Asymptotic behavior of non-autonomous lattice systems

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Abstract

The asymptotic behavior of non-autonomous infinite-dimensional lattice systems is studied. It is shown that the non-autonomous lattice reaction–diffusion system has a compact uniform attractor. The uniform asymptotic compactness of the system is established by showing that the tails of the solutions are uniformly small when time goes to infinity. The upper semicontinuity of uniform attractors is also obtained when the infinite-dimensional reaction–diffusion system is approached by a family of finite-dimensional systems.

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1. Introduction

This paper is concerned with the long time behavior of non-autonomous lattice dynamical systems, which contain the following reaction–diffusion system as a special case:

$$\dot{u}_i(t) - \nu(u_{i-1} - 2u_i + u_{i+1})(t) + \lambda u_i(t) + f(u_i(t)) = g_i(t), \quad i \in \mathbb{Z}, \ t > \tau, \ \tau \in \mathbb{R},$$

(1.1)

where \( \mathbb{Z} \) is the integer set, \( \nu \) and \( \lambda \) are positive constants, for each \( t \in \mathbb{R}, \ g(t) = (g_i(t))_{i \in \mathbb{Z}} \) is given, \( f \) is a smooth function satisfying a dissipative condition.

The lattice system (1.1) arises from many applications in biology and circuit models; see, for example, [7,8,19,20] and references therein. In particular, the system is a model for the prop-

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agation of nerve pulses in myelinated axons where the membrane is excitable only at spatially
 discrete sites. In this case, $u_i$ represents the potential at the $i$th active site.

Lattice differential equations have been extensively studied in the literature. The traveling
 wave solutions of such equations were investigated in [1,4,5,11,14–16,27]. The chaotic prop-
erties of solutions were examined in [10,12,13]. For the asymptotic behavior of lattice systems,
we refer the reader to [6,18,26,28]. In this paper, we are interested in the existence of uni-
form attractors for the non-autonomous reaction–diffusion system (1.1). In the autonomous case,
the existence of compact global attractors for system (1.1) was established in [6] in a standard
$l^2$ space, and in [26] in a weighted $l^2$ space. Since the lattice system is defined in the unbounded
integer set $\mathbb{Z}$, verifying the asymptotic compactness of the solution operators is a major step
towards proving the existence of attractors for the system. Such compactness in [6,26] was ob-
tained by the uniform estimates on the tails of solutions with a bounded set of initial data when
$t \to \infty$. In the present case, system (1.1) is non-autonomous and therefore it defines a family of
processes, instead of a semigroup in the autonomous case. The goal of this paper is to establish
the existence of uniform attractors for these processes by extending the “tail ends” method to the
non-autonomous lattice systems. In this case, the estimates on the tails of the solutions must be
uniform with respect to initial data in a bounded set as well as all translations of the external term
involved in the system.

This paper is organized as follows. In the next section, we recall some basic facts of almost
periodic functions and processes associated with non-autonomous lattice systems. We will also
present our main results in that section. Section 3 is devoted to the existence of a family of processes
for the lattice reaction–diffusion system (1.1). We will show that these processes are
continuous with respect to initial data as well as the external terms. In Section 4, we establish
the existence of uniform bounded absorbing sets for the processes, and derive uniform estimates
on the tails of the solutions when $t \to \infty$. The uniform asymptotic compactness of the processes
is given in Section 5. Based on the existence of uniform absorbing sets and asymptotic compact-
ness, we show that the lattice system (1.1) has a compact uniform attractor which uniformly
attracts any bounded subset of the phase space. In the last section, we consider the upper semi-
continuity of uniform attractors. We define a family of finite-dimensional approximation systems
for the infinite-dimensional lattice system (1.1), and prove that the uniform attractors of these ap-
proximation systems converge to the uniform attractor of the original system.

2. Main results

In this section, we describe our main results. We first establish the existence of uniform at-
tractors for the non-autonomous lattice systems, and then present the upper semicontinuity of
uniform attractors when an infinite-dimensional lattice system is approximated by a family of
finite-dimensional systems.

For each $\tau \in \mathbb{R}$, consider the non-autonomous reaction–diffusion system for $u(t) =
(u_i(t))_{i \in \mathbb{Z}^N}$:

$$\dot{u}_i(t) + \nu(Au(t))_i + \lambda u_i(t) = -f(u_i(t)) + g_i(t), \quad i \in \mathbb{Z}^N, \quad t > \tau,$$

where $\mathbb{Z}^N$ is the product of $N$ integer sets, $\nu$ and $\lambda$ are positive constants. For each $t \in \mathbb{R}$,
$g(t) = (g_i(t))_{i \in \mathbb{Z}^N}$ is a given sequence. $f$ is a given smooth nonlinear function. The symbol $A$
in (2.1) is a linear operator defined by, for every $u = (u_i)_{i \in \mathbb{Z}^N}$ and $i = (i_1, i_2, \ldots, i_N)$,

$$(Au)_i = 2Nu_{i_1}u_{i_2} \ldots u_{i_N} - u_{(i_1, i_2, \ldots, i_N)} - u_{(i_1, i_2 - 1, \ldots, i_N)} - \cdots - u_{(i_1, i_2, \ldots, i_N - 1)}
- u_{(i_1 + 1, i_2, \ldots, i_N)} - u_{(i_1, i_2 + 1, \ldots, i_N)} - \cdots - u_{(i_1, i_2, \ldots, i_N + 1)}.$$
System (2.1) is supplemented with the initial data
\[ u_i(\tau) = u_{\tau,i}, \quad i \in \mathbb{Z}^N. \tag{2.2} \]
Throughout this paper, we assume \( f \) satisfies
\[ f(s)s \geq 0, \quad \text{for all } s \in \mathbb{R}. \tag{2.3} \]
Since \( f \) is smooth, (2.3) implies that \( f(0) = 0 \).

We intend to study the asymptotic behavior of system (2.1)–(2.2). For that purpose, it is necessary to specify the space where the external term \( g \) lives. In the sequel, we assume that \( g_0(t) = (g_0,i(t))_{i \in \mathbb{Z}^N} \) is an almost periodic function in \( t \in \mathbb{R} \) with values in \( l^2 \), where
\[ l^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}^N} : \sum_{i \in \mathbb{Z}^N} u_i^2 < \infty \right\}. \]
The norm and inner product of \( l^2 \) are denoted by \( \| \cdot \| \) and \( (\cdot, \cdot) \), respectively. It is clear that the operator \( A \) defined above is a bounded linear operator on \( l^2 \). Since an almost periodic function is bounded and uniformly continuous on \( \mathbb{R} \) (see, e.g., [22]), it follows that \( g_0 \in C_b(\mathbb{R}, l^2) \), where \( C_b(\mathbb{R}, l^2) \) is the space of bounded continuous functions on \( \mathbb{R} \) with the norm \( \| g \|_{C_b(\mathbb{R}, l^2)} = \sup_{t \in \mathbb{R}} \| g(t) \|_2 \) for \( g \in C_b(\mathbb{R}, l^2) \). Further, by Bochner’s criterion in [22], whenever \( g_0 : \mathbb{R} \to l^2 \) is almost periodic, the set of all translations \( \{ g_0(\cdot + h) : h \in \mathbb{R} \} \) is precompact in \( C_b(\mathbb{R}, l^2) \). Let \( \mathcal{H}(g_0) \) be the closure of this set in \( C_b(\mathbb{R}, l^2) \). Then, for any \( g \in \mathcal{H}(g_0) \), \( g \) is almost periodic and \( \mathcal{H}(g) = \mathcal{H}(g_0) \). For each \( h \in \mathbb{R} \), denote by \( T(h) \) the translation on \( \mathcal{H}(g_0) \) with \( T(h)g = g(\cdot + h) \) for all \( g \in \mathcal{H}(g_0) \). It is evident that \( \{ T(h) \}_{h \in \mathbb{R}} \) is a continuous translation group on \( \mathcal{H}(g_0) \) that leaves \( \mathcal{H}(g_0) \) invariant:
\[ T(h)\mathcal{H}(g_0) = \mathcal{H}(g_0), \quad \text{for all } h \in \mathbb{R}. \]
The translation group \( \{ T(h) \}_{h \in \mathbb{R}} \) will be used to define a semigroup for the non-autonomous lattice system (2.1)–(2.2).

