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Integrable representations of toroidal Lie algebras co-ordinatized by rational quantum tori

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ABSTRACT

For any positive integers d, n with $d > 1$ and $n > 1$, we fix an n by n complex matrix $q = (q_{ij})$ satisfying $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$ with all q_{ij} roots of unity. Let $\widetilde{\tau}(d, q)$ be the universal central extension of the Lie subalgebra $\mathfrak{sl}_d(\mathbb{C}_q)$ of $\mathfrak{gl}_d(\mathbb{C}_q)$ with trace in $[\mathbb{C}_q, \mathbb{C}_q]$, where $\mathbb{C}_q$ is the rational quantum torus associated to q, and let $\widehat{\tau}(d, q)$ be the Lie algebra by adding the n degree derivations to $\widetilde{\tau}(d, q)$ with respect to the n non-commuting variables in $\mathbb{C}_q$. The Lie algebra $\widehat{\tau}(d, q)$, called the toroidal Lie algebra co-ordinatized by the rational quantum torus $\mathbb{C}_q$, has an n-dimensional center C. In this paper, we obtain a classification of irreducible integrable modules with finite dimensional weight spaces and with non-zero center action over the toroidal Lie algebra $\widehat{\tau}(d, q)$.

0. Introduction

Toroidal Lie algebras are universal central extensions of iterated loop algebras with more than one variable. They are natural generalization of affine Lie algebras. They were introduced by Moody, Rao and Yokonuma in [EMY] and since then have attracted considerable mathematical attention. Toroidal Lie algebras have also appeared in mathematical physics. Inami, Kanno, Ueno and Xiong [IKUX] have shown that a toroidal Lie algebra with $n = 2$ arises as a symmetry algebra in a four-dimensional non-linear sigma model with a Wess–Zumino like term introduced by Donaldson [D], much in the same way as affine Lie algebras arise as symmetry algebras in the Wess–Zumino–Witten model. Inami, Kanno and Ueno [IKU] have generalized this construction to arbitrary n.

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In recent years, extended affine Lie algebras (EALAs) have been studied in great detail. EALAs are Lie algebras which have a non-degenerate invariant form, a self-centralizing finite dimensional ad-diagonalizable abelian subalgebra (i.e., a Cartan subalgebra), a discrete irreducible root system, and ad-nilpotency of non-isotropic root spaces (see [AABGP,BGK,ABGP] for definitions and structure theory). Toroidal Lie algebras which are universal central extension of $\mathcal{G} \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ (where $\mathcal{G}$ is a simple finite dimensional Lie algebra and $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is a Laurent polynomial algebra in n commuting variables) are examples of cores of EALAs and are studied in [E1,E2,E3,E4,E5,E6,EM,EMY, GL,BB] among others. There are many EALAs which allow not only the Laurent polynomial algebras as co-ordinate algebras but also quantum tori, Jordan tori and Octonion tori as co-ordinate algebras depending on the type of algebras (see [AABGP,BGK,BGKN,MP,Y1,Y2,Y3]). For instances, EALAs of type A_{d-1} are tied up with the Lie algebra $\mathfrak{gl}_d(\mathbb{C}_q)$. Quantum tori defined in [MP] and [M] are non-commutative analogue of Laurent polynomial algebras. To get an EALA one has to form appropriate central extension of $\mathfrak{gl}_d(\mathbb{C}_q)$ and add certain outer derivations (just like one obtains an affine Kac–Moody Lie algebra from a loop algebra by forming a one-dimensional central extension and then adding the degree derivation).

As a first step we classify irreducible integrable modules for the universal central extension of $\mathfrak{gl}_d(\mathbb{C}_q)$ with finite dimensional weight spaces (we assume at least one zero degree central operator acts non-trivially). Representations of the universal central extension of $\mathfrak{gl}_d(\mathbb{C}_q)$ are studied in [GL, G1,G2,G3,E3,E6,BZK,EB] in two-variable case. In fact explicit construction of representations for the quantum torus case through the use of vertex operators is done in [GL,G1,G2,BZK].

We now explain in detail the results of this paper. In Section 1 we collect basic notation and definitions. Throughout the paper we assume that the quantum torus $\mathbb{C}_q$ is rational, i.e., all q_{ij} are roots of unity (see [CP,N]). We now consider $M_d(\mathbb{C}) \otimes \mathbb{C}_q$ as an associative algebra and we denote the induced Lie algebra by $\mathfrak{gl}_d(\mathbb{C}_q)$. Inside this algebra we consider the Lie subalgebra $\tau(d, q) = \mathfrak{sl}_d(\mathbb{C}_q)$ of matrices whose trace is in $[\mathbb{C}_q, \mathbb{C}_q]$. The universal central extension of $\tau(d, q)$ is denoted by $\widetilde{\tau}(d, q)$ and we note in our case that $d \geqslant 2$ and $n \geqslant 2$. The universal center is infinite dimensional and has a natural $\mathbb{Z}^n$-gradation. To reflect the natural $\mathbb{Z}^n$-gradation on $\widetilde{\tau}(d, q)$, we add degree derivation $d_1, \ldots, d_n$ with respect to the variables $t_1, \ldots, t_n$ and denote the resultant Lie algebra by $\widehat{\tau}(d, q)$. The Lie algebra $\widehat{\tau}(d, q)$ has an n-dimensional center C. First we note that the co-ordinate in $\mathbb{C}_q$ can be changed by an element of $\mathrm{GL}(n, \mathbb{Z})$ as explained in Theorem 1.14 (also see [MP]). We also note that the affine Lie algebra corresponding to $\mathfrak{sl}_d(\mathbb{C}) \otimes \mathbb{C}[t_n, t_n^{-1}]$ is a subalgebra of $\widehat{\tau}(d, q)$ and let Δ_{aff} be the corresponding root system. The roots of $\widehat{\tau}$ of the form $\Delta_{\mathrm{aff}}^+ + \mathbb{Z}^{n-1}$ are denoted by Δ_1^+ and the corresponding sum of root spaces, a subalgebra, denoted by B (see 5.10). A weight module for $\widehat{\tau}(d, q)$ is called highest weight module if there exists a vector v killed by B and v generates the module (see 5.5). In Section 2 we determine the action of central elements in $\widehat{\tau}(d, q)$ on an irreducible $\widehat{\tau}(d, q)$-module V with finite dimensional weight spaces.

We prove in Theorem 3.5 that for an irreducible integrable module V over $\widehat{\tau}(d, q)$ with finite dimensional weight spaces if some zero degree central operator acts non-trivially then V is a highest weight module for a suitable choice of co-ordinates. The study of highest weight modules over $\widehat{\tau}(d, q)$ is reduced to the study of finite dimensional irreducible modules for $\tau_{n-1}(d, q)$ (considering the finite $n-1$ variable without any center or degree derivations). Also in Section 6 we have indicated how to get back to the original module by the method of induced modules first to non-graded version $\tau(d, q)$ and then the graded version $\widehat{\tau}(d, q)$.

In Section 4 we prove that certain semisimple finite dimensional Lie algebras (all components are of type A) are quotients of $\tau_{n-1}(d, q)$ and hence any finite dimensional irreducible module lifts to $\tau_{n-1}(d, q)$. In Sections 5 and 6 we proved that any irreducible finite dimensional module has to come from the construction given in Section 4. There we have easily classified finite dimensional irreducible modules for $\tau_{n-1}(d, q)$ (Theorem 6.6), which generalizes the main result in [Z1]. M. Lau got a classification for all finite dimensional simple modules over all multi-loop Lie algebras [L]. We do not use his result since our solution is more direct and simple for our case.

Section 5 is devoted to a classification of irreducible integrable highest weight modules over $\widetilde{\tau}^n(d, q)$ with finite dimensional weight spaces. Some of these modules were studied in [EZ]. We could have simplified the proof where we have proved certain polynomial vanishing on the module, by considering the $\tau_{n-1}(d,q)$-module. But then we would not have got the information on the center which will be used in Section 6.

Eventually we proved that the irreducible integrable modules over $\widehat{\tau}(d,q)$ with non-zero central action have to come from finite dimensional irreducible modules of $\tau_{n-1}(d,q)$ in Section 6. We have not answered the converse. Namely given finite dimensional irreducible module for $\tau_{n-1}(d,q)$ does there exist an integrable module for $\widehat{\tau}(d,q)$? This question will be addressed in a subsequent paper.

The question where all the degree zero central operators acts trivially is not answered. In this case we believe the full center acts trivially. Thus the problem is reduced to the Lie algebra $\tau(d,q)$ with derivations added. From this paper it is easy to see that a large number of irreducible integrable modules can be constructed. It is an open problem (even in two-variables case) to prove that they exhaust all irreducible integrable modules. The problem is to reduce the module for $\tau(d,q)$ with derivations to a module of $\tau(d,q)$ without these derivations. For some representations of full toroidal Lie algebras, see [EJ] and [FK].

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{C}$ the sets of all integers, non-negative integers, positive integers and complex numbers, respectively.

1. Notation and known results

We first recall the definition of a quantum torus from [BGK]. Fix positive integers $n, d \geqslant 2$. Let $q = (q_{ij})$ be an $n \times n$ matrix where q_{ij} are non-zero complex numbers such that $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$. The quantum torus associated to q is the non-commutative Laurent polynomial algebra $\mathbb{C}_q = \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}]$ with defining relations $t_i t_j = q_{ij} t_j t_i$. Clearly $\mathbb{C}_q$ is $\mathbb{Z}^n$-graded and each graded component is one-dimensional.

