right) \leqslant \operatorname{Log} V$$

and, by (69), (45), for any $\xi > \xi_m$,

$$\operatorname{Log}\left(\int_X e^{-u}\, dV\right) \leqslant C_1(\xi) + \xi I(u) - \bar{u} \leqslant C_1 + (m+1)\, \xi J(u) - \bar{u}.$$

Therefore,

$$C_2 \leqslant (m+1)\, \xi J(u) - \bar{u}$$

and we can write

$$\mathfrak{J}_+(u) = J(u) - V\bar{u} - V \geqslant [1-(m+1)\, \xi V]\, J(u) + C_3. \tag{71}$$

Suppose

$$1-(m+1)\, \xi_m V > 0 \tag{72}$$

and pick ξ sufficiently close to ξ_m in order that

$$\delta = 1-(m+1)\, \xi V > 0.$$

Now inequality (71) and non-negativity of $J(u)$ yield the boundedness from below of $\mathfrak{J}_+$ on $\mathfrak{A} \cap \Sigma$:

$$\mathfrak{J}_+(u) \geqslant \delta J(u) + C_4(\delta) \geqslant C_4(\delta). \tag{73}$$

Since ω belongs to $C_1(X)$, by (10), $V = (\pi^m/m!) \int_X \overset{m}{\Lambda} C_1$ and (70) shows that the geometric condition (72) can be written under the form (63).

Let (u_k) be a minimizing sequence for $\mathfrak{J}_+|_{\mathfrak{A} \cap \Sigma}$. Then $\mathfrak{J}_+(u_k) \leqslant C_5$ and, by (57), (73),

$$(m+1)^{-1} \|\nabla u_k\|_2^2 \leqslant J(u_k) \leqslant C_6 = \delta^{-1}(C_5 - C_4).$$

Therefore $\|\nabla u_k\|_2$, as well as

$$|\overline{u_k}| = V^{-1} |J(u_k) - \mathfrak{J}_+(u_k) - V|,$$

is bounded independently of k. Consequently, $\sup_k \|u_k\|_{W_{1,2}} < \infty$, and thus $\{u_k\}$ has in $W_{1,2}(X)$ a weakly convergent subsequence.

(iv) The assertions concerning minimizing sequences are discussed above, in each case. In the case $C_1(X) > 0$, we conjecture that the weak limit $\bar{u}$ of the minimizing sequence $\{u_k\}$ described above in (iii) under (63) is the desired critical point corresponding to a Kähler–Einstein metric.

We now turn to the boundedness of weak solutions (from a unified point of view).

PROPOSITION 4. *There exists a real number $c(m)$, depending only on m, such that any weak solution $u \in \overline{c\mathfrak{A}} = \overline{\mathfrak{A}_c}$ $(1 \leqslant c < c(m))$ of deformation equations belongs to $L_\infty(X)$.*

Proof. We use a well-known iterative procedure for L_p-norms.

In the remark closing Section II, we noticed the existence of a constant $c(m) > 1$ with the following property: if $1 \leqslant c < c(m)$, one can find a scalar $D(c)$ such that, for any Lipschitzian increasing function h: $\mathbb{R} \to \mathbb{R}$ and $u \in \overline{\mathfrak{A}_c}$,

$$\langle dJ(u), h(u) \rangle \geqslant D^{-1} \int_X h'(u) |\nabla u|^2 \, dV. \tag{74}$$

Now if we suppose u is a weak solution of deformation equations, by definition (44), u verifies

$$\langle dJ(u), \xi \rangle = \int_X K(x, u)\, \xi \, dV, \qquad \xi \in W_{1,2m}(X), \tag{75}$$

where $K(x, u) = 1 - e^{-\delta u + f}$ $(\delta = 0, 1, \text{ or } -1)$ belongs to $\bigcap_{q \geqslant 1} L_q(X)$. Thus (74), (75) yield

$$\int_X h'(u) |\nabla u|^2 \, dV \leqslant D \int_X K(x, u)\, h(u) \, dV. \tag{76}$$