In the next section, we will show that for every \( g \in \mathcal{H}(g_0) \), \( \tau \in \mathbb{R} \) and \( u_{\tau} = (u_{\tau,i})_{i \in \mathbb{Z}^N} \in l^2 \), system (2.1)–(2.2) has a unique global solution \( u = (u_i(\cdot))_{i \in \mathbb{Z}^N} \in C([\tau, \infty), l^2) \), based on which one can associate a family of operators with problem (2.1)–(2.2). Given \( g \in \mathcal{H}(g_0) \), \( \tau \in \mathbb{R} \) and \( \tau \geq \tau \), define a mapping \( U^g(\tau, \tau) \) from \( l^2 \) into itself such that, for each \( u_{\tau} \in l^2 \), \( U^g(\tau, \tau)u_{\tau} = u(\tau) \), the state of the solution \( u \) of system (2.1)–(2.2) at time \( \tau \). The family of mappings \( \{ U^g(\tau, \tau) : \tau \geq \tau, \tau \in \mathbb{R} \} \) is called the process corresponding to system (2.1)–(2.2) with time symbol \( g \in \mathcal{H}(g_0) \). The first result of this paper claims the existence of a uniform attractor with respect to \( g \in \mathcal{H}(g_0) \) for the family of processes \( \{ U^g(\tau, \tau) \}_{g \in \mathcal{H}(g_0)} \), which will be proved in Section 5.

**Theorem 2.1.** Suppose \( g_0 \in C_b(\mathbb{R}, l^2) \) is almost periodic and (2.3) holds. Then the family of processes \( \{ U^g(\tau, \tau) \}_{g \in \mathcal{H}(g_0)} \) has a compact uniform attractor \( \mathcal{A} \) in \( l^2 \) with respect to \( g \in \mathcal{H}(g_0) \). More precisely, \( \mathcal{A} \) is the minimal compact subset of \( l^2 \) which satisfies, for every bounded set \( B \subset l^2 \):
\[ \lim_{t \to \infty} \sup_{g \in \mathcal{H}(g_0)} \text{dist}_{l^2}(U^g(\tau, \tau)B, \mathcal{A}) = 0, \quad \text{for all } \tau \in \mathbb{R}, \]
where
\[ \text{dist}_{l^2}(U^g(\tau, \tau)B, \mathcal{A}) = \sup_{u \in B} \text{dist}_{l^2}(U^g(\tau, \tau)u, \mathcal{A}). \]
We now consider finite-dimensional approximations to the infinite-dimensional system (2.1)–(2.2) on finite lattices. For each \( i = (i_1, i_2, \ldots, i_N) \in \mathbb{Z}^N \), denote by \( |i| = \max\{|i_1|, |i_2|, \ldots, |i_N|\} \). Given \( n \geq 1 \), consider the \((2n + 1)^N\)-dimensional system of ordinary differential equations on the lattices \( \{i \in \mathbb{Z}^N : |i| \leq n\} \):

\[
\dot{u}_i + v(Au)_i + \lambda u_i = - f(u_i) + g(t), \quad t > \tau, \quad \tau \in \mathbb{R},
\]

with the periodic boundary conditions

\[
\begin{align*}
\dot{u}(i_1 + 2n + 1, i_2, \ldots, i_N) = & \cdots = u(i_1, i_2, \ldots, i_{N-1}, i_N + 2n + 1) \\
= & u(i_1, i_2, \ldots, i_N),
\end{align*}
\]

and the initial data

\[
\begin{align*}
\dot{u}_i(\tau) = & \ u_{\tau,i}, \quad |i| \leq n, \quad \tau \in \mathbb{R}.
\end{align*}
\]

In the last section, we will show that for each \( n \geq 1 \), the finite-dimensional approximation system has a uniform attractor \( \mathcal{A}_n \) in \( \mathbb{R}^{(2n+1)^N} \) which can be naturally embedded into \( l^2 \) by zero extension. Further, we will prove that these uniform attractors are upper semicontinuous when \( n \to \infty \). In other words, the following result will be established in Section 6.

**Theorem 2.2.** Suppose \( g_0 \in C_b(\mathbb{R}, l^2) \) is almost periodic and (2.3) holds. Then for each \( n \geq 1 \), system (2.4)–(2.6) has a compact uniform attractor \( \mathcal{A}_n \). Further, \( \mathcal{A}_n \) is upper semicontinuous to \( \mathcal{A} \) when \( n \to \infty \), that is,

\[
\lim_{n \to \infty} d_{l^2}(\mathcal{A}_n, \mathcal{A}) = 0,
\]

where

\[
d_{l^2}(\mathcal{A}_n, \mathcal{A}) = \sup_{a \in \mathcal{A}_n} \text{dist}_{l^2}(a, \mathcal{A}).
\]

### 3. Processes associated with non-autonomous lattice systems

In this section, we prove the existence and uniqueness of global solutions for problem (2.1)–(2.2), and then define a family of continuous processes in \( l^2 \) associated with the non-autonomous lattice system.

We now reformulate problem (2.1)–(2.2) as an abstract ordinary differential equation in \( l^2 \). For each sequence \( u = (u_i)_{i \in \mathbb{Z}^N} \) and \( j = 1, 2, \ldots, N \), define the linear operators \( B_j \) and \( B_j^* \) on \( l^2 \) by, for every \( i = (i_1, i_2, \ldots, i_N) \in \mathbb{Z}^N \),

\[
\begin{align*}
(B_j u)_i &= u(i_1, \ldots, i_j + 1, \ldots, i_N) - u(i_1, \ldots, i_j, i_N), \\
(B_j^* u)_i &= u(i_1, \ldots, i_j - 1, \ldots, i_N) - u(i_1, \ldots, i_j, i_N).
\end{align*}
\]