For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ let $t^a = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} \in \mathbb{C}_q$. We define the following maps, $\sigma, f : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{C}^*$ by

$$\sigma(a,b) = \prod_{1 \leqslant i \leqslant j \leqslant n} q_{ji}^{a_j b_i}, \tag{1.1}$$

$$f(a,b) = \sigma(a,b)\sigma(b,a)^{-1}. \tag{1.2}$$

It is easy to check the following for any $a, b \in \mathbb{Z}^n$, $k \in \mathbb{Z}$:

$$f(a,b) = \prod_{i,j=1}^{n} q_{ji}^{a_j b_i}, \tag{1.3}$$

$$f(a,b) = f(b,a)^{-1},$$

$$f(ka,a) = f(a,ka) = 1,$$

$$f(a+b,c) = f(a,c) f(b,c),$$

$$f(a,b+c) = f(a,b) f(a,c),$$

$$\sigma(a,b+c) = \sigma(a,b)\sigma(a,c),$$

$$t^a t^b = \sigma(a,b) t^{a+b}, \qquad t^a t^b = f(a,b) t^b t^a. \tag{1.4}$$

The *radical of* f is defined by

$$\operatorname{rad} f = \{ a \in \mathbb{Z}^n \mid f(a,b) = 1, \ \forall b \in \mathbb{Z}^n \}. \tag{1.5a}$$

Note that $\operatorname{rad} f$ is a subgroup of $\mathbb{Z}^n$. Further,

$$m \in \operatorname{rad} f \quad \Leftrightarrow \quad f(r,s) = 1, \quad \forall r, s \in \mathbb{Z}^n \text{ with } r + s = m. \tag{1.5b}$$

Proof. $m \in \operatorname{rad} f$, iff $f(m,s) = 1$ for all $s \in \mathbb{Z}^n$, iff $f(r+s,s) = 1$ for r, s with $r+s = m \in \operatorname{rad} f$, iff $f(r,s) = 1$ for all r, s with $r + s = m$ as $f(s,s) = 1$. □

1.6. Proposition. *(See Proposition 2.44 [BGK].)*

1. *The center $Z(\mathbb{C}_q)$ of $\mathbb{C}_q$ has a basis consisting of monomials t^a, $a \in \operatorname{rad} f$.*
2. *The Lie subalgebra $[\mathbb{C}_q, \mathbb{C}_q]$ of $\mathbb{C}_q$ has a basis consisting of monomial t^a, $a \in \mathbb{Z}^n \setminus \operatorname{rad} f$.*
3. $\mathbb{C}_q = [\mathbb{C}_q, \mathbb{C}_q] \oplus Z(\mathbb{C}_q)$.

Fix a positive integer $d \geqslant 2$ and let $M_d(\mathbb{C})$ be the matrix algebra with unit matrices E_{ij}. We denote the corresponding Lie algebra as $\mathfrak{gl}_d(\mathbb{C})$. Let $\mathfrak{sl}_d(\mathbb{C})$ be the simple subalgebra of $\mathfrak{gl}_d(\mathbb{C})$ of trace zero matrices.

1.7. Let $M_d(\mathbb{C}_q)$ be the associative matrix algebra of $d \times d$ matrices with entries in $\mathbb{C}_q$. It is easy to see that

$$M_d(\mathbb{C}_q) \cong M_d(\mathbb{C}) \otimes \mathbb{C}_q$$

and the matrix multiplication is given by $(X \otimes t^a) \cdot (Y \otimes t^b) = XY \otimes t^a t^b$. We denote the corresponding Lie algebra by $\mathfrak{gl}_d(\mathbb{C}_q)$. Here the Lie bracket is given by

$$\left[X \otimes t^a, Y \otimes t^b\right]_0 = XY \otimes t^a t^b - YX \otimes t^b t^a.$$

Define $\mathfrak{sl}_d(\mathbb{C}_q) = \{X \in M_d(\mathbb{C}_q) \mid \operatorname{Tr}(X) \in [\mathbb{C}_q, \mathbb{C}_q]\}$ where trace $X = \sum X_{ii}$ for $X = (X_{ij})$.

1.8. We now consider the following Lie subalgebra inside $M_d(\mathbb{C}) \otimes \mathbb{C}_q$

$$\tau(d,q) = \left(I \otimes [\mathbb{C}_q, \mathbb{C}_q]\right) \oplus \left(\mathfrak{sl}_d(\mathbb{C}) \otimes \mathbb{C}_q\right).$$

Note that $\tau(d,q)$ is actually the quotient algebra of $\mathfrak{gl}_d(\mathbb{C}_q)$ by its center. The Lie bracket in $\tau(d,q)$ can be also written as

$$\begin{aligned}
\left[I \otimes \left[t^a, t^b\right], X \otimes t^c\right] &= X \otimes \left[\left[t^a, t^b\right], t^c\right], \\
\left[I \otimes \left[t^a, t^b\right], I \otimes \left[t^c, t^d\right]\right] &= I \otimes \left[\left[t^a, t^b\right], \left[t^c, t^d\right]\right], \\
\left[X \otimes t^a, Y \otimes t^b\right] &= [X,Y]_f \otimes \frac{t^a \circ t^b}{2} + (X \circ Y) \otimes \frac{1}{2}\left[t^a, t^b\right] + \frac{1}{d}\operatorname{Tr}(XY) I \otimes \left[t^a, t^b\right],
\end{aligned}$$

where

$$\begin{aligned}
t^a \circ t^b &= t^a t^b + t^b t^a, \\
[X,Y]_f &= XY - YX, \\
X \circ Y &= XY + YX - \frac{2}{d}\operatorname{Tr}(XY) I, \\
\left[t^a, t^b\right] &= t^a t^b - t^b t^a.
\end{aligned}$$

It is easy to see that (also mentioned in [BGK]):

1.9. Proposition. $\tau(d,q) = \mathfrak{sl}_d(\mathbb{C}_q)$.

1.10. We note that $\mathfrak{gl}_d(\mathbb{C}) \otimes \mathbb{C}_q = \mathfrak{sl}_d(\mathbb{C}_q) \oplus (I \otimes Z(\mathbb{C}_q))$, and $I \otimes Z(\mathbb{C}_q)$ is central. Our interest in this paper is $\tau(d,q) = \mathfrak{sl}_d(\mathbb{C}_q)$ and its universal central extension but most often we work with $\mathfrak{gl}_d(\mathbb{C}) \otimes \mathbb{C}_q$ and its extension.

1.11. We will now recall the universal central extension of $\tau = \tau(d,q)$ from [BGK]. Let J be the linear span of

$$x \otimes y + y \otimes x, \quad xy \otimes z + yz \otimes x + zx \otimes y$$

inside $\mathbb{C}_q \otimes \mathbb{C}_q$ for all $x, y, z \in \mathbb{C}_q$. Let $\langle x, y\rangle_0$ denote the element $x \otimes y + J$ in $(\mathbb{C}_q \otimes \mathbb{C}_q)/J$.

Define

$$HC_1(\mathbb{C}_q) = \left\{ \sum_i \langle x_i, y_i\rangle_0 \;\Big|\; \sum_i [x_i, y_i] = 0 \right\}.$$

Let $\widetilde{\tau}(d,q) = \tau \oplus HC_1(\mathbb{C}_q)$ where Lie brackets are given by

$$\left[X \otimes t^a, Y \otimes t^b\right] = \left[X \otimes t^a, Y \otimes t^b\right]_0 + \mathrm{Tr}(XY)\left\langle t^a, t^b\right\rangle_0 \delta_{a+b,\mathrm{rad}\, f}$$

where

$$\delta_{a,\mathrm{rad}\, f} = \begin{cases} 1 & \text{if } a \in \mathrm{rad}\, f, \\ 0 & \text{if } a \notin \mathrm{rad}\, f. \end{cases}$$

From now on we simply write $\langle t^a, t^b\rangle = \delta_{a+b,\mathrm{rad}\, f} \langle t^a, t^b\rangle_0$. From [BGK] we have

1.12. Proposition. *$\widetilde{\tau}(d,q)$ is the universal central extension of $\tau(d,q)$.*

Clearly $\widetilde{\tau}(d,q)$ is $\mathbb{Z}^n$-graded and to reflect this fact we add degree derivation. Let D be the linear span of $d_1, \dots, d_n$. Let $\widehat{\tau}(d,q) = \widetilde{\tau}(d,q) \oplus D$ and extend the Lie bracket as

$$\begin{aligned} \left[d_i, X \otimes t^a\right] &= a_i X \otimes t^a, \\ \left[d_i, I \otimes t^a\right] &= a_i I \otimes t^a, \\ \left[d_i, \left\langle t^a, t^b\right\rangle\right] &= (a_i + b_i)\left\langle t^a, t^b\right\rangle, \\ [d_i, d_j] &= 0. \end{aligned}$$

Then the center C of $\widehat{\tau}(d,q)$ is spanned by $C_i = \langle t_i, t_i^{-1}\rangle$ for $i = 1, 2, \dots, n$. We recall the following statements which are contained in Lemmas 3.18 and 3.19 of [BGK].

1.13. Lemma.

(1) $\langle 1, t^a\rangle = 0$ *for all* $a \in \mathbb{Z}^n$.
(2) $\langle t^b, t^a (t^b)^{-1}\rangle = \sum_{i=1}^d b_i \langle t_i, t^a t_i^{-1}\rangle$.

(3) $\langle t^a, t^b\rangle = \sigma(a,b)\sum_{i=1}^d a_i\langle t_i, t^{a+b}t_i^{-1}\rangle$.
(4) $(t^b)^{-1} = \sigma(b,b)t^{-b}$.

(5)
$$\dim HC_1(\mathbb{C}_q)_a = \begin{cases} 0 & \text{if } a \notin \operatorname{rad} f, \\ n-1 & \text{if } a \in \operatorname{rad} f \setminus \{0\}, \\ n & \text{if } a = 0. \end{cases}$$

Our aim in this paper is to classify irreducible integrable modules for $\widehat{\tau}(d,q)$ with finite dimensional weight spaces and with non-zero center action.

We will now indicate a certain internal symmetry in τ which is based on an observation in [MP]. Let $\mathrm{GL}(n,\mathbb{Z})$ be the group of $n\times n$ matrices with integer entries and determinant ± 1. $\mathrm{GL}(n,\mathbb{Z})$ acts naturally on $\mathbb{Z}^n$ consisting of all column vectors with entries in $\mathbb{Z}$. Let $e_1,\dots,e_n$ be the standard basis (unit column vectors) of $\mathbb{Z}^n$. Then $Ae_i = \sum_k a_{ki}e_k$ for $A = (a_{ij}) \in \mathrm{GL}(n,\mathbb{Z})$. The following result is Lemma 6.2(a) in [Z2].