Let $p \geqslant 2$ and $l > 0$. Choosing, in (76), $h = h_l$, where h_l is the odd real valued mapping defined by

$$h_l(t) = t^{p-1} \text{ when } 0 \leqslant t \leqslant l \qquad \text{and} \qquad h_l(t) = l^{p-1} \text{ if } t \geqslant l,$$

we obtain

$$\int_X h_l'(u) |\nabla u|^2 \, dV = \frac{4(p-1)}{p^2} \int_{|u| \leqslant l} |\nabla |u|^{p/2}|^2 dV \leqslant D \int_X |K(x, u)\, h_l(u)| \, dV;$$

letting l tend to infinity, we find

$$\int_X |\nabla |u|^{p/2}|^2 \, dV \leqslant C_1 \, p \int_X |K(x, u)| \, |u|^{p-1} \, dV. \tag{77}$$

According to Sobolev's embedding theorem, $W_{1,2}(X) \subset L_{2\alpha}(X)$ $(\alpha = m/(m-1))$,

$$\|u\|_{\alpha p}^{p} = \| \, |u|^{p/2} \|_{2\alpha}^{\alpha} \leqslant C_2(\|\nabla |u|^{p/2}\|_2^2 + \| \, |u|^{p/2} \|_2^2). \tag{78}$$

On the other hand, picking $\beta \in \,]1, \alpha[$ and noting that $p\beta/(p(\beta-1)+1) \leqslant \beta/(\beta-1)$, by Hölder's inequality, we have

$$\begin{aligned} \int_X |K(x, u) \, u^{p-1}| \, dV &\leqslant \|K(x, u)\|_{p\beta/(p(\beta-1)+1)} \, \| \, |u|^{p-1} \|_{p\beta/(p-1)} \\ &\leqslant C(\beta, V) \|(K(x, u)\|_{\beta/(\beta-1)} \|u\|_{p\beta}^{p-1} \leqslant C_3 \|u\|_{p\beta}^{p-1}. \end{aligned} \tag{79}$$

Taking into account (77), (78), (79), we get

$$\|u\|_{\alpha p}^{p} \leqslant C_4(p \|u\|_{\beta p}^{p-1} + \|u\|_p^p). \tag{80}$$

Either $\lim_{q \to \infty} \|u\|_q \leqslant 1$ and then $\|u\|_\infty \leqslant 1$, or else, since the mapping $q \to V^{-1/q} \|u\|_q$ is increasing, for $p \geqslant p_0$, $\|u\|_p > 1$ and thus, by (80), $\|u\|_{\alpha p}^{p} \leqslant C_5 \, p \|u\|_{\beta p}^{p}$; setting $\gamma = \alpha/\beta > 1$ and $q_0 = \beta p_0$, this gives

$$\|u\|_{\gamma q} \leqslant (C_6 q)^{\beta/q} \|u\|_q \qquad \text{when} \quad q \geqslant q_0,$$

and, by iteration, for all integers n,

$$\begin{aligned} \|u\|_{\gamma^n q} &\leqslant (C_6 q)^{(\beta/q)[\gamma^{1-n} + \cdots + \gamma^{-k} + \cdots + \gamma^{-1} + 1]} \gamma^{(\beta/q)[(n-1)\gamma^{1-n} + \cdots k\gamma^{-k} + \cdots \gamma^{-1}]} \|u\|_q \\ &\leqslant \max\{1, (C_6 q)^{(\beta/q)(\gamma/(\gamma-1))}\} \, \gamma^{(\beta/q)(\gamma/(\gamma-1)^2)} \|u\|_q = C_7 \|u\|_q, \end{aligned}$$

i.e., since $\lim_{n \to \infty} \gamma^n q = +\infty$, $\|u\|_\infty \leqslant$ Const., the required L_∞ estimate.