Then we find that

\[
(B_j^* u, v) = (u, B_j v), \quad \text{for all } u, v \in l^2,
\]

and

\[
A = \sum_{j=1}^N B_j B_j^* = \sum_{j=1}^N B_j^* B_j.
\]
where $A$ is the linear operator defined in Section 2. We now define an operator $\tilde{f}$ on $l^2$ which is associated with $f$. For each $u = (u_i)_{i \in \mathbb{Z}^N}$, let $\tilde{f}(u) = (f(u_i))_{i \in \mathbb{Z}^N}$. Since $f$ is smooth and $f(0) = 0$, it is easy to verify that $\tilde{f}$ maps $l^2$ into itself. To simplify notations, we identify $\tilde{f}$ with $f$ and use the same symbol $f$ to denote them. Then problem (2.1)–(2.2) is equivalent to the equation in $l^2$:

$$\dot{u} + \nu Au + \lambda u = -f(u) + g(t), \quad \text{for } t > \tau,$$

(3.1)

and

$$u(\tau) = u_\tau \in l^2,$$

(3.2)

where $\tau \in \mathbb{R}$ and $g \in \mathcal{H}(g_0)$. Next, we show that problem (3.1)–(3.2) is well-posed in $l^2$. By $f(0) = 0$, after simple computations, we find that $f$ is locally Lipschitz continuous on $l^2$, that is, for every bounded set $Y$ in $l^2$, there exists a constant $C$ depending only on $Y$ such that

$$\|f(u) - f(v)\| \leq C \|u - v\|, \quad \text{for all } u, v \in Y.$$

(3.3)

Then it follows from the standard theory of ordinary differential equations that there exists a unique local solution $u$ for problem (3.1)–(3.2) such that $u \in C([\tau, \tau + T_0], l^2)$ for some $T_0 > 0$. The following estimates show that this local solution $u$ is actually defined globally.

**Lemma 3.1.** Suppose $g_0 \in C_b(\mathbb{R}, l^2)$ is almost periodic and (2.3) holds. Let $g \in \mathcal{H}(g_0)$, $\tau \in \mathbb{R}$ and $u_\tau \in l^2$ with $\|u_\tau\| \leq R$. Then the solution $u$ of problem (3.1)–(3.2) defined on $[\tau, \tau + T)$ with $T > 0$ satisfies

$$\|u(t)\| \leq C, \quad \text{for all } \tau \leq t < \tau + T,$$

where $C$ is a constant depending on $\lambda$, $R$ and $\|g_0\|_{C_b(\mathbb{R}, l^2)}$. In particular, $C$ is independent of $\tau$ and $g \in \mathcal{H}(g_0)$.

In the sequel, we denote by $C$ any positive constant which may change value from line to line.

**Proof.** Taking the inner product of (3.1) with $u$ in $l^2$, we find that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \sum_{j=1}^N \|B_j u\|^2 + \lambda \|u\|^2 = -(f(u), u) + (g, u).$$

(3.4)

Since

$$|(g, u)| \leq \|g\| \|u\| \leq \frac{1}{2} \lambda \|u\|^2 + \frac{1}{2\lambda} \|g\|^2,$$

by (2.3) we get

$$\frac{d}{dt} \|u\|^2 + 2 \nu \sum_{j=1}^N \|B_j u\|^2 + \lambda \|u\|^2 \leq \frac{1}{\lambda} \|g_0\|^2_{C_b(\mathbb{R}, l^2)}.$$  

(3.5)

Note that $g \in \mathcal{H}(g_0)$, and therefore $\|g\|_{C_b(\mathbb{R}, l^2)} = \|g_0\|_{C_b(\mathbb{R}, l^2)}$. By (3.5) we have

$$\frac{d}{dt} \|u\|^2 + \lambda \|u\|^2 \leq \frac{1}{\lambda} \|g_0\|^2_{C_b(\mathbb{R}, l^2)}.$$
Multiplying the above by $e^{\lambda t}$ and then integrating between $\tau$ and $t$, we find
\[ \|u(t)\|^2 \leq e^{-\lambda (t-\tau)} \|u_\tau\|^2 + \frac{C}{\lambda^2}, \]  \tag{3.6}
which implies Lemma 3.1. \qed

It follows from Lemma 3.1 that the solution $u$ of problem (3.1)–(3.2) is defined for all $t \geq \tau$. Therefore, one can associate a family of processes $\{U^g(t, \tau) : t \geq \tau, \ \tau \in \mathbb{R}\}_{g \in \mathcal{H}(g_0)}$ on $l^2$ with system (3.1)–(3.2) such that for given $g \in \mathcal{H}(g_0)$, $\tau \in \mathbb{R}$ and $t \geq \tau$, if $u_\tau \in l^2$, then $U^g(t, \tau)u_\tau = u(t)$, the state of the solution $u$ of system (3.1)–(3.2) at time $t$. By the unique solvability of problem (3.1)–(3.2), the family of processes $\{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)}$ satisfies the multiplicative properties:
\[ U^g(t, s)U^g(s, \tau) = U^g(t, \tau), \quad \text{for all } t \geq s \geq \tau, \ \tau \in \mathbb{R}, \]
\[ U^g(\tau, \tau) = I, \quad \tau \in \mathbb{R}, \]
where $I$ is the identity operator. Further the following translation identity holds for the processes and the translation group $\{T(h)\}_{h \in \mathbb{R}}$:
\[ U^{g+h}(t, \tau + h) = U^T(h)g(t, \tau), \quad \text{for all } h \in \mathbb{R}, \ t \geq \tau, \ \tau \in \mathbb{R}, \]
where $T(h)g = g(\cdot + h)$ for $g \in \mathcal{H}(g_0)$.

This paper is concerned with the uniform asymptotic behavior of the family of processes $\{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)}$ with respect to $g \in \mathcal{H}(g_0)$ when $t \to \infty$. To that end, it is necessary to establish the continuity of the processes with respect to $g \in \mathcal{H}(g_0)$ and initial data.

**Lemma 3.2.** Suppose $g_0 \in C_b(\mathbb{R}, l^2)$ is almost periodic and (2.3) holds. Let $g, g_n \in \mathcal{H}(g_0)$ and $w, w_n \in l^2$. If $g_n \to g$ and $w_n \to w$ as $n \to \infty$, then for any $t \geq \tau$,
\[ \|U^{g_n}(t, \tau)w_n - U^g(t, \tau)w\| \to 0, \quad \text{as } n \to \infty. \]

**Proof.** Let $u_n(t, \tau) = U^{g_n}(t, \tau)w_n, u(t, \tau) = U^g(t, \tau)w$ and $v_n(t, \tau) = u_n(t, \tau) - u(t, \tau)$. Then it follows from (3.1) that
\[ \frac{d}{dt}v_n + \nu A v_n + \lambda v_n = f(u) - f(u_n) + g_n - g. \]  \tag{3.7}

Taking the inner product of the above with $v_n$ in $l^2$, we get
\[ \frac{1}{2} \frac{d}{dt} \|v_n\|^2 + \nu \sum_{j=1}^N \|B_j v_n\|^2 + \lambda \|v_n\|^2 = (f(u) - f(u_n), v_n) + (g_n - g, v_n). \]  \tag{3.8}

By Lemma 3.1, it follows that $u_n$ is bounded in $l^2$. Therefore by (3.3) we have the following estimates on the first term on the right-hand side of (3.7):
\[ \left| (f(u) - f(u_n), v_n) \right| \leq \|f(u) - f(u_n)\| \|v_n\| \leq C \|v_n\|^2. \]  \tag{3.9}

By (3.7) and (3.8), we obtain
\[ \frac{d}{dt} \|v_n\|^2 \leq C \|v_n\|^2 + C \|g_n - g\|^2, \]
which along with Gronwall’s inequality yields
\[ \| v_n(t, \tau) \|^2 \leq e^{C(t-\tau)} \| v_n(\tau, \tau) \|^2 + \int_{\tau}^{t} e^{C(t-s)} \| g_n(s) - g(s) \|^2 \, ds \]
\[ \leq e^{C(t-\tau)} \| w_n - w \|^2 + \frac{e^{C(t-\tau)}}{C} \| g_n - g \|_{C_b(\mathbb{R}, l^2)} \to 0, \quad \text{as } n \to \infty. \]

The proof is complete. \( \square \)

4. Absorbing sets and estimates on tails of solutions

In this section, we establish uniform estimates with respect to \( g \in \mathcal{H}(g_0) \) for the solutions of the lattice systems. We will show that the family of processes \( \{ U^g(t, \tau) \}_{g \in \mathcal{H}(g_0)} \) has a uniform absorbing set in \( l^2 \). We will also derive the uniform estimates on the tails of solutions, which will play a critical role for proving the asymptotic compactness of the processes.