1.14. Theorem. *$\mathbb{C}_q \cong \mathbb{C}_{q'}$ iff there exists $A \in \mathrm{GL}(n,\mathbb{Z})$ such that $q'_{ij} = \prod_{k,l} q_{kl}^{a_{ki}a_{lj}}$.*

We will extend this to our Lie algebra $\widehat{\tau}(d,q)$. Let $A \in \mathrm{GL}(n,\mathbb{Z})$ and define q' as above. Now by Theorem 1.14 $\tau(d,q) \cong \tau(d,q')$ and hence their universal central extensions are isomorphic. So $\widehat{\tau}(d,q) \cong \widehat{\tau}(d,q')$. Now the co-ordinates for $\mathbb{C}_{q'}$ are given by $s_i = t^{Ae_i}$.

From Lemma 1.13(2), we have

1.15. Lemma. $\langle s_i, s_i^{-1}\rangle = \langle t^{Ae_i}, (t^{Ae_i})^{-1}\rangle = \sum_k a_{ki}\langle t_k, t_k^{-1}\rangle$.

2. Action of central elements

Recall that the n-dimensional center C of $\widehat{\tau}(d,q)$ is spanned by $C_i = \langle t_i, t_i^{-1}\rangle$ for $i = 1,2,\dots,n$. Let $\dot{\mathfrak{h}}$ be the Cartan subalgebra of $\mathfrak{sl}_d(\mathbb{C})$ spanned by the elements $E_{ii} - E_{i+1,i+1}$ for $1 \leqslant i \leqslant d-1$. Let $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \sum_{i=1}^n \mathbb{C}C_i \oplus D$ which we call Cartan subalgebra of $\widehat{\tau}$. Throughout this section we fix an irreducible weight module V for $\widehat{\tau}(d,q)$ with finite dimensional weight spaces relative to the Cartan subalgebra $\mathfrak{h}$.

2.1. Definition. Let $m = (m_1,\dots,m_n) \in \mathbb{Z}^n$. A linear map $z: V \to V$ is called *a central operator of degree m* if z commutes with the action of $\widetilde{\tau}(d,q)$ and $d_i z - z d_i = m_i z$.

For example $\langle t^r, t^s\rangle$ is a central operator of degree $r+s$ if $r+s \in \operatorname{rad} f$.

2.2. Lemma. *Let z be a central operator of degree $m \in \mathbb{Z}^n$.*

(a) *If $zv \neq 0$ for some v, then $zw \neq 0$ for all non-zero w in V.*
(b) *If z is a non-zero central operator, then there exists another non-zero central operator T of degree $-m$ such that $Tz = zT = \mathrm{Id}$.*
(c) *If z_1, z_2 are non-zero central operators of degree m, then there exists a scalar λ such that $z_1 = \lambda z_2$.*

Proofs are identical to those of Lemmas 1.7 and 1.8 in [E5].

2.3. Let $L = \{r \in \operatorname{rad} f \mid \langle t^m, t^{r-m}\rangle \neq 0 \text{ on } V \text{ for some } m\}$. Let S be the subgroup of $\mathbb{Z}^n$ generated by L. Note that if $m \in S$ then there exists a non-zero central operator of degree m. Let k be the rank of S and l be the rank of $\operatorname{rad} f$. Clearly $k \leqslant l$. Now by standard basis theorem there exists a $\mathbb{Z}$-basis $m^{(1)},\dots,m^{(n)}$ of $\mathbb{Z}^n$ such that $p_1 m^{(1)},\dots,p_k m^{(k)}$ is a $\mathbb{Z}$-basis of S for some non-zero integers $p_1,\dots,p_k$. It is also standard fact that there exists $A \in \mathrm{GL}(n,\mathbb{Z})$ such that $Am^{(i)} = e_i$.

Thus after a co-ordinate change, as explained in Theorem 1.14, we can assume that there exists non-zero central operators $z_1,\dots,z_k$ of degree $(p_1,0,\dots,0),\dots,(0,\dots,0,p_k,0,\dots,0)$ respectively. We now state the following important

2.4. Proposition. *Let V be an irreducible module for $\widehat{\tau}(d,q)$ with finite dimensional weight spaces. After a suitable co-ordinate change for $\mathbb{C}_q$ we have the following on V:*

(1) *There exist a non-negative integer k and non-zero central operators $z_1,\dots,z_k$ of degree $(p_1,0,\dots),\dots,(0,\dots p_k,\dots 0)$ for some positive integers $p_1,\dots,p_k$.*
(2) *$k<n$ if $\mathbb{C}_q$ is rational.*
(3) *$\langle t^r t_i^{-1},t_i\rangle\neq 0$ on V implies $i\geqslant k+1$ and $r_{k+1}=\dots=r_n=0$.*
(4) *There exists a proper submodule W for $\widetilde{\tau}(d,q)\oplus D_k$ (D_k is the linear span of $d_{k+1},\dots,d_n$) such that V/W has finite dimensional weight spaces with respect to $\dot{h}\oplus\sum_i\mathbb{C}C_i\oplus D_k$.*

Proof. We have already seen (1). For (4) we take $W=\{v-z_iv\mid v\in V,\ 1\leqslant i\leqslant k\}$. The proof of (4) is identical to the proof of Theorem 4.5(5) in [E4].

To prove (3), suppose $\langle t^r t_i^{-1},t_i\rangle\neq 0$ on V such that $r_{k+j}\neq 0$ for some $j>0$. Then $r\in\operatorname{rad}(f)$, and the rank of S is at least $k+1$, which is a contradiction to the choice of k. Thus $\langle t^r t_i^{-1},t_i\rangle\neq 0$ on V implies $r_{k+1}=\dots=r_n=0$. Now let $r\in\operatorname{rad} f$ and assume $r_{k+1}=\dots=r_n=0$ and that $\langle t^r t_i,t_i^{-1}\rangle\neq 0$ on V for a fixed i: $1\leqslant i\leqslant k$. Take h in $\dot{\mathfrak{h}}$ such that $(h,h)\neq 0$. Consider the nilpotent subalgebra H_i spanned by $ht^r t_i^{-1}$, ht_i and $\langle t^r t_i^{-1},t_i\rangle$. Note that $r,p_ie_i\in S$, and that weight spaces $V_{\alpha+m}$ and V_α for any $m\in S$ are considered the same in V/W. Then $M=(W+\sum_{j=1}^{p_i}V_{\lambda+je_i})/W$ for any weight λ is a finite dimensional non-zero subspace of V/W which is a module for H_i. Then $H_i|_M$ is a finite dimensional nilpotent algebra acting on M. By Lie's theorem it has a common eigenvector say v. It is easy to see that $[ht^r t_i^{-1},ht_i]$ is zero on v. Hence $z=\langle t^r t_i^{-1},t_i\rangle$ is zero on M, that is, there exists v in V, $v\notin W$ such that $zv\in W$. Suppose $z\neq 0$ then z^{-1} exists. We also note that from the definition of W, z and z^{-1} leaves W-invariant.

Thus $v=z^{-1}zv\in z^{-1}W\subseteq W$, which is a contradiction. So $z=0$. In particular $\langle t^r t_i^{-1},t_i\rangle=0$ for $1\leqslant i\leqslant k$. This completes the proof of (3).

Now we prove (2). Since $\mathbb{C}_q$ is rational, then $l=n$. Suppose $k=n$. From (3) we see that $\langle t^{r+s}t_i^{-1},t_i\rangle=0$ for $r,s\in\mathbb{Z}^n$ with $r+s\in\operatorname{rad} f$. Then

$$\langle t^r,t^s\rangle=\sigma(r,s)\sum r_i\langle t_i,t^{r+s}t_i^{-1}\rangle=0,$$

for $r,s\in\mathbb{Z}^n$ with $r+s\in\operatorname{rad} f$. Consequently $k=0$, which is a contradiction since we have assumed that $k=l=n>1$. □

Next we will prove that $k=n-1$ if $\mathbb{C}_q$ is rational and $S\neq 0$ for V. In the process we record some interesting results which are of independent interest. Let H be the Lie algebra of diagonal matrices inside $\mathfrak{gl}_d(\mathbb{C})$. Now consider the quantum torus $\mathbb{C}_{\bar q}=\mathbb{C}[t_1^{\pm1},t_2^{\pm1}]$ with the relations $t_1t_2=\bar q t_2t_1$ with $\bar q^N=1$ for some integer N. Consider the Lie algebra

$$\widehat{\mathfrak{h}}=(H\otimes\mathbb{C}_{\bar q})\oplus\mathbb{C}C_1\oplus\mathbb{C}C_2$$

with Lie bracket

$$\left[h_1t_1^{r_1}t_2^{r_2},h_2t_1^{s_1}t_2^{s_2}\right]=h_1h_2\left(t_1^{r_1}t_2^{r_2}t_1^{s_1}t_s^{s_2}-t_1^{s_1}t_2^{s_2}t_1^{r_1}t_2^{r_2}\right)+(h_1,h_2)\delta_{r_1+s_1,0}\delta_{r_2+s_2,0}(r_1C_1+r_2C_2),$$

where C_1, C_2 are central and $(h_1,h_2)=\operatorname{Tr}(h_1h_2)$. Consider the subalgebra

$$\widehat{\dot{\mathfrak{h}}}=(\dot{\mathfrak{h}}\otimes\mathbb{C}_{\bar q})+\left(I\otimes[\mathbb{C}_{\bar q},\mathbb{C}_{\bar q}]\right)\oplus\mathbb{C}C_1\oplus\mathbb{C}C_2.$$

Clearly $\widehat{\dot{\mathfrak{h}}}$ is a $\mathbb{Z}^2$-graded Lie algebra with $\deg C_1=\deg C_2=0$.

2.5. Lemma. *There does not exist a $\mathbb{Z}^2$-graded module for $\widehat{\dot{\mathfrak{h}}}$ with finite dimensional components and with non-zero central action.*

Proof. Consider the subalgebra

$$\dot{\mathfrak{h}} \otimes \mathbb{C}\big[t_1^{\pm N}, t_2^{\pm N}\big] \oplus \mathbb{C}C_1 \oplus \mathbb{C}C_2$$

and restrict any module of $\widehat{\dot{\mathfrak{h}}}$ onto it. Now the torus is commutative and we can apply Proposition 1.11 of [E5] to conclude that C_i acts trivially for $i = 1, 2$. □

We will now record an interesting corollary which is of independent interest. Recall the Lie algebra τ with Lie bracket $[\cdot,\cdot]_0$. Now define a Lie structure on $\tau_0 = \tau \oplus \sum \mathbb{C}C_i \oplus \sum \mathbb{C}d_i$ by $[X \otimes t^r, Y \otimes t^s] = [X \otimes t^r, Y \otimes t^s]_0 + (X, Y)\delta_{r+s,0} \sum_i r_i C_i$. The remaining brackets are same. The action of d_i's are to measure the degree. Note that τ_0 is the quotient of $\widehat{\tau}$.