Summary of Conjectures. Here we summarize the conjectures made in Proposition 3 in the three cases (i), (ii), and (iii). In each case we conjecture that the weak limit $\bar{u}$ of the appropriately restricted (weakly convergent) minimizing sequences for the associated functionals $\mathfrak{J}$ on $\mathfrak{A}$ (with added constraints) corresponds to the desired Kähler–Einstein metric. These conjectures divide into two parts. First, it is necessary to show that the weak limit $\bar{u}$ is a critical point of the associated functional in each case. Second, it is necessary to show that $\bar{u} \in \mathfrak{A}$ and is sufficiently smooth to generate the desired Kähler deformation. In future papers, we intend to pursue these research directions.

APPENDIX: DIFFERENTIAL FORMS NOTATION

Here we rewrite some of the key equations of this article in terms of differential forms, hoping it will facilitate reading for those more familiar with this formalism. The analogous of Eq. (x) will be noted (x')

(1) The complex Monge–Ampère operator $M(\varphi)$ (13) is the ratio of the $2m$-forms $\omega_\varphi^m = (\omega + (1/4i\pi)\, dd^c\varphi)^m$ and ω^m, where ω is defined in (3). Taking into account the commutativity of even exterior algebra, we have

$$\omega_\varphi^m = M(\varphi)\,\omega^m = \sum_{k=0}^{m} (4i\pi)^{-k} C_m^k (dd^c\varphi)^k \wedge \omega^{m-k}. \tag{16'}$$

Since $dd^c\varphi$ and ω are closed 2-forms, we see that $\omega_\varphi^m - \omega^m$ is the exact $2m$-form

$$\omega_\varphi^m - \omega^m = d\left[\sum_{k=1}^{m} (4i\pi)^{-k} C_m^k\, d^c\varphi \wedge (dd^c\varphi)^{k-1} \wedge \omega^{m-k}\right]. \tag{17'}$$

(2) For any functional H, let us set $\bar{H} = (m!/\pi^m)\,H$. Recalling $dV = (\pi^m/m!)\,\omega^m$ (4), as regards J and I defined in (27) and (28), we can write

$$\bar{J}(\varphi) = \int_X \int_0^1 \varphi(\omega^m - \omega_{s\varphi}^m)\, ds \tag{27'}$$

$$\bar{I}(\varphi) = \int_X \varphi(\omega^m - \omega_\varphi^m). \tag{28'}$$

Developing $\omega_{s\varphi}^m$, we obtain

$$\bar{I} = \sum_{k=1}^{m} \bar{I}_k, \qquad \bar{J} = \sum_{k=1}^{m} \frac{\bar{I}_k}{k+1} \tag{29'}$$

with $\bar{I}_k(\varphi) = \bar{H}_k(\varphi, ..., \varphi)$ and

$$\bar{H}_k(u_1, ..., u_{k+1}) = -(4i\pi)^{-k} C_m^k \int_X u_{k+1}\, dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \omega^{m-k}. \tag{30'}$$

Since $dd^c u_k$ and ω are closed forms, an integration by parts, using Stokes theorem, yields

$$\begin{aligned}\bar{H}_k&(u_1, ..., u_{k+1})\\ &= -(4i\pi)^{-k} C_m^k \int_X dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge d^c u_k \wedge du_{k+1} \wedge \omega^{m-k}.\end{aligned} \tag{81}$$

Noting that the forms

$$d'd''u_1 \wedge \cdots \wedge d'd''u_{k-1} \wedge d'u_k \wedge d'u_{k+1} \wedge \omega^{m-k},$$
$$d'd''u_1 \wedge \cdots \wedge d'd''u_{k-1} \wedge d''u_k \wedge d''u_{k+1} \wedge \omega^{m-k}$$