**Lemma 4.1.** Suppose \( g_0 \in C_b(\mathbb{R}, l^2) \) is almost periodic and (2.3) holds. Then there exists a constant \( M \) such that any solution \( u \) of problem (3.1)–(3.2) satisfies
\[ \| u(t) \| \leq M, \quad \text{for all } t \geq \tau + T, \]
where \( M \) depends only on \( \lambda \) and \( g_0 \); \( T \) depends only on \( \lambda, g_0 \) and \( R \) when \( \| u_\tau \| \leq R \). In particular, both \( M \) and \( T \) are independent of \( \tau \) and \( g \in \mathcal{H}(g_0) \).

**Proof.** It follows from (3.6) that, whenever \( \| u_\tau \| \leq R \),
\[ \| u(t) \|^2 \leq e^{-\lambda(t-\tau)} R^2 + \frac{C}{\lambda^2} \leq \frac{2C}{\lambda^2}, \quad \text{for } t - \tau \geq T, \]
where \( T = \frac{1}{\lambda} \ln \frac{R^2 \lambda^2}{C} \). The proof is complete. \( \square \)

In the sequel, we denote by \( B \) the bounded set in \( l^2 \):
\[ B = \{ u \in l^2 : \| u \| \leq M \}. \tag{4.1} \]
where \( M \) is the constant in Lemma 4.1. Then it follows from Lemma 4.1 that \( B \) is a uniform absorbing set for the family of processes \( \{ U^g(t, \tau) \}_{g \in \mathcal{H}(g_0)} \); that is, for any \( \tau \in \mathbb{R} \) and bounded set \( X \) in \( l^2 \), there exists a constant \( T \) depending only on \( X, \lambda \) and \( g_0 \) such that
\[ U^g(t, \tau)X \subseteq B, \quad \text{for all } g \in \mathcal{H}(g_0) \text{ and } t - \tau \geq T. \]

Next, we establish the uniform estimates on the tails of solutions when \( t \to \infty \), which are crucial for verifying the uniform asymptotic compactness of the family of processes. These estimates are uniform with respect to bounded initial data in \( l^2 \) as well as all elements \( g \in \mathcal{H}(g_0) \).

**Lemma 4.2.** Suppose \( g_0 \in C_b(\mathbb{R}, l^2) \) is almost periodic and (2.3) holds. Let \( u \) be the solution of problem (3.1)–(3.2) with \( \| u_\tau \| \leq R \). Then for every \( \epsilon > 0 \), there exist constants \( K(\epsilon) \) and \( T(\epsilon, R) \) such that, if \( t - \tau \geq T(\epsilon, R) \),
\[ \sum_{|i| \geq K(\epsilon)} |u_i(t)|^2 \leq \epsilon, \]
where \( K(\epsilon) \) depends only on \( \epsilon, \lambda \) and \( g_0 \); \( T(\epsilon, R) \) depends only on \( \epsilon, \lambda, g_0 \) and \( R \). In particular, \( K(\epsilon) \) and \( T(\epsilon, R) \) are independent of \( \tau \) and \( g \in \mathcal{H}(g_0) \).
Proof. We use an idea of cut-off functions to establish the uniform estimates on the tails of the solutions as in [25] for autonomous equations.

Let $\theta$ be a smooth cut-off function satisfying $0 \leq \theta(s) \leq 1$ for $s \geq 0$ and

$$
\theta(s) = 0 \quad \text{for } 0 \leq s < 1, \quad \theta(s) = 1 \quad \text{for } s \geq 2.
$$

Let $k$ be a fixed integer which will be specified later, and set $v = (v_i)_{i \in \mathbb{Z}^N}$ with $v_i = \theta(\frac{|i|}{k})u_i$.

Then taking the inner product of (3.1) with $v$ in $l^2$, we find

$$
\frac{1}{2} \frac{d}{dt} \sum_{i} \theta \left( \frac{|i|}{k} \right) |u_i(t)|^2 + v \sum_{j=1}^{N} (B_j u(t), B_j v(t)) + \lambda \sum_{i} \theta \left( \frac{|i|}{k} \right) |u_i(t)|^2
$$

$$
= - \sum_{i} \theta \left( \frac{|i|}{k} \right) f(u_i(t))u_i(t) + \sum_{i} \theta \left( \frac{|i|}{k} \right) u_i(t)g_i(t).
$$

(4.2)

Let us first estimate the last term on the right-hand side of the above. Note that

$$
\sum_{i} \theta \left( \frac{|i|}{k} \right) g_i(t)u_i(t) = \sum_{|i| \geq k} \theta \left( \frac{|i|}{k} \right) g_i(t)u_i(t)
$$

$$
\leq \frac{1}{2} \lambda \sum_{|i| \geq k} \theta^2 \left( \frac{|i|}{k} \right) |u_i(t)|^2 + \frac{1}{2\lambda} \sum_{|i| \geq k} |g_i(t)|^2
$$

$$
\leq \frac{1}{2} \lambda \sum_{|i| \geq k} \theta \left( \frac{|i|}{k} \right) |u_i|^2 + \frac{1}{2\lambda} \sum_{|i| \geq k} |g_i(t)|^2.
$$

(4.3)

Since $g_0$ is almost periodic, it is known that any function $g \in \mathcal{H}(g_0)$ is also almost periodic. Therefore, for fixed $g \in \mathcal{H}(g_0)$, the set $\{(g_i(t))_{i \in \mathbb{Z}^N}: \ t \in \mathbb{R}\}$ is precompact in $l^2$, which implies that for given $\epsilon > 0$, there exists a constant $C(g, \epsilon)$ depending on $g$ and $\epsilon$ such that when $k \geq C(g, \epsilon)$,

$$
\frac{1}{\lambda} \sum_{|i| \geq k} |g_i(t)|^2 \leq \frac{\epsilon}{4}, \quad \forall t \in \mathbb{R}.
$$

Because the set $\mathcal{H}(g_0)$ is compact in $C_b(\mathbb{R}, l^2)$, it follows from the above that there exists $K_1(\epsilon)$, depending only on $\epsilon$ but independent of $g \in \mathcal{H}(g_0)$, such that for all $k \geq K_1(\epsilon)$,

$$
\frac{1}{\lambda} \sum_{|i| \geq k} |g_i(t)|^2 \leq \frac{\epsilon}{2}, \quad \text{for all } t \in \mathbb{R} \text{ and } g \in \mathcal{H}(g_0).
$$