2.6. Corollary. *Assume there exist non-zero integers N_{ij} such that $q_{ij}^{N_{ij}} = 1$. Then there does not exist a module for τ_0 with finite dimensional weight spaces and non-zero central action.*

Proof. Restrict the module to $\dot{\mathfrak{h}} \otimes \mathbb{C}[t_i^{\pm N_{ij}}, t_j^{\pm N_{ij}}]$ for suitable i and j and apply the argument as in the proof of the lemma above. □

2.7. Theorem. *Suppose $\mathbb{C}_q$ is rational. Let V be an irreducible module for $\widehat{\tau}$ with finite dimensional weight spaces. Let k be as defined in* 2.3. *If $k > 0$, then $k = n - 1$.*

Proof. Note that the rank of $\operatorname{rad} f$ equals $l = n$ under the assumptions of the theorem. We may assume Proposition 2.4. By Lemma 1.13(3), we can assume that $\langle t^m t_i^{-1}, t_i\rangle$ is non-zero on V for some $m \in \operatorname{rad} f$ and $1 \leqslant i \leqslant n$. Now by Proposition 2.4(3) we can assume that $i \geqslant k + 1$ and $m_{k+1} = \cdots = m_n = 0$. We already know that $k < n$. Suppose $k < n - 1$ then $k + 1, k + 2 \leqslant n$ we can assume that $\langle t^m t_{k+1}^{-1}, t_{k+1}\rangle$ is non-zero on V. Consider W as in Proposition 2.4(4) and note that V/W is a module for $\widetilde{\tau} \oplus D_k$ with finite dimensional weight spaces with respect to $\dot{\mathfrak{h}} \oplus \sum_i \mathbb{C}C_i \oplus D_k$. Let $q_{k+1,k+2} = q$ and $N_{k+1,k+2} = N$. Consider the subspace (not subalgebra) $\mathcal{H}$ spanned by

$$\begin{aligned}
&(1)\quad \dot{\mathfrak{h}} t^m t_{k+1}^{rN} t_{k+2}^{sN}; \quad s > 0,\ \text{or}\ s = 0 \text{ and } r > 0;\\
&(2)\quad \dot{\mathfrak{h}} t_{k+1}^{rN} t_{k+2}^{sN}; \quad s < 0,\ \text{or}\ s = 0 \text{ and } r < 0;\\
&(3)\quad \big\langle t^m t_{k+1}^{-1}, t_{k+1}\big\rangle \quad \text{and} \quad \big\langle t^m t_{k+2}^{-1}, t_{k+2}\big\rangle.
\end{aligned}$$

Because of Proposition 2.4(3), we know that

$$\big[h_1 t^m t_{k+1}^{r_1 N} t_{k+2}^{s_1 N}, h_2 t_{k+1}^{r_2 N} t_{k+2}^{s_2 N}\big]\big|_V = (h_1, h_2)\big\langle t^m t_{k+1}^{r_1 N} t_{k+2}^{s_1 N}, t_{k+1}^{r_2 N} t_{k+2}^{s_2 N}\big\rangle\big|_V,$$

and all other brackets on V are zero. Then

$$\big[h_1 t^m t_{k+1}^{r_1 N} t_{k+2}^{s_1 N}, h_2 t_{k+1}^{r_2 N} t_{k+2}^{s_2 N}\big]\big|_V = \delta_{r_1+r_2,0}\delta_{s_1+s_2,0}(h_1, h_2)N\big(r_2\big\langle t^m t_{k+1}^{-1}, t_{k+1}\big\rangle + s_2\big\langle t^m t_{k+2}^{-1}, t_{k+2}\big\rangle\big)\big|_V.$$

Then $\mathcal{H}|_V$ is a Lie algebra in Proposition 1.11 of [E5], and W is an $\mathcal{H}|_V$ submodule of V. The action of $\langle t^m t_{k+1}, t_{k+1}^{-1}\rangle$ on V/W is still invertible since it is invertible on W. From Proposition 2.4(4), we know that V/W has finite dimensional weight spaces with respect to $\dot{h} \oplus \sum_i \mathbb{C}C_i \oplus D_k$. Then there exists a submodule X of V/W with finite dimensional graded components. Note that the central elements:

$$\big\langle t^m t_{k+1}, t_{k+1}^{-1}\big\rangle \quad \text{and} \quad \big\langle t^m t_{k+2}, t_{k+2}^{-1}\big\rangle$$

are of degree zero with respect to D_k. Thus from Lemma 2.5 it follows that $\langle t^m t_{k+1}, t_{k+1}^{-1}\rangle$ acts trivially on X, which is a contradiction. Thus we have proved that $k = n - 1$. □

3. Integrable modules are highest weight modules

In this section we will define integrable modules over $\widehat{\tau}(d,q)$ and prove that an irreducible integrable module for $\widehat{\tau}(d,q)$ with finite dimensional weight spaces admits a "highest weight" space. For this we need to define real roots, null roots and Weyl group. The point is that the Weyl group is the same as in the commutative torus case. The set of weights of an irreducible integrable module with finite dimensional weight spaces carries same properties as in the commuting case and hence the same conclusions can be drawn. Thus we closely follow the notation of Section 1 in [E4].

Recall that $\dot{\mathfrak{h}}$ is the Cartan subalgebra consisting of trace zero diagonal matrices of $\mathfrak{sl}_d(\mathbb{C})$. Thus $\dim \dot{\mathfrak{h}} = d-1$. Let $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$ be the standard simple roots of $\dot{\mathfrak{h}}$. Let $\alpha_1^\vee, \dots, \alpha_{d-1}^\vee$ be the co-roots. Recall $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \sum_{i=1}^n \mathbb{C}C_i \oplus \sum_{i=1}^n \mathbb{C}d_i$ where $C_i = \langle t_i, t_i^{-1}\rangle$. Hence $\dim \mathfrak{h} = d-1+2n$. Define $\delta_i \in \mathfrak{h}^*$, $1 \leqslant i \leqslant n$, such that $\delta_i(\dot{\mathfrak{h}}) = 0$, $\delta_i(d_j) = \delta_{ij}$ and $\delta_i(C_j) = 0$. Let $w_1, \dots, w_n$ in $\mathfrak{h}^*$ such that $w_i(\dot{\mathfrak{h}}) = 0$, $w_i(d_j) = 0$ and $w_i(C_j) = \delta_{ij}$. It is easy to see that $\alpha_1, \dots, \alpha_{d-1}, \delta_1, \dots, \delta_n, w_1, \dots, w_n$ forms a basis of $\mathfrak{h}^*$. Let $(\,,)$ be the standard symmetric bilinear form on $\dot{\mathfrak{h}}^*$ (i.e., $(\alpha_i, \alpha_i) = 2$). We now extend to $\mathfrak{h}^*$ as non-degenerate symmetric bilinear form as $(\dot{\mathfrak{h}}^*, \delta_i) = 0$, $(\dot{\mathfrak{h}}^*, w_i) = 0$, $(\delta_i, \delta_j) = 0$, $(w_i, w_j) = 0$ and $(w_i, \delta_j) = \delta_{ij}$. Similarly we can define a non-degenerate symmetric bilinear form on $\mathfrak{h}$ by choosing the standard form on $\dot{\mathfrak{h}}$ and extending as $(\dot{\mathfrak{h}}, C_i) = (\dot{\mathfrak{h}}, d_i) = 0$, $(C_i, C_j) = 0$, $(d_i, d_j) = 0$ and $(C_i, d_j) = \delta_{ij}$.

For m in $\mathbb{Z}^n$ define $\delta_m = \sum m_i \delta_i$ and note that $(\delta_m, \delta_n) = 0$. We call δ_m's null roots. Let $\dot{\Delta}$ be the root system for $\mathfrak{sl}_d(\mathbb{C})$. Let $\gamma = \alpha + \delta_m$ for $\alpha \in \dot{\Delta}$. Then γ is called real root. Define $\gamma^\vee = \alpha^\vee + \sum m_i C_i$ and note that $\gamma(\gamma^\vee) = 2$ and $(\gamma, \gamma) = 2$. Let Δ^{re} be the set of real roots and let $\Delta = \{\alpha + \delta_m, \delta_m \mid \alpha \in \Delta^{re}, \delta_m \text{ is a null root}\}$. Let $\dot{\mathcal{G}} = \mathfrak{sl}_d(\mathbb{C})$ and let $\dot{\mathcal{G}} = \bigoplus_{\alpha \in \dot{\Delta} \cup \{0\}} \dot{\mathcal{G}}_\alpha$ be the root space decomposition.

Define $\widetilde{\tau}_{\alpha+\delta_m} = \dot{\mathcal{G}}_\alpha \otimes \mathbb{C}t^m$, and

$$\widetilde{\tau}_{\delta_m} = \begin{cases} \dot{\mathfrak{h}} \otimes \mathbb{C}t^m \oplus I \otimes \mathbb{C}t^m & \text{if } m \in \mathbb{Z}^n \setminus \operatorname{rad} f, \\ \dot{\mathfrak{h}} \otimes \mathbb{C}t^m \oplus HC_1(\mathbb{C}_q)_m & \text{if } m \in \operatorname{rad} f \setminus \{0\}. \end{cases}$$

Then $\widehat{\tau} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \widetilde{\tau}_\alpha$ is a root space decomposition of $\widehat{\tau}$.

For a real root γ we define the reflection r_γ on $\mathfrak{h}^*$ by

$$r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma, \quad \lambda \in \mathfrak{h}^*.$$

The group generated by r_γ, $\gamma \in \Delta^{re}$ is called Weyl group and is denoted by $\mathcal{W}$. It is easy to verify that the form on $\mathfrak{h}^*$ is $\mathcal{W}$-invariant.