(which are respectively of type $(m+2, m-1)$, $(m-1, m+2)$) vanish identically, (81) becomes

$$\bar{H}_k(u_1, ..., u_{k+1}) = \left(\frac{i}{2\pi}\right)^k \frac{C_m^k}{2} \int_X d'd''u_1 \wedge \cdots \wedge d'd''u_{k-1} \wedge (d'u_k \wedge d''u_{k+1} + d'u_{k+1} \wedge d''u_k) \wedge \omega^{m-k}, \quad (82)$$

which implies $\bar{H}_k$ is symmetric in its arguments. Also, the right-hand side of (82) is the sum of two integrals which are real and equal (taking the complex conjugate of one of them, we get the other), thus

$$\bar{H}_k(u_1, ..., u_{k+1}) = \int_X E_k(u_1, ..., u_{k+1})\, \omega^m, \quad (32')$$

where

$$E_k(u_1, ..., u_{k+1})\, \omega^m = \left(\frac{i}{2\pi}\right)^k C_m^k\, d'd''u_1 \wedge \cdots \wedge d'd''u_{k-1} \wedge d'u_k \wedge d''u_{k+1} \wedge \omega^{m-k}. \quad (37')$$

In the same vein, Eq. (48) defining the functional $\tilde{H}_k$ takes the form

$$\bar{\bar{H}}_k(u_1, ..., u_{k+1}) = \left(\frac{i}{2\pi}\right)^k C_m^k \int_X \left[\bigwedge_{h=1}^{k-1} (g_{\lambda\bar{\mu}} + \partial_{\lambda\bar{\mu}} u_h)\, dz^\lambda \wedge dz^{\bar{\mu}}\right] \wedge d'u_k \wedge d''u_{k+1} \wedge \omega^{m-k}. \quad (48')$$

(3) The proof of inequality (38) in Proposition 1,

$$|E_k(u_1, ..., u_{k+1})| \leqslant C(m, k) \left[\prod_{h=1}^{k-1} (\nabla^{\alpha\beta} u_h \nabla_{\alpha\beta} u_h)^{1/2}\right] |\nabla u_k|\, |\nabla u_{k+1}|, \quad (38')$$

is obtained by expanding the right-hand side of (37′) in a g-orthonormal frame, where $\omega = (i/2\pi) \sum_{\lambda=1}^m dz^\lambda \wedge dz^{\bar{\lambda}}$ and $\omega^m = (i/2\pi)^m\, m!\, dz^1 \wedge dz^{\bar{1}} \wedge \cdots \wedge dz^m \wedge dz^{\bar{m}}$, using exterior calculus.

On the other hand, Eq. (46), expressing the second derivative of J, becomes

$$(d^2\bar{J}(u))(v, w) = \sum_{k=1}^m \left(\frac{i}{2\pi}\right)^k C_m^k \int_X (d'd''u)^{k-1} \wedge d'v \wedge d''w \wedge \omega^{m-k}. \quad (46')$$

Writing $(d^2\bar{J}(u))\ (v, v) = \int_X \mathfrak{A}\omega^m$, we get the equality $A = |\nabla' v|^2 M(u)$ by computing $A(P)$ in a frame adapted to u; $A(P)$ is equal to the coefficient of $\prod_1^m (dz^\lambda \wedge dz^{\bar{\lambda}})$ in the development of the $2m$-form

$$\sum_{k=1}^{m} \frac{kC_m^k}{m!} \left(\sum_\lambda a_\lambda\, dz^\lambda \wedge dz^{\bar{\lambda}}\right)^{k-1} \wedge \left(\sum_\mu \partial_\mu v\, dz^\mu\right) \wedge \left(\sum_\nu \partial_{\bar{\nu}} v\, dz^{\bar{\nu}}\right) \wedge \left(\sum_\rho dz^\rho \wedge dz^{\bar{\rho}}\right)^{m-k}$$

and thus

$$A(P) = \sum_{k=1}^{m} \sum_{\lambda_1 \neq \cdots \lambda_k} \frac{1}{(k-1)!} a_{\lambda_1} \cdots a_{\lambda_{k-1}} \partial_{\lambda_k} v\, \partial_{\bar{\lambda}_k} v.$$

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