(4.4)

On the other hand, by simple computations, we obtain the following estimates on the second term on the left-hand side of (4.2):

$$
\nu \sum_{j=1}^{N} (B_j u(t), B_j v(t)) \geq \nu \sum_{j=1}^{N} \sum_{i \in \mathbb{Z}^N} \left( \theta \left( \frac{|i|}{k} \right) |(B_j u)_i|^2 \right) - \frac{C_1}{k} \|u(t)\|^2,
$$

which along with Lemma 4.1 shows that there exists $T_1(R)$ such that for all $\tau \in \mathbb{R}$, $g \in \mathcal{H}(g_0)$ and $t - \tau \geq T_1(R)$,

$$
\nu \sum_{j=1}^{N} (B_j u(t), B_j v(t)) \geq \nu \sum_{j=1}^{N} \sum_{i \in \mathbb{Z}^N} \left( \theta \left( \frac{|i|}{k} \right) |(B_j u)_i|^2 \right) - \frac{C_2}{k}.
$$
By choosing $K_2(\epsilon)$ large enough such that $\frac{C_2}{K} \leq \frac{\epsilon}{4}$ for $k \geq K_2(\epsilon)$, we get from the above that
\begin{equation}
\nu \sum_{j=1}^{N} (B_j u(t), B_j v(t)) \geq \nu \sum_{j=1}^{N} \sum_{i \in \mathbb{Z}^N} \left( \theta \left( \frac{|i|}{k} \right) |(B_j u)_i|^2 \right) - \frac{\epsilon}{4}.
\end{equation}

Let $K(\epsilon) = \max\{K_1(\epsilon), K_2(\epsilon)\}$. By the estimates in (4.3)–(4.5), it follows from (2.3) and (4.2) that for all $k \geq K(\epsilon)$ and $t - \tau \geq T_1(R)$,
\begin{equation*}
\frac{d}{dt} \sum_{i \in \mathbb{Z}^N} \theta \left( \frac{|i|}{k} \right) |u_i(t)|^2 + \lambda \sum_{i \in \mathbb{Z}^N} \theta \left( \frac{|i|}{k} \right) |u_i(t)|^2 \leq \epsilon,
\end{equation*}
which along with Gronwall’s lemma yields
\begin{equation*}
\sum_{|i| \geq 2k} |u_i(t)|^2 \leq \sum_{i \in \mathbb{Z}^N} \theta \left( \frac{|i|}{k} \right) |u_i(t)|^2 \leq \frac{2\epsilon}{\lambda},
\end{equation*}
which implies Lemma 4.2. The proof is complete. \(\square\)

5. Uniform attractors for lattice systems

In this section, we establish the existence of uniform attractors for the non-autonomous lattice system (3.1)–(3.2). We first define a semigroup of nonlinear operators for the family of processes \(\{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)}\) in an extended phase space as in [9]. Then we show that the semigroup is point dissipative and asymptotically compact, and hence it has a global attractor. The existence of a uniform attractor for the processes will be obtained from the global attractor of the associated semigroup.

For the reader’s convenience, we recall the definition of uniform attractors for a family of processes (see, e.g., [9]).

**Definition 5.1.** A closed set \(\mathcal{A}\) of \(l^2\) is said to be the uniform attractor of the family of processes \(\{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)}\) with respect to \(g \in \mathcal{H}(g_0)\) if \(\mathcal{A}\) satisfies

(i) \(\mathcal{A}\) attracts every bounded set in \(l^2\) uniformly with respect to \(g \in \mathcal{H}(g_0)\); that is, for any bounded \(X \subset l^2\),
\begin{equation*}
\lim_{t \to \infty} \sup_{g \in \mathcal{H}(g_0)} \text{dist}(U^g(t, \tau)X, \mathcal{A}) = 0, \quad \text{for all } \tau \in \mathbb{R}.
\end{equation*}

(ii) \(\mathcal{A}\) is minimal among all closed subsets of \(l^2\) satisfying property (i); that is, if \(\tilde{\mathcal{A}}\) is any closed subset of \(l^2\) satisfying property (i), then \(\mathcal{A} \subseteq \tilde{\mathcal{A}}\).

If the family of processes \(\{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)}\) has a uniform attractor, then it must be unique. To prove the existence of such a uniform attractor, it is convenient to transfer the family of processes
to a semigroup of nonlinear operators, and then use the semigroup theory to investigate the uniform attractor of the processes. As in [9], we define a nonlinear semigroup \( \{S(t)\}_{t \geq 0} \) acting on the extended phase space \( l^2 \times \mathcal{H}(g_0) \) by the following formula, for every \( t \geq 0 \), \( u \in l^2 \) and \( g \in \mathcal{H}(g_0) \),

\[
S(t)(u, g) = (U^g(t, 0)u, T(t)g),
\]

where \( T \) is the translation group given by \( T(t)g = g(\cdot + t) \) for each \( g \in \mathcal{H}(g_0) \). By the translation identity and the multiplicative properties of the processes discussed in Section 3, it is clear that \( \{S(t)\}_{t \geq 0} \) satisfies the semigroup identities:

\[
S(t)S(s) = S(t+s), \quad S(0) = I, \quad \forall t \geq s \geq 0.
\]

The relations between the dynamics of a general family of processes and its associated semigroup has been extensively studied in [9]. In our case, we know that if \( \{S(t)\}_{t \geq 0} \) has a global attractor in the extended phase space \( l^2 \times \mathcal{H}(g_0) \), then the family of the processes \( \{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)} \) possesses a uniform attractor in the phase space \( l^2 \), which is actually the projection onto \( l^2 \) of the global attractor of \( \{S(t)\}_{t \geq 0} \). To describe more details about the structure of the uniform attractor, the following concepts are needed.

**Definition 5.2.** Given \( g \in \mathcal{H}(g_0) \), a curve \( t \to u(t) \in l^2 \) is said to be a complete solution for the process \( U^g(t, \tau) \) if it satisfies

\[
U^g(t, \tau)u(\tau) = u(t), \quad \forall \tau \in \mathbb{R} \quad \text{and} \quad t \geq \tau.
\]

(5.1)

The kernel of the process \( U^g(t, \tau) \) is the collection \( \mathcal{K}_g \) of all its bounded complete solutions; that is,

\[
\mathcal{K}_g = \{ u(\cdot) \in C_b(\mathbb{R}, l^2) : u(\cdot) \text{ satisfies (5.1)} \}.
\]

The kernel section of the process \( U^g(t, \tau) \) at time \( s \in \mathbb{R} \) is the set

\[
\mathcal{K}_g(s) = \{ u(s) : u(\cdot) \in \mathcal{K}_g \}.
\]

Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be the projectors from \( l^2 \times \mathcal{H}(g_0) \) onto \( l^2 \) and \( \mathcal{H}(g_0) \), respectively; that is, for every \( (u, g) \in l^2 \times \mathcal{H}(g_0) \),

\[
\mathcal{F}_1(u, g) = u \quad \text{and} \quad \mathcal{F}_2(u, g) = g.
\]

Then it follows from the uniform attractors theory in [9], we have the following proposition for the family of processes \( \{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)} \). We also refer the reader to [2,3,17,21,23,24] for the attractors theory of semigroups.