3.1. Definition. A module V over $\widehat{\tau}(d,q)$ is called *integrable* if

(1) $V = \bigoplus V_\lambda$ where $V_\lambda = \{v \in V \mid hv = \lambda(h)v, \ \forall h \in \mathfrak{h}\}$,
(2) for all $\alpha \in \dot{\Delta}$, $m \in \mathbb{Z}^n$ and v in V there exists an integer $k = k(\alpha, m, v)$ such that $(\dot{\mathcal{G}}_\alpha \otimes t^m)^k v = 0$.

For a weight module V let $P(V)$ denote the set of all weights of V. Then the following is standard.

3.2. Lemma. *Let V be an irreducible integrable module for $\widehat{\tau}(d,q)$. Then*:

(1) *$P(V)$ is $\mathcal{W}$-invariant*;
(2) $\dim V_\lambda = \dim V_{w\lambda}$ *for all $w \in \mathcal{W}$ and $\lambda \in P(V)$*;
(3) $\lambda(\alpha^\vee) \in \mathbb{Z}$ *for all $\alpha \in \Delta^{re}$, $\lambda \in P(V)$*;
(4) *for $\alpha \in \Delta^{re}$, $\lambda \in P(V)$ with $\lambda(\alpha^\vee) > 0$, then $\lambda - \alpha \in P(V)$*;
(5) *$\lambda(C_i)$ is an integer for all $\lambda \in P(V)$.*

Proof. Let $\alpha \in \dot{\Delta}^+$ and then let $X_\alpha = E_{ij} \in \dot{\mathcal{G}}$, $i < j$. Let $Y_\alpha = E_{ji}$ so that $\alpha^\vee = E_{ii} - E_{jj}$. It is standard fact that $[\alpha^\vee, X] = 2X$, $[\alpha^\vee, Y] = -2Y$ and $[X, Y] = \alpha^\vee$. So that X, Y, $\alpha^\vee$ is an $\mathfrak{sl}_2$-copy.

Let $\gamma = \alpha + \delta_m$ and $X_\gamma = \sigma(m, -m)^{-1} X_\alpha \otimes t^m$, $Y_\gamma = Y_\alpha \otimes t^{-m}$, $\gamma^\vee = \alpha^\vee + \sum m_i C_i$. Then it is easy to check that X_γ, Y_γ, $\gamma^\vee$ is an $\mathfrak{sl}_2$-copy.

Now from standard $\mathfrak{sl}_2$ integrable module theory (1), (2), (3) and (4) follows.

From 2.3 it follows that C_i acts as scalar. From (2) we know that $\lambda(\alpha^\vee + C_i)$ is an integer and hence $\lambda(C_i)$ is an integer. □

Let

$$\Delta_1^+ = \{\alpha + \delta_m, \delta_{m'} \mid m_n > 0,\ m'_n > 0,\ \text{or}\ m_n = 0\ \text{and}\ \alpha > 0\},$$
$$\Delta_1^- = \{\alpha + \delta_m, \delta_{m'} \mid m_n < 0,\ m'_n < 0,\ \text{or}\ m_n = 0\ \text{and}\ \alpha < 0\},$$
$$\Delta_1^0 = \{\delta_m \mid m \in \mathbb{Z}^n,\ m \neq 0,\ m_n = 0\}.$$

Clearly $\Delta = \Delta_1^+ \cup \Delta_1^0 \cup \Delta_1^-$.

3.3. Theorem. *Let V be an irreducible integrable module for $\widehat{\tau}(d,q)$. Suppose $C_1, \dots, C_{n-1}$ acts trivially and C_n acts as a positive integer. Then there exists a $\lambda \in P(V)$ such that $\lambda + \gamma \notin P(V)$ for all $\gamma \in \Delta_1^+$.*

Proof. The proof follows from the proof of Proposition 2.4 of [E4]. Just note that the Weyl group of A type in the commuting torus case is the same as the one defined in this section. Further the arguments for the proof of Proposition 2.4 of [E4] use the properties of weight system of integrable modules and they are all available in the present situation. □

3.4. Note that on an irreducible integrable module we can assume that $C_n \neq 0$ and $C_i = 0$ for $1 \leqslant i \leqslant n-1$ after suitable change of co-ordinates by Proposition 2.4.

Suppose all q_{ij}'s are roots of unity and then Theorem 2.7 is available. Thus in this case after suitable change of co-ordinates we have C_i acts trivially for $1 \leqslant i \leqslant n-1$, C_n acts non-trivially (assuming some zero degrees center acts non-trivial). By symmetry we can as well assume $C_n > 0$ and hence we have Theorem 3.3.

3.5. Theorem. *Let V be an irreducible integrable module for $\widehat{\tau}(d,q)$ with finite dimensional weight spaces. Assume that some zero degree central operator C_i acts on V non-trivially. Then after a suitable change of co-ordinates there exists a weight $\lambda \in P(V)$ such that $\lambda + \gamma \notin P(V)$ for all $\gamma \in \Delta_1^+$ (or $\lambda - \gamma \notin P(V)$ for all $\gamma \in \Delta_1^+$).*

Proof. Follows from 3.4 and Theorem 3.3. □

4. Constructing finite dimensional τ-modules

This section is devoted to the study of finite dimensional representations of $\tau(d,q)$ assuming that $\mathbb{C}_q$ is rational. In this case $\tau(d,q)$ has finite dimensional semisimple quotient Lie algebras and hence a large class of finite dimensional modules. We will classify finite dimensional irreducible modules for $\tau(d,q)$. This classification will be used in the final classification of irreducible integrable modules for $\widehat{\tau}(d,q)$, in the sense that the classification of irreducible integrable modules of $\widehat{\tau}(d,q)$ will be reduced to the classification of finite dimensional irreducible modules for $\tau(d,q)$ with $n-1$ variables.

For convenience we denote $\Gamma = \operatorname{rad} f$. Let N_{ij} be the least positive integer such that $q_{ij}^{N_{ij}} = 1$. From [N, Theorem III.4], after changing co-ordinates as in Theorem 1.14 if necessary we may assume that

$$N_{2s-1,2s} = N_{2s,2s-1}, \quad N_{2s-3,2s-2} = N_{2s-2,2s-3}, \quad \dots, \quad N_{1,2} = N_{2,1},$$

where each integer is a factor of the next, for some s and all other $N_{ij} = 1$. Combining this with the result in [Z1] we have the following:

4.1. Lemma.

(1) *There is a natural associative algebra isomorphism*

$$\pi : \mathbb{C}_q/J \simeq \bigotimes_{i=1}^{s} M_{N_{2i-1,2i}}(\mathbb{C}) = M_N(\mathbb{C})$$

where $J = \langle t_1^{N_{1,2}} - 1, t_2^{N_{1,2}} - 1, \ldots, t_{2s-1}^{N_{2s-1,2s}} - 1, t_{2s}^{N_{2s-1,2s}} - 1, t_{2s+1} - 1, \ldots, t_n - 1\rangle$ *and* $N = N_{1,2}N_{3,4}\cdots N_{2s-1,2s}$.

(2) $\Gamma = N_{1,2}(\mathbb{Z}\oplus\mathbb{Z}) \oplus N_{3,4}(\mathbb{Z}\oplus\mathbb{Z}) \oplus \cdots \oplus N_{2s-1,2s}(\mathbb{Z}\oplus\mathbb{Z}) \oplus \mathbb{Z}^{n-2s}$. *We simply write it as* $\Gamma = N_1\mathbb{Z} \oplus N_2\mathbb{Z} \oplus \cdots \oplus N_n\mathbb{Z}$.

Clearly, N_i is the minimum positive integer such that $t_i^{N_i} \in Z(\mathbb{C}_q)$, and $\pi(t^m) = I_N$, the identity matrix, for all $m \in \Gamma$.

4.2. For each i: $1 \leqslant i \leqslant n$, let M_i be a positive integer. Let $\underline{a}_i = (a_{i1}, \ldots, a_{iM_i})$ be non-zero complex numbers. Let $K = M_1M_2\cdots M_n$. For $T = (i_1, \ldots, i_n)$ where $1 \leqslant i_j \leqslant M_j$, and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, define $a_T^m = a_{1i_1}^{m_1}\cdots a_{ni_n}^{m_n}$. Let φ be a homomorphism defined by

4.3. $\varphi : \mathbb{C}_q \to \bigoplus_{K\text{-copies}} M_N(\mathbb{C}) = \mathcal{G}(K)$, $t^m \mapsto \bigoplus_{T\in\mathcal{I}} a_T^m \pi(t^m)$ where $\mathcal{I} = \{(i_1, \ldots, i_n) \mid 1 \leqslant i_j \leqslant M_j\}$ and π is given in Lemma 4.1. Let $T_1, \ldots, T_K$ be all elements of $\mathcal{I}$.

Consider $P_j(t_j) = \prod_{k=1}^{M_j}(t_j^{N_j} - a_{jk}^{N_j})$. It is not too difficult to see that $\ker\varphi$ contains $P_j(t_j)$, $1 \leqslant j \leqslant n$. Let J be the ideal generated by $P_j(t_j)$ inside $\mathbb{C}_q$ clearly $\ker\varphi \supseteq J$.

4.4. Lemma. *Assume* $(a_{i1}^{N_i}, \ldots, a_{iM_i}^{N_i})$ *are distinct complex numbers for each i. Then*

(1) φ *is surjective*;
(2) $\ker\varphi = J$.

