ELSEVIER Discrete Mathematics 160 (1996) 279-281

DISCRETE MATHEMATICS

Note

On nearly regular co-critical graphs

Tibor Szabó[a, b]

[a, *] Department of Mathematics, The Ohio State University, 231 W 18th Ave, Columbus, OH 43210, USA
[b] Department of Computer Science, Eötvös Loránd University, Budapest, Hungary

Received 18 January 1994; revised 19 January 1995

Abstract

A graph G is called (K_3, K_3)-co-critical if the edges of G can be coloured with two colours without getting a monochromatic triangle, but adding any new edge to the graph, this kind of 'good' colouring is impossible. In this short note we construct (K_3, K_3)-co-critical graphs of maximal degree $O(n^{3/4})$.

1. Introduction

In [2] Galluccio, Simonovits and Simonyi dealt with the concept of (K_3, K_3)-co-critical graphs. They called a graph G (K_3, K_3)-co-critical (or just simply co-critical) if the edges of G can be coloured with two colours (say RED and BLUE) without getting a monochromatic triangle, but adding any arbitrary new edge to the graph, this kind of 'good' colouring is impossible.

Among several other results they looked for co-critical graphs with low edge-density. It is easy to construct (K_3, K_3)-co-critical graphs with a linear number of edges (see [2]). But those examples all have vertices of degree $\geqslant cn$. The natural question arises: what about the 'nearly regular' co-critical graphs with low edge-density or what is the smallest possible maximal degree a co-critical graph can have. In [2], using a random graph construction, the authors proved the existence of (K_3, K_3)-co-critical graphs of maximal degree $O(n^{3/4}\log n)$.

In this note we give a simpler and constructive example of co-critical graphs of maximal degree $O(n^{3/4})$. Unfortunately, we still have a big gap between the trivial lower bound $c\sqrt{n}$ and this new upper bound. The lower bound follows from the fact

* Corresponding address.

SSDI 0012-365X(95)00121-2

that a co-critical graph must contain a K_3-saturated graph and that in a K_3-saturated graph the distance of any two points is $\leqslant 2$ (see [1]). (By a K_3-saturated graph we mean a graph G which does not contain a triangle, but adding any new edge to G results in a triangle.) A drawback of our construction is that locally it has a lot of edges, so it can not answer any other of the open problems discussed at the end of [2].

2. The construction

Following the notation of [2] if G and H are two graphs, we denote by $G \otimes H$ the graph what we obtain by joining a copy of G to a copy of H completely (each vertex of G to each vertex of H). To keep this note self-contained we prove a variant of a lemma used in [2].

Lemma. *Let C be a non-bipartite graph. If we colour the edges of $K_3 \otimes C$ with two colours, then we get a monochromatic K_3.*

Proof. Let $V(K_3) = \{p, q, r\}$.

Suppose to the contrary, that there is a 2-colouring of $K_3 \otimes C$ without monochromatic triangles.

Thus we can assume that two of the edges of the K_3 have the same colour (say RED) and the third one is different (BLUE). Let, say, r be the common endpoint of the two RED edges.

Suppose that one of the edges between r and C, say $\{r, c_1\}$, is RED. Then either one of the edges $\{c_1, p\}$, $\{c_1, q\}$, say $\{c_1, p\}$, is RED and r, c_1 and p form a RED triangle or both $\{c_1, p\}$ and $\{c_1, q\}$ are BLUE making c_1, p and q a BLUE triangle.

But this is impossible, thus we can assume that all the edges between C and r are BLUE. Therefore all the edges of C must be RED, otherwise we would have a BLUE triangle $\{r, c, d\}$, for some $c, d \in C$.

The neighbours of p in C, which are connected by a RED edge to p must form an independent set. The BLUE neighbours are independent also, since they must be the subset of the RED neighbours of q. (If not, there would be a vertex in $V(C)$ which together with p and q would form a BLUE triangle.)

This is a contradiction, since we partitioned $V(G)$ into two independent subsets. □

Definition. Let G and H be two graphs. We define their *or*-product $G \vee H$ by the following:

$V(G \vee H) = V(G) \times V(H)$ and

$\{(g, h), (g', h')\} \in E(G \vee H)$ if either $\{g, g'\} \in E(G)$ or $\{h, h'\} \in E(H)$.

Theorem. *If G and H are non-bipartite K_3-saturated graphs then $G \vee H$ is* (K_3, K_3)*-co-critical.*

Proof. We can give a trivial good colouring of the edges of $G \vee H$ by colouring an edge RED if the first coordinates of the two vertices formed an edge in G and colouring BLUE all the remaining edges.

Let us add a new edge to $G \vee H$: $\{(g, h), (g', h')\}$. Originally it is not an edge of $G \vee H$ which means $\{g, g'\} \notin E(G)$ and $\{h, h'\} \notin E(H)$. Since G and H are *maximal* K_3-free graphs there exist $g_1 \in V(G)$ and $h_1 \in V(H)$ such that $\{g, g_1\}$, $\{g_1, g'\} \in E(G)$ and $\{h, h_1\}$, $\{h_1, h'\} \in E(H)$.

So (g, h), (g', h') and (g_1, h) form a triangle and also each of these points are connected to every vertex of the form (x, h_1), $x \in V(G)$. Thus, by our Lemma, this subgraph can not be coloured with two colours without getting a monochromatic triangle. □

Theorem. *There exists an infinite sequence of* (K_3, K_3)*-co-critical graphs with*

$$d_{\max}(G_n) = O(n^{3/4}).$$

Proof. Füredi and Seress [1] constructed K_3-saturated graphs with maximal degree $2/\sqrt{3}\ \sqrt{n} + O(n^{7/24})$. This quantity is sufficiently small for our purposes. Taking the or-product of two such graphs we get a (K_3, K_3)-co-critical graph by the previous theorem. The maximal degree of the product graph is $O(n^{3/4})$. □

Acknowledgements

I would like to thank Ákos Seress for help and encouragement, Miklós Simonovits for improving the presentation of this paper.

References

[1] Z. Füredi and Á. Seress, Maximal triangle-free graphs with restrictions on the degrees, J. Graph Theory 18 (1994) 11–24.

[2] A. Galluccio, M. Simonovits and G. Simonyi, On the structure of co-critical graphs, in: Y. Alavi and A. Schwenk, eds., Graph Theory, Combinatorics and Applications, Volume 2, Proc. of the 7th Quadrennial Internat. Conf. on the Theory and Applications of Graphs, Western Michigan University (1992) 1053–1071.