**Proposition 5.3.** If the semigroup \( S(t) \) is continuous, point dissipative and asymptotically compact, then it has a compact global attractor \( \mathcal{A}_S \) in \( l^2 \times \mathcal{H}(g_0) \). Further, if \( \mathcal{A} \) is the projection of \( \mathcal{A}_S \) onto \( l^2 \), i.e., \( \mathcal{A} = \mathcal{F}_1 \mathcal{A}_S \), then \( \mathcal{A} \) is the compact uniform attractor for the family of processes \( \{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)} \). In addition,

1. \( \mathcal{A}_S = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_g(0) \times \{g\} \),
2. \( \mathcal{A} = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_g(0) \),
3. \( \mathcal{F}_2 \mathcal{A}_S = \mathcal{H}(g_0) \).
In order to apply Proposition 5.3, we need to establish the asymptotic compactness of the semigroup \( \{S(t)\}_{t \geq 0} \), which is stated as follows.

**Lemma 5.4.** Suppose \( g_0 \in C_b(\mathbb{R}, l^2) \) is almost periodic and (2.3) holds. Then the semigroup \( \{S(t)\}_{t \geq 0} \) is asymptotically compact, that is, if \( \{(u_n, g_n)\}_{n=1}^{\infty} \) is bounded in \( l^2 \times H(g_0) \), and \( t_n \to \infty \), then \( \{S(t_n)(u_n, g_n)\}_{n=1}^{\infty} \) is precompact in \( l^2 \times H(g_0) \).

**Proof.** Notice that the set \( H(g_0) \) is compact and \( T(t_n)g_n \in H(g_0) \). So, without loss of generality, we can assume that there exists \( \tilde{g} \in H(g_0) \) such that

\[
T(t_n)g_n \to \tilde{g} \quad \text{in} \quad H(g_0), \quad \text{as} \quad n \to \infty. \tag{5.2}
\]

By the boundedness of \( \{u_n\}_{n=1}^{\infty} \) in \( l^2 \), it follows from Lemma 4.1 that the set \( \{U^{g_n}(t_n, 0)u_n\}_{n=1}^{\infty} \) is bounded in \( l^2 \), and hence weakly compact. Then there exists \( \tilde{u} \in l^2 \) and a subsequence of \( \{U^{g_n}(t_n, 0)u_n\}_{n=1}^{\infty} \) (still denoted by \( \{U^{g_n}(t_n, 0)u_n\}_{n=1}^{\infty} \)) such that

\[
U^{g_n}(t_n, 0)u_n \to \tilde{u} \quad \text{weakly in} \quad l^2, \quad \text{as} \quad n \to \infty. \tag{5.3}
\]

In what follows, we prove that the weak convergence is actually strong, that is, we will show that for every \( \epsilon > 0 \), there exists \( K(\epsilon) \) such that when \( n \geq K(\epsilon) \),

\[
\|U^{g_n}(t_n, 0)u_n - \tilde{u}\| \leq \epsilon. \tag{5.4}
\]

By Lemma 4.2, we find that there exist \( I_1(\epsilon) \) and \( K_1(\epsilon) \) such that for \( n \geq K_1(\epsilon) \),

\[
\sum_{|i| \geq I_1(\epsilon)} \left| \left( U^{g_n}(t_n, 0)u_n \right)_i \right|^2 \leq \frac{\epsilon^2}{8}. \tag{5.5}
\]

On the other hand, since \( \tilde{u} \in l^2 \), there exists \( I_2(\epsilon) \) such that

\[
\sum_{|i| \leq I_2(\epsilon)} |\tilde{u}_i|^2 \leq \frac{\epsilon^2}{8}. \tag{5.6}
\]

Let \( I(\epsilon) = \max\{I_1(\epsilon), I_2(\epsilon)\} \), by the weak convergence (5.3) we have

\[
\left( \left( U^{g_n}(t_n, 0)u_n \right)_i \right)_{|i| \leq I(\epsilon)} \to (\tilde{u}_i)_{|i| \leq I(\epsilon)} \quad \text{in} \quad \mathbb{R}^{2I(\epsilon)+1}, \quad \text{as} \quad n \to +\infty,
\]

which implies that there exists \( K_2(\epsilon) \) such that when \( n \geq K_2(\epsilon) \),

\[
\sum_{|i| \geq I(\epsilon)} \left| \left( U^{g_n}(t_n, 0)u_n \right)_i - \tilde{u}_i \right|^2 \leq \frac{\epsilon^2}{2}. \tag{5.7}
\]

Setting \( K(\epsilon) = \max\{K_1(\epsilon), K_2(\epsilon)\} \), it follows from (5.5)–(5.7) that, for \( n \geq K(\epsilon) \),

\[
\|U^{g_n}(t_n, 0)u_n - \tilde{u}\|^2 \leq \sum_{|i| \leq I(\epsilon)} \left| \left( U^{g_n}(t_n, 0)u_n \right)_i - \tilde{u}_i \right|^2 + \sum_{|i| \geq I(\epsilon)} \left| \left( U^{g_n}(t_n, 0)u_n \right)_i - \tilde{u}_i \right|^2 \leq \frac{\epsilon^2}{2} + 2 \sum_{|i| \geq I(\epsilon)} \left( \left| \left( U^{g_n}(t_n, 0)u_n \right)_i \right|^2 + |\tilde{u}_i|^2 \right) \leq \epsilon^2,
\]

which implies (5.4), and therefore

\[
U^{g_n}(t_n, 0)u_n \to \tilde{u} \quad \text{in} \quad l^2. \tag{5.8}
\]

By (5.2) and (5.8), we obtain

\[
S(t_n)(u_n, g_n) = \left( U^{g_n}(t_n, 0)u_n, T(t_n)g_n \right) \to (\tilde{u}, \tilde{g}), \quad \text{as} \quad n \to \infty,
\]

which concludes the proof. \( \square \)
We are now ready to prove the main result of this section, that is, the existence of uniform attractors for the processes \( \{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)} \).

**Proof of Theorem 2.1.** By Lemma 3.2 and the continuity of the translation group \( \{T(t)\}_{t \in \mathbb{R}} \), we find that the semigroup \( \{S(t)\}_{t \geq 0} \) is continuous in \( l^2 \times \mathcal{H}(g_0) \). Let \( B_S = B \times \mathcal{H}(g_0) \), where \( B \) is the uniform absorbing set of the processes \( \{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)} \) defined in (4.1). Then \( B_S \) is a bounded absorbing set for \( \{S(t)\}_{t \geq 0} \), and hence \( \{S(t)\}_{t \geq 0} \) is point dissipative. In addition, \( \{S(t)\}_{t \geq 0} \) is asymptotically compact as proved in Lemma 5.4. Applying Proposition 5.3, we conclude that the family of processes \( \{U^g(t, \tau)\}_{g \in \mathcal{H}(g_0)} \) has a uniform attractor \( \mathcal{A} \) with respect to \( g \in \mathcal{H}(g_0) \). Further \( \mathcal{A} \) is the union of all bounded trajectories of the processes; that is,

\[
\mathcal{A} = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_g(0),
\]

where \( \mathcal{K}_g(0) \) is the kernel section at time \( t = 0 \). The proof is complete. \( \square \)

### 6. Upper semicontinuity of attractors

In this section, we approximate the infinite-dimensional lattice system (3.1)–(3.2) by a family of finite-dimensional systems, and study the relations between the asymptotic behavior of the original system and the approximate systems. More precisely, we will show that the uniform attractors of these systems are upper semicontinuous.