Proof. Consider the $K \times K$ matrix

$$X = \left(a_{1i_1}^{m_1N_1}\cdots a_{ni_n}^{m_nN_n}\right)_{0\leqslant m_i\leqslant M_i-1,\, 1\leqslant i_j\leqslant M_j}.$$

The index m determines rows and the index $(i_1, i_2, \ldots, i_n)$ determines the columns. It follows from Lemma 3.11 of [E4] that X is invertible. Now consider the map in 4.3 for all $m \in \operatorname{rad} f$ which will take the following form as $\pi(t^m) = I$:

$$t^m \mapsto \bigoplus_{T\in\mathcal{I}} a_T^m I.$$

Since the matrix X is invertible it follows that $(I, 0, \ldots, 0), \ldots, (0, \ldots, I)$ belongs to $\mathcal{G}(K)$ (recall that $\operatorname{Image}\pi = M_N(\mathbb{C})$). Now we will modify the map in the following way. Fix an index $T = (i_1, \ldots, i_n)$, $1 \leqslant i_j \leqslant M_j$ and consider the map

$$t^m \overset{\pi_T}{\mapsto} a_T^m \pi\left(t^m\right).$$

As in the case of π it is easy to see that $\ker\pi_T$ is generated by $t_j^{N_i} - a_{ij}^{N_j}$, $1 \leqslant j \leqslant n$. Further $\operatorname{Image}\pi_T = M_N(\mathbb{C})$. Now for $Y \in M_N(\mathbb{C})$ let $P(t) \in \mathbb{C}_q$ such that $\pi_{T_1}(P(t)) = Y$. Let $X \in \mathbb{C}_q$ such that $\varphi(X) = (I, 0, \dots, 0)$. Now it is easy to verify that

$$(Y, 0, \dots, 0) = \varphi(X)\varphi\big(P(t)\big) = \varphi\big(XP(t)\big).$$

This argument holds for any index set T. Hence it proves that φ is surjective. First note that $J \subseteq \ker\varphi$. Further $\{t^m,\ 0 \leqslant m_i < M_i N_i\}$ is a spanning set for $\mathbb{C}_q/J$ and further the cardinality of the above set is equal to the dimension of $\mathrm{Image}\,\varphi = \mathcal{G}(K)$. It is easy to conclude that $J = \ker\varphi$ since $KN^2 = \dim\mathbb{C}_q/\ker\varphi \leqslant \dim\mathbb{C}_q/J \leqslant KN^2$. □

4.5. Consider the map

$$\widetilde{\varphi} : M_d(\mathbb{C}) \otimes \mathbb{C}_q \mapsto \bigoplus_{K\text{-copies}} M_d(\mathbb{C}) \otimes M_N(\mathbb{C}) \cong \bigoplus_{K\text{-copies}} M_{dN}(\mathbb{C}),$$
$$\widetilde{\varphi}\big(X \otimes t^m\big) = X \otimes \varphi\big(t^m\big).$$

Clearly $\widetilde{\varphi}$ is surjective and $\ker\widetilde{\varphi}$ is the ideal generated by $M_d(\mathbb{C}) \otimes P_j(t_j)$, $1 \leqslant j \leqslant n$. Our interest is to classify irreducible finite dimensional modules for τ. We will first note that

$$M_d(\mathbb{C}) \otimes \mathbb{C}_q = \tau \oplus \big(I \oplus Z(\mathbb{C}_q)\big), \tag{4.6}$$

$$\widetilde{\varphi}(\tau) = \bigoplus_{K\text{-copies}} M_{dN}(\mathbb{C}), \tag{4.7}$$

$$\widetilde{\varphi}\big(I \otimes Z(\mathbb{C}_q)\big) = \bigoplus_{(K\text{-copies})} \mathbb{C}I. \tag{4.8}$$

In order to obtain modules we generally construct modules for $M_d(\mathbb{C}) \otimes \mathbb{C}_q$ and restrict to $\tau(d, q)$. Conversely any τ-module can be extended to $M_d(\mathbb{C}) \otimes \mathbb{C}_q$ by letting $I \otimes Z(\mathbb{C}_q)$ act as scalars.

4.9. Any finite dimensional irreducible module for $\bigoplus_{K\text{-copies}} M_{dN}(\mathbb{C})$ is a module for $\tau(d, q)$ by the map $\widetilde{\varphi}$. It remains irreducible for $\tau(d, q)$ as the additional space $I \otimes Z(\mathbb{C}_q)$ acts by scalar. In the next section we will prove the converse in the sense that any finite dimensional irreducible module for $\tau(d, q)$ comes from above.

5. Highest weight integrable modules over $\widetilde{\tau}^n(d, q)$

Let A be a unital associative algebra, not necessarily commutative. Let $M_d(A)$ be the matrix algebra with entries in A. Let $\mathfrak{sl}_d(A) = \{X \in M_d(A) \mid \mathrm{Tr}\, X \in [A, A]\}$. We will first construct ideals for $\mathfrak{sl}_d(A)$ from ideals in A. Then we specialize to the case $A = \mathbb{C}_q$. As earlier we assume $d \geqslant 2$.

5.1. Lemma. *Let J be an ideal of A. Then $\widetilde{J} = \sum_{j \neq i} JE_{ij} \oplus \sum_{j \neq i} [AE_{ij}, JE_{ji}]$ is an ideal in $\mathfrak{sl}_d(A)$.*

Proof. One can directly show that $\widetilde{J}$ is an ideal of $M_d(A)$. Consequently it is an ideal of $\mathfrak{sl}_d(A)$. We omit the details. □

5.2. Lemma. *Let I be the identity matrix of $M_d(A)$. Let J be an ideal of A and let $\widetilde{J}$ be as before. Then $\sum_{i \neq j} [AE_{ij}, JE_{ji}] = \dot{\mathfrak{h}} \otimes J + I \otimes [A, J]$.*

Proof. Recall that $\dot{\mathfrak{h}}$ is the Cartan subalgebra of $\mathfrak{sl}_d(\mathbb{C})$ spanned by vector $E_{ii} - E_{jj}$. Let

$$\mathcal{S} = \sum_{i \neq j} [AE_{ij}, IE_{ji}].$$

We first note that $\dot{\mathfrak{h}} \otimes J \subseteq \mathcal{S}$ by considering $[E_{ij}, bE_{ji}] = b(E_{ii} - E_{jj})$ for $b \in J$. Now for $a \in A$, $b \in J$ consider

$$[aE_{ij}, bE_{ji}] = abE_{ii} - baE_{jj} = [a, b]E_{ii} + ba(E_{ii} - E_{jj}).$$

This proves $[a, b]E_{ii} \in \mathcal{S}$. Which means we have proved that $\dot{\mathfrak{h}} \otimes J \oplus I \otimes [A, J] \subseteq \mathcal{S}$. From the above formula we easily see that $\mathcal{S} \subseteq \dot{\mathfrak{h}} \otimes J \oplus I \otimes [A, J]$, which completes the proof. □

We now specialize for $A = \mathbb{C}_q$ and construct ideals for $\widetilde{\tau}_q$. Let J be an ideal of $\mathbb{C}_q$. Define

$$\widetilde{J}_c = \widetilde{J} \oplus \langle \mathbb{C}_q, J \rangle.$$

It is easy to check that $\widetilde{J}_c$ is an ideal in $\widetilde{\tau}(d, q)$.

5.3. Let $N_i \in \mathbb{N}$ be minimal such that $t_i^{N_i} \in Z(\mathbb{C}_q)$. (This is different from the definition in Section 4 since our variables may not be the same as in Section 4.) For $\ell_j \in \mathbb{N}$, $1 \leqslant j \leqslant n-1$, let $Q_j(t_j) = \prod_{k=1}^{\ell_j}(t_j^{N_j} - a_{jk})$, where $a_{jk} \in \mathbb{C}$. Let Q be the ideal of $\mathbb{C}_q$ generated by $Q_j(t_j)$ for $j = 1, \ldots, n-1$. Recall that D_k is the linear span of the derivations $d_{k+1}, \ldots, d_n$ and $\mathfrak{h}_k = \dot{\mathfrak{h}} \oplus \sum_{i=1}^n \mathbb{C}C_i \oplus D_k$.

5.4. Let $\widetilde{\tau}^{\pm} = \bigoplus_{\alpha \in \Delta_1^{\pm}} \widetilde{\tau}_\alpha$, $\widetilde{\tau}^0 = \mathfrak{h}_n \bigoplus_{\alpha \in \Delta_1^0} \widetilde{\tau}_\alpha$, $\widetilde{\tau}^{0,n} = \mathfrak{h}_{n-1} \bigoplus_{\alpha \in \Delta_1^0} \widetilde{\tau}_\alpha$. Then

$$\widetilde{\tau} = \widetilde{\tau}^+ \oplus \widetilde{\tau}^- \oplus \widetilde{\tau}^0.$$

Let $\widetilde{\tau}^n = \widetilde{\tau}^- \oplus \widetilde{\tau}^{0,n} \oplus \widetilde{\tau}^+ = \widetilde{\tau} \oplus \mathbb{C}d_n$. Then $\widehat{\tau} = \widetilde{\tau}^n \oplus \sum_{i=1}^{n-1} \mathbb{C}d_i$.

5.5. A weight module for $\widehat{\tau}(d, q)$ (respectively for $\widetilde{\tau}^n$) is called a *highest weight module* if there exists a weight vector v such that $\widetilde{\tau}^+ v = 0$ and v generates the module. We have already seen that an integrable irreducible module V for $\widehat{\tau}(d, q)$ with finite dimensional weight spaces (with reference to $\mathfrak{h}$) is a highest weight module if part of the center acts non-trivially up to a choice of co-ordinates (Theorem 3.5). Let V be an irreducible integrable module over $\widehat{\tau}(d, q)$ with finite dimensional weight spaces. Now we see that there exists a proper submodule W over $\widetilde{\tau}^n(d, q)$ such that V/W has finite dimensional weight spaces with respect to $\mathfrak{h}_{n-1}$ (Proposition 2.4(4) and Theorem 2.7). Clearly V/W is a highest weight module for the Lie algebra $\widetilde{\tau}^n$.

5.6. Proposition. *Let V be an irreducible integrable module over $\widehat{\tau}(d, q)$ with finite dimensional weight spaces. Then the $\widetilde{\tau}^n$-module V/W has an irreducible quotient module which is still a highest weight module.*

Proof. Recall that there exist non-zero central operators $z_1, \ldots, z_{n-1}$ of degree $(p_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, p_{n-1}, 0)$ as W is defined as $\{v - z_i v \mid v \in V,\ 1 \leqslant i \leqslant n-1\}$.

Let $U(\widetilde{\tau}^n)_0 = \{x \in U(\widetilde{\tau}^n) \mid [\mathfrak{h}_{n-1}, x] = 0\}$. As $U(\widetilde{\tau}^n)_0$-module, $(V/W)_\lambda$ has a proper maximal submodule $(V/W)'_\lambda$ over $U(\widetilde{\tau}^n)_0$ since it is finite dimensional, which follows from finite dimensional Lie algebra theory. Take a maximal weight $\widetilde{\tau}^n$-submodule V'/W of V/W with $(V'/W)_\lambda \cap (V/W)_\lambda \subset (V/W)'_\lambda$. Then V/V' is an irreducible quotient of V/W. □

In general, the module V/W is reducible, which will be seen in Theorem 6.12. We will now classify irreducible integrable highest weight modules for $\widetilde{\tau}^n$ with finite dimensional weight spaces with respect to $\mathfrak{h}_{n-1}$. In the last section we will indicate how to recover the original $\widehat{\tau}(d, q)$-module V from this classification.