For every integer \( n \geq 1 \), consider the following \((2n + 1)^N\)-dimensional system of ordinary differential equations, for each \( i = (i_1, i_2, \ldots, i_N) \in \mathbb{Z}^N \) with \( |i| \leq n \),

\[
\dot{u}_i + v(Au)_i + \lambda u_i = -f(u_i) + g_i(t), \quad t > \tau, \; \tau \in \mathbb{R},
\]

with the periodic boundary conditions

\[
\begin{align*}
    u(i_1 + 2n + 1, i_2, \ldots, i_N) &= \cdots = u(i_1, i_2, \ldots, i_{N-1}, i_N + 2n + 1) \\
    &= u(i_1, i_2, \ldots, i_N),
\end{align*}
\]

and the initial data

\[
u_i(\tau) = u_{\tau,i}, \quad |i| \leq n, \; \tau \in \mathbb{R}.
\]

(6.3)

Similar to system (3.1)–(3.2), we can show that problem (6.1)–(6.3) is well-posed in \( \mathbb{R}^{(2n+1)^N} \); that is, for every \( u_\tau = (u_{\tau,i})_{|i| \leq n} \in \mathbb{R}^{(2n+1)^N} \), problem (6.1)–(6.3) possesses a unique solution \( u = (u_i(\cdot))_{|i| \leq n} \in C([\tau, +\infty), \mathbb{R}^{(2n+1)^N}) \), which continuously depends on initial data. Therefore we can associate a family of continuous processes \( \{U^g_n(t, \tau)\}_{g \in \mathcal{H}(g_0)} \) with problem (6.1)–(6.3); for every \( g \in \mathcal{H}(g_0), \; \tau \in \mathbb{R}, \) and \( t \geq \tau \), \( U^g_n(t, \tau) \) maps \( \mathbb{R}^{(2n+1)^N} \) into itself such that, for each \( u_\tau = (u_{\tau,i})_{|i| \leq n} \in \mathbb{R}^{(2n+1)^N}, \; U^g_n(t, \tau)u_\tau = u(t) \), the state of system (6.1)–(6.3) at time \( t \). For system (6.1)–(6.3), we have the following uniform estimates on solutions, which are analogous to system (3.1)–(3.2).

**Lemma 6.1.** Suppose \( g_0 \in C_b(\mathbb{R}, l^2) \) is almost periodic and (2.3) holds. Then there exists a constant \( M \) such that any solution \( u \) of problem (6.1)–(6.3) satisfies

\[
\|u(t)\|_{\mathbb{R}^{(2n+1)^N}} \leq M, \quad \text{for all } t \geq \tau + T,
\]

where \( M \) depends only on \( \lambda \) and \( g_0 \); \( T \) depends only on \( \lambda, \; g_0 \) and \( R \) when \( \|u_\tau\|_{\mathbb{R}^{(2n+1)^N}} \leq R \). In particular, both \( M \) and \( T \) are independent of \( \tau \), \( g \in \mathcal{H}(g_0) \) and \( n \).
Lemma 6.2. Suppose \( g_0 \in C_b(\mathbb{R}, l^2) \) is almost periodic and (2.3) holds. Let \( u \) be the solution of problem (3.1)–(3.2) with \( \|u_0\|_{\mathbb{R}^{(2n+1)N}} \leq R \). Then for every \( \epsilon > 0 \), there exist constants \( K(\epsilon) \) and \( T(\epsilon, R) \) such that, if \( t - \tau \geq T(\epsilon, R) \),

\[
\sum_{K(\epsilon) \leq |i| \leq n} |u_i(t)|^2 \leq \epsilon,
\]

where \( K(\epsilon) \) depends only on \( \epsilon, \lambda \) and \( g_0; T(\epsilon, R) \) depends only on \( \epsilon, \lambda, g_0 \) and \( R \). In particular, \( K(\epsilon) \) and \( T(\epsilon, R) \) are independent of \( \tau \), \( g \in \mathcal{H}(g_0) \) and \( n \).

The uniform estimates in Lemmas 6.1 and 6.2 can be derived in a manner similar to that in Lemmas 4.1 and 4.2, and therefore we will not pursue the details here again. Denote by \( B_n \) the ball in \( \mathbb{R}^{(2n+1)N} \):

\[
B_n = \left\{ u \in \mathbb{R}^{(2n+1)N} : \|u\|_{\mathbb{R}^{(2n+1)N}} \leq M \right\},
\]

where \( M \) is the constant in Lemma 6.1. Then Lemma 6.1 shows that \( B_n \) is a bounded absorbing set for the family of processes \( \{U^g_n(t, \tau)\}_{g \in \mathcal{H}(g_0)} \), that is, for any \( \tau \in \mathbb{R} \) and bounded set \( X \) in \( \mathbb{R}^{(2n+1)N} \), there exists a constant \( T \) depending only on \( X, \lambda \) and \( g_0 \) such that

\[
U^g_n(t, \tau)X \subseteq B_n, \quad \text{for all } g \in \mathcal{H}(g_0) \text{ and } t - \tau \geq T.
\]

Since \( \mathbb{R}^{(2n+1)N} \) is finite-dimensional, the bounded set \( B_n \) is precompact. Therefore, it follows that the family of processes \( \{U^g_n(t, \tau)\}_{g \in \mathcal{H}(g_0)} \) has a uniform attractor \( A_n \) in \( \mathbb{R}^{(2n+1)N} \). Further \( A_n \) is the union of all bounded complete trajectories of the processes; that is,

\[
A_n = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_{n,g}(0),
\]

where \( \mathcal{K}_{n,g}(0) \) is the kernel section at 0 of the process \( U^g_n(t, \tau) \) for \( g \in \mathcal{H}(g_0) \):

\[
\mathcal{K}_{n,g}(0) = \{ u(0) \in \mathbb{R}^{(2n+1)N} : u(\cdot) \in C_b(\mathbb{R}, \mathbb{R}^{(2n+1)N}) \text{ is a complete solution of } U^g_n(t, \tau) \}. \tag{6.6}
\]

The goal of this section is to show that the attractors \( A_n \) converge to the uniform attractor \( A \) of the infinite-dimensional system (3.1)–(3.2). To that end, given an element \( u \in \mathbb{R}^{(2n+1)N} \), we extend it as an element of \( l^2 \) such that \( u_i = 0 \) for \( |i| > n \). If no confusion arises, we shall denote such an extended element in \( l^2 \) still by the same notation \( u \). Then we have the following result which is crucial to the proof of the upper semicontinuity of uniform attractors.

Lemma 6.3. Suppose \( g_0 \in C_b(\mathbb{R}, l^2) \) is almost periodic and (2.3) holds. If \( u_{0,n} \in A_n \ (n = 1, 2, \ldots) \), then there exists a subsequence \( u_{0,n_k} \) of \( u_{0,n} \) and \( u_0 \in A \) such that \( u_{0,n_k} \) converges to \( u_0 \) in \( l^2 \).