5.7. Proposition. *Let V be an irreducible highest weight module for $\widetilde{\tau}^n$ with finite dimensional weight spaces. Then there exists an ideal $\widetilde{Q}_c$ as defined in* 5.3 *such that $\widetilde{Q}_c V = 0$.*

We will first develop several lemmas before proving this proposition.

5.8. Lemma. *Let $m \in \mathbb{Z}^n \setminus \{0\}$. Then $\widetilde{\tau}_{\delta_m}$ is spanned by $A(i,m) = E_{ii}t^m - E_{i+1i+1}t^m$, $1 \leqslant i \leqslant d-1$ and $h(r,s) = \sigma(r,s)E_{11}t^m - \sigma(s,r)E_{dd}t^m + \langle t^r, t^s\rangle$ for all $r,s \in \mathbb{Z}^n$ with $r+s=m$.*

Proof. Let S_m be the linear span of the above vectors which is clearly contained in $\widetilde{\tau}_{\delta_m}$. Suppose $m \notin \operatorname{rad} f$. Then we know $\langle t^r, t^s\rangle = 0$ and further $E_{ii}t^m$ spans $\widetilde{\tau}_{\delta_m}$ for $1 \leqslant i \leqslant d$. Now from (1.5b) there exist $r,s \in \mathbb{Z}^n$ such that $r+s=m$ and $f(r,s) \neq 1$ which implies $\sigma(r,s) \neq \sigma(r,s)$. Thus it is elementary to see that $E_{11}t^m$ in S_m which in turn prove that $E_{ii}t^m$ in S_m for all i. Thus $S_m = \widetilde{\tau}_{\delta_m}$.

Now suppose $m \in \operatorname{rad} f$. From (1.5b) it follows that $\sigma(r,s) = \sigma(s,r)$ for all $r,s \in \mathbb{Z}^n$ such that $r+s=m$. This proves that $\langle t^r, t^s\rangle \in S_m$. Now the lemma follows from the definition of $\widetilde{\tau}_{\delta_m}$ in Section 3. □

5.9. Lemma. *Let $m \in \mathbb{Z}^n$ such that $m_n = 0$. Then*:

(a) *For $p > 1$, $m + pe_n \in \operatorname{rad} f$ iff $f(r+\ell e_n, s+ke_n) = 1$ for all $r,s \in \mathbb{Z}^n$ with $r_n = s_n = 0$ and $r+s=m$ and for all $\ell,k \in \mathbb{Z}$, $\ell > 0$, $k > 0$ and $\ell + k = p$.*
(b) *$m + e_n \in \operatorname{rad} f$ iff $f(r, s+e_n) = 1$ for all $r,s \in \mathbb{Z}^n$ with $r_n = s_n = 0$, $r+s=m$.*
(c) *$m \in \operatorname{rad} f$ iff $f(r-e_n, s+e_n) = 1$ for all $r,s \in \mathbb{Z}^n$ with $r_n = s_n = 0$, $r+s=m$.*

Proof. This is easy to verify by using properties of f listed in (1.3)–(1.4) and $f(a,b)^k = f(ka,b) = f(a,kb)$. □

5.10. Let $\mathcal{S}$ be the linear span of $E_{ii+1}t^m$ for $1 \leqslant i \leqslant d-1$, $m \in \mathbb{Z}^n$ with $m_n = 0$, and $E_{d1}t^m$ for $m \in \mathbb{Z}^n$ with $m_n = 1$. Let B be the linear span of

$$\text{(B1)}: \quad E_{ij}t^m; \quad i<j,\ m_n \geqslant 0 \text{ or } j<i,\ m_n \geqslant 1,$$

$$\text{(B2)}: \quad \sigma(r,s)E_{1,1}t^{r+s} - \sigma(s,r)E_{dd}t^{r+s} + \langle t^r, t^s\rangle; \quad r+s=m,\ m_n > 0,$$

$$\text{(B3)}: \quad E_{ii}t^m - E_{jj}t^m; \quad m_n > 0.$$

Let $\langle \mathcal{S}\rangle$ be the subalgebra generated by $\mathcal{S}$ inside $\widetilde{\tau}(d,q)$.

5.11. Lemma. *We have $B = \langle \mathcal{S}\rangle = \widetilde{\tau}^+$.*

Proof.

Claim 1. *$\langle t^r, t^s\rangle \in B$ for $r+s=m$, $m_n > 0$.*

From (B3) we see that $E_{11}t^{r+s} - E_{dd}t^{r+s} \in B$.

Now by multiplying $\sigma(r,s)$ and subtracting from (B2), we have $(\sigma(r,s) - \sigma(s,r))E_{dd}t^{r+s} + \langle t^r, t^s\rangle \in B$. Suppose $m \in \operatorname{rad} f$ then from (1.5b) we have $\sigma(r,s) = \sigma(s,r)$ and hence $\langle t^r, t^s\rangle \in B$. Suppose $m \notin \operatorname{rad} f$ then $\langle t^r, t^s\rangle = 0$. In any case we have the claim.

Now we can see that $B = \widetilde{\tau}^+$ and in particular B is a subalgebra. Further note that the spanning set of S belongs to B and hence $\langle \mathcal{S}\rangle \subseteq B$.

We need only to show that $\mathcal{S}$ generates $\widetilde{\tau}^+$. Within the Lie algebra τ it is easy to see that S generates τ^+ (the definition for τ^+ is obvious), that is,

$$\langle \mathcal{S}\rangle \bmod \big(\langle \mathbb{C}_q, \mathbb{C}_q\rangle\big) = \widetilde{\tau}^+ \bmod \big(\langle \mathbb{C}_q, \mathbb{C}_q\rangle\big).$$

It will be enough to show that

$$\big\langle t^r, t^s\big\rangle \in \langle S\rangle; \quad \text{for } r,s \in \mathbb{Z}^n,\ r+s=m \in \operatorname{rad} f,\ m_n > 0.$$

This follows by computing

$$\begin{aligned}&\left[E_{12}t^r+c_1, E_{21}t^s+c_2\right]=\sigma(r,s)\left(E_{11}t^{r+s}-E_{22}t^{r+s}\right)+\left\langle t^r,t^s\right\rangle\in\langle\mathcal{S}\rangle,\\&\left[E_{12}+c_1', E_{21}t^{r+s}+c_2'\right]=E_{11}t^{r+s}-E_{22}t^{r+s}\in\langle\mathcal{S}\rangle,\end{aligned}$$

where $E_{12}t^r+c_1, E_{21}t^s+c_2, E_{12}+c_1', E_{21}t^{r+s}+c_2'\in\langle\mathcal{S}\rangle$. This completes the proof. □

Proof of Proposition 5.7. Recall that the Cartan subalgebra for $\widetilde{\tau}^n$ is $\mathfrak{h}_{n-1}=\dot{\mathfrak{h}}\oplus\sum_i\mathbb{C}\langle t_i,t_i^{-1}\rangle\oplus\mathbb{C}d_n$ and V is a weight module for $\widetilde{\tau}^n$ with finite dimensional weight spaces relative to the Cartan subalgebra $\mathfrak{h}_{n-1}$.

Let v be a weight vector of highest weight $\lambda\in\mathfrak{h}_{n-1}^*$. Then any non-zero vector in the weight space V_λ is also a highest weight vector for weight reasons.

Let $r\in\mathbb{Z}^n$ such that $r_n=0$ and $0\leqslant r_i<N_i$ for all i where N_i are defined as in 5.3. We fix a simple root α of $\mathfrak{sl}_d(\mathbb{C})$ and k such that $1\leqslant k\leqslant n-1$. Let E_{ii+1} be the corresponding root vector for the root α. Consider

$$\left\{E_{i+1,i}t^rt_k^{\ell N_k}v,\ \ell\in\mathbb{Z}^+\right\}\subseteq V_{\lambda-\alpha}.$$

Since all weight spaces are finite dimensional there exists a non-zero polynomial $P_{k,\alpha,r}$ in $t_k^{N_k}$ such that

$$\left(E_{i+1,i}t^rP_{k,\alpha,r}\left(t_k^{N_k}\right)\right)v=0.$$

For $m\in\mathbb{Z}^n$ with $m_n=0$ we have

$$\begin{aligned}0&=\left(E_{i,i+1}t^m\right)\left(E_{i+1,i}t^rP_{k,\alpha,r}\left(t_k^{N_k}\right)\right)v=\left(E_{i+1,i}t^rP_{k,\alpha,r}\left(t_k^{N_k}\right)\right)\left(E_{ii+1}t^m\right)v\\&\quad+\left(E_{ii}t^mt^rP_{k,\alpha,r}\left(t_k^{N_k}\right)\right)v-\left(E_{i+1,i+1}t^rt^mP_{k,\alpha,r}\left(t_k^{N_k}\right)\right)v+\left\langle t^m,t^rP_{k,\alpha,r}\left(t_k^{N_k}\right)\right\rangle v.\end{aligned}$$

Since v is a highest weight vector then the first term is zero. We have $\langle t^m,t^rP_{k,\alpha,r}\rangle v=0$ as there is no nth term in t^m. This follows from Proposition 2.4(3), Theorem 2.7 and Lemma 1.13(3). Thus we have proved for all i, $1\leqslant i\leqslant d-1$ and for all $m\in\mathbb{Z}^n$ with $m_n=0$,

$$\left(E_{ii}t^mt^rP_{k,\alpha,r}\left(t_k^{N_k}\right)\right)v-\left(E_{i+1,i+1}t^rt^mP_{k,\alpha,r}\left(t_k^{N_k}\right)\right)v=0. \tag{5.12}$$

As before let $r\in\mathbb{Z}^n$ such that $r_n=0$ and $0\leqslant r_i<N_i$ for all i. Let k be such that $1\leqslant k\leqslant n$. Consider $\{E_{1d}t^rt_k^{\ell N_k}t_n^{-1}v,\ \ell\in\mathbb{Z}^+\}\subseteq V_{\lambda-\alpha_0}$ where α_0 is the additional simple root for the affine $\mathfrak{sl}_d(\mathbb{C})$ so that $\alpha_0=\delta_n-\beta$ and β is the maximal root in $\mathfrak{sl}_d(\mathbb{C})$ which corresponds to the root vector E_{1d}.