Proof. Since \( u_{0,n} \in A_n \), it follows from (6.5) and (6.6) that there exist \( g_n \in \mathcal{H}(g_0) \) and a bounded complete solution \( u_n(\cdot) \in C_b(\mathbb{R}, \mathbb{R}^{(2n+1)N}) \) for \( U^g_n(t, \tau) \) such that

\[
u_{0,n} = u_n(0), \quad u_n(t) \in A_n, \quad \text{for all } t \in \mathbb{R} \text{ and } n = 1, 2, \ldots \tag{6.7}
\]

Since the set \( \mathcal{H}(g_0) \) is compact, there exist \( g \in \mathcal{H}(g_0) \) and a subsequence of \( \{g_n\}_{n=1}^{\infty} \) (still denoted by \( \{g_n\}_{n=1}^{\infty} \)) such that

\[
g_n \to g \quad \text{in } C_b(\mathbb{R}, l^2), \quad \text{as } n \to \infty. \tag{6.8}
\]
Note that \( u_n(t) \in \mathcal{A}_n \subset B_n \) where \( B_n \) is the bounded set given in (6.4). Therefore, we have
\[
\| u_n(t) \| \leq M, \quad \text{for all } t \in \mathbb{R} \text{ and } n = 1, 2, \ldots, \tag{6.9}
\]
which along with (6.1) implies
\[
\| \dot{u}_n(t) \| \leq C, \quad \text{for all } t \in \mathbb{R} \text{ and } n = 1, 2, \ldots, \tag{6.10}
\]
Let \( J_k \) \((k = 1, 2, \ldots)\) be a sequence of compact intervals of \( \mathbb{R} \) such that \( J_k \subset J_{k+1} \) and \( \bigcup_k J_k = \mathbb{R} \). In what follows, by Ascoli’s theorem we shall prove that \( \{ u_n \}_{n=1}^\infty \) is precompact in \( C(J_k, l^2) \) for each positive integer \( k \). For that purpose, we have to show that \( \{ u_n \}_{n=1}^\infty \) is equicontinuous in \( C(J_k, l^2) \) for each \( k \), and \( \{ u_n(t) \}_{n=1}^\infty \) is precompact in \( l^2 \) for each \( t \in J_k \).

By (6.10) we have
\[
\| u_n(s) - u_n(t) \| \leq \| \dot{u}_n \| | s - t | \leq C | s - t |,
\]
which implies the equicontinuity of \( \{ u_n \}_{n=1}^\infty \). To prove the precompactness of \( \{ u_n(t) \}_{n=1}^\infty \), we notice that for each \( t \in J_k \), \( \{ u_n(t) \} \) is bounded in \( l^2 \) by (6.9), and hence there exists a subsequence of \( \{ u_n(t) \} \) (still denoted by \( \{ u_n(t) \} \)) and \( w_t \in l^2 \) such that
\[
u_n(t) \rightharpoonup w_t \quad \text{weakly in } l^2.
\]
Then, using Lemma 6.2 and proceeding as in the proof of Lemma 5.4, one can show that the weak convergence is actually strong convergence, and therefore \( \{ u_n(t) \} \) is precompact in \( l^2 \) for fixed \( t \in J_k \).

Now, we have already proved the equicontinuity of \( \{ u_n \} \) in \( C(J_k, l^2) \) and the precompactness of \( \{ u_n(t) \} \) for \( t \in J_k \). It thus follows from Ascoli’s theorem that \( \{ u_n \} \) is precompact in \( C(J_k, l^2) \) for each positive integer \( k \). Then we infer that there exists a subsequence \( \{ u_{n_k} \} \) of \( \{ u_n \} \) and \( u \in C(J, l^2) \) such that \( \{ u_n(t) \} \) converges to \( u \) in \( C(J, l^2) \). Using Ascoli’s theorem again, we can show, by induction, that there is a subsequence \( \{ u_{n_k+1} \} \) of \( \{ u_{n_k} \} \) such that \( \{ u_{n_k+1} \} \) converges to \( u \) in \( C(J_{k+1}, l^2) \). Finally, taking a diagonal subsequence in the usual way, we find that there exist a subsequence \( \{ u_{n_k} \} \) of \( \{ u_n \} \) and \( u \in C(\mathbb{R}, l^2) \) such that
\[
u_{n_k} \to u \quad \text{in } C(J, l^2) \quad \text{for any compact interval } J \subset \mathbb{R}. \tag{6.11}
\]
By (6.9) we have
\[
u(t) \leq C, \quad \text{for all } t \in \mathbb{R}. \tag{6.12}
\]
Next, we show that \( u \) solves Eq. (3.1). For convenience, we denote by \( \{ u_n \} \) the sequence \( \{ u_{n_k} \} \) in (6.11). Then it follows from (6.10) that
\[
\dot{u}_n \rightharpoonup \dot{u} \quad \text{in } L^\infty(\mathbb{R}, l^2) \quad \text{weak star}. \tag{6.13}
\]
For fixed \( i \in \mathbb{Z}^N \), let \( n > |i| \). Since \( u_n(\cdot) \) is the solution of system (6.1)–(6.3) with \( g_n \in \mathcal{H}(g_0) \), we have
\[
\dot{u}_{n,i} + v(Au_n)_i + \lambda u_{n,i} = - f(u_{n,i}) + g_{n,i}(t), \quad t \in \mathbb{R}.
\]
Then for each \( \psi \in C_0^\infty(J) \), we get
\[
\int_J \dot{u}_{n,i}(t) \psi(t) \, dt + \nu \int_J (A(u_n))_i \psi(t) \, dt + \lambda \int_J u_{n,i} \psi(t) \, dt = - \int_J f(u_{n,i}) \psi(t) \, dt + \int_J g_{n,i}(t) \psi(t) \, dt.
\]
Taking the limit as $n \to \infty$, and using (6.8), (6.11) and (6.13) we obtain
\[ \dot{u}_i + \nu (A(u))_i + \lambda u_i = -f(u_i) + g_i(t), \quad \text{for all } t \in J \text{ and } i \in \mathbb{Z}^N. \]
Since $J$ is arbitrary, by (6.12) we find that $u$ is a bounded complete solution of problem (3.1)–(3.2), and hence $u(0) \in \mathcal{A}$. By (6.11) we conclude that
\[ u_{n_k}(0) \to u(0) \in \mathcal{A}. \]
The proof is complete. \hfill \Box

We are now ready to prove the upper semicontinuity of the uniform attractors as claimed in Theorem 2.2.

**Proof of Theorem 2.2.** As indicated above, the compact set $\mathcal{A}_n$ given by (6.5) is the uniform attractor for system (2.4)–(2.6). We now prove the upper semicontinuity of these attractors by contradiction arguments. If (2.7) is not true, then there exist a sequence $u_{n_k} \in \mathcal{A}_{n_k}$ and $\delta > 0$ such that
\[ d_{l_2}(u_{n_k}, \mathcal{A}) \geq \delta > 0. \tag{6.14} \]
On the other hand, by Lemma 6.3 there exists a subsequence $u_{n_{k_m}}$ of $u_{n_k}$ such that
\[ d_{l_2}(u_{n_{k_m}}, \mathcal{A}) \to 0, \]
which contradicts (6.14). The proof is complete. \hfill \Box

**References**