Since $V_{\lambda-\alpha_0}$ is finite dimensional there exists a polynomial $P_{k,\alpha_0,r}$ in $t_k^{N_k}$ such that

$$\left(E_{1d}t^rP_{k,\alpha_0,r}\left(t_k^{N_k}\right)t_n^{-1}\right)v=0.$$

For $m\in\mathbb{Z}^n$, $m_n=0$, we have

$$\begin{aligned}0&=\left(E_{d1}t^mt_n\right)\left(E_{1d}t^rP_{k,\alpha_o,r}\left(t_k^{N_k}\right)t_n^{-1}\right)v=\left(E_{1d}t^rt_n^{-1}P_{k,\alpha_0,r}\left(t_k^{N_k}\right)\right)\left(E_{d1}t^mt_n\right)v\\&\quad+\left(E_{dd}t^mt_nt^rP_{k,\alpha_0,r}\left(t_k^{N_k}\right)t_n^{-1}\right)v-\left(\left(E_{11}t^rP_{k,\alpha_0,r}\left(t_k^{N_k}\right)t_n^{-1}\right)t^mt_n\right)v+\left\langle t^mt_n,t^rP_{k,\alpha_0,r}\left(t_k^{N_k}\right)t_n^{-1}\right\rangle v.\end{aligned}$$

The first term is zero as v is a highest weight vector. Thus we proved

$$\big(E_{dd}t^m t_n t^r P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1}\big)v - \big(\big(E_{11}t^r P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1}\big)t^m t_n\big)v$$
$$+ \big\langle t^m t_n, t^r P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1}\big\rangle v = 0. \tag{5.13}$$

Noting that components of $P_{k,\alpha_0,r}(t_k^{N_k})$ are in the center of $\mathbb{C}_q$, so they can be moved to the left or to the right. Let

$$Y_i = E_{i+1i}t^s t^r P_{k,\alpha,r}\big(t_k^{N_k}\big), \quad 1 \leqslant i \leqslant d-1, \qquad Y_d = E_{1d}t^s t^r P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1},$$

where r is chosen as earlier and $s \in \operatorname{rad} f$ and $s_n = 0$.

5.14. Claim 1. *$Y_i v = 0$ for $1 \leqslant i \leqslant d$.*

It is sufficient to prove that $Y_i v$ is a highest weight vector. Indeed the weight of $Y_i v$ is strictly less than λ and hence generates a proper submodule as V is irreducible it has no proper submodules and hence $Y_i v = 0$.

In view of Lemma 5.11 it is sufficient to prove that $Y_i v$ is killed by $E_{p,p+1}t^m$ and $E_{d1}t^m t_n$ for all $m \in \mathbb{Z}^n$ with $m_n = 0$. For $1 \leqslant p, i \leqslant d-1$ consider

$$X = \big(E_{pp+1}t^m\big)Y_i v = Y_i\big(E_{pp+1}t^m\big)v + \big[E_{pp+1}t^m, Y_i\big]v.$$

The first term is zero as $E_{pp+1}t^m$ is a positive root vector. The second term is zero unless $p = i$ in which case we have

$$X = \big(E_{ii}t^m t^s t^r P_{k,\alpha,r}\big(t_k^{N_k}\big)\big)v - \big(E_{i+1i+1}t^s t^r t^m P_{k,\alpha,r}\big(t_k^{N_k}\big)\big)v.$$

Since $s \in \operatorname{rad} f$ we have $\sigma(m,s) = \sigma(s,m)$. From (5.12) we know that

$$\sigma(m,s)^{-1}X = \big(E_{ii}t^{m+s}t^r P_{k,\alpha,r}\big(t_k^{N_k}\big)\big)v - \big(E_{i+1,i+1}t^r t^{m+s} P_{k,\alpha,r}\big(t_k^{N_k}\big)\big)v = 0.$$

For $1 \leqslant i \leqslant d-1$ we have

$$\big(E_{d1}t^m t_n\big)Y_i v = Y_i\big(E_{d1}t^m t_n\big)v + \big[E_{d1}t^m t_n, Y_i\big]v.$$

The first term is zero as $E_{d1}t^m t_n$ is a positive root vector. For the second term the commutator itself is zero. Similarly one can see that $E_{pp+1}t^m Y_d v = 0$ for all $1 \leqslant p \leqslant d-1$. Thus we are left with

$$\big(E_{d1}t^m t_n\big)Y_d v = Y_d\big(E_{d1}t^m t_n\big)v + \big[E_{d1}t^m t_n, Y_d\big]v.$$

The first term is zero as $E_{d1}t^m t_n$ is a positive root vector. Consider

$$Z = \big[E_{d1}t^m t_n, E_{1d}t^s t^r P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1}\big]v = \big(E_{dd}t^m t_n t^s t^r P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1}\big)v$$
$$- E_{11}t^s t^r t_n^{-1} t^m t_n P_{k,\alpha_0,r}\big(t_k^{N_k}\big)v + \big\langle t^m t_n, t^s t^s P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1}\big\rangle v.$$

Now consider

$$\big\langle t^m t_n, t^s t^r P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1}\big\rangle v = -\big\langle t^s, t^r P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1}t^m t_n\big\rangle v - \big\langle t^r P_{k,\alpha_0,r}\big(t_k^{N_k}\big)t_n^{-1}, t^m t_n t^s\big\rangle v.$$

Now the first term is zero as s does not contain nth term. Hence we have

$$\sigma(m,-s)Z = E_{dd}t^{m+s}t_nt^rt_n^{-1}P_{k,\alpha_0,r}\big(t_k^{N_k}\big)v - E_{11}t^rt_n^{-1}t^{m+s}t_nP_{k,\alpha_0,r}\big(t_k^{N_k}\big)v$$
$$+\big\langle t^{m+s}t_n, t^rt_n^{-1}P_{k,\alpha_0,r}\big(t_k^{N_k}\big)\big\rangle v,$$

which is zero by (5.13) where we take $m+s$ instead of m. Claim 1 follows.

For $1 \leqslant k \leqslant n-1$ let

$$Q_k = \prod_{\substack{\alpha \text{ simple} \\ 0\leqslant r_i<N_i \\ r_m=0}} P_{k,\alpha,r}\big(t_k^{N_k}\big).$$

Notice that it is a finite product and hence Q_k is a polynomial in $t_k^{N_k}$.

5.15. Claim 2. *Let $m, a, b \in \mathbb{Z}^n$ with $m_n = 0$, $a_n = b_n = 0$ and $m = a + b$. Then*:

(a) $E_{i+1i}t^mQ_kv = 0$.
(b) $E_{1d}t^mt_n^{-1}Q_kv = 0$.
(c) $\sigma(a+e_n, b-e_n)E_{dd}t^mQ_kv - \sigma(a-e_n, b+e_n)E_{11}t^mQ_kv + \langle t^at_n, t^bt_n^{-1}Q_k\rangle v = 0$.
(d) $(E_{ii}t^mQ_k - E_{jj}t^mQ_k)v = 0$.
(e) *If $m \notin \operatorname{rad} f$, then $E_{ii}t^mQ_kv = 0$.*
(f) *If $m \in \operatorname{rad} f$, then $\langle t^at_n^k, t^bt_n^\ell Q_k\rangle v = 0$ for all $\ell, k \in \mathbb{Z}$.*

For any $m \in \mathbb{Z}^n$ with $m_n = 0$ there exists $s \in \Gamma$ with $s_n = 0$ and $r \in \mathbb{Z}^n$ with $r_n = 0$ such that $0 \leqslant r_i < N_i$ and $m = s + r$. We write $t^mQ_k = \sigma(r,s)^{-1}t^rt^sQ_k$. Now (a) and (b) follow from Claim 1. To see (c), consider the following from (b),

$$0 = \big(E_{d1}t^at_n\big)\big(E_{1d}t^bt_n^{-1}Q_k\big)v = \big(E_{1d}t^bt_n^{-1}Q_k\big)\big(E_{d1}t^at_n\big)v$$
$$+\sigma(a+e_n, b-e_n)E_{dd}t^{a+b}Q_kv - \sigma(b-e_n, a+e_n)E_{11}t^{a+b}Q_kv + \big\langle t^at_n, t^bt_n^{-1}Q_k\big\rangle v.$$

The first term is zero as $E_{d1}t^bt_n$ is a positive root vector. Thus (c) is proved.

To see (d), we consider

$$0 = E_{ii+1}\big(E_{i+1i}t^mQ_k\big)v = \big(E_{i+1i}t^mQ_k\big)E_{ii+1}v + E_{ii}t^mQ_k - E_{i+1,i+1}t^mQ_kv.$$

The first term is zero as E_{ii+1} is a positive root vector. Thus we have (d).

For (e), we have $m \notin \operatorname{rad} f$. Then by Lemma 5.9(c) there exist $a, b, \in \mathbb{Z}^n$ such that $a_n = b_n = 0$, $a + b = m$ and $f(a+e_n, b-e_n) \neq 1$ which means $\sigma(a+e_n, b-e_n) \neq \sigma(b-e_n, a+e_n)$. Thus in (c) combining with (d) we can deduce that $E_{11}t^mQ_kv = 0$. Using (5.12) we have (e).

For (f), from (d) and (c) we have $\langle t^at_n, t^bt_n^{-1}Q_k\rangle v = 0$. By induction on $l > 1$ and using

$$\big\langle t^at_n^\ell, t^bt_n^{-\ell}Q_k\big\rangle v = -\big\langle\big(t^bt_n^{-\ell}Q_k\big)\big(t^at_n^{\ell-1}\big), t_n\big\rangle v - \big\langle t^bt_n^{-\ell}Q_kt_n, t^at_n^{\ell-1}\big\rangle v = 0,$$

we deduce (f) for $\ell + k = 0$.

Let $X = \langle t^at_n^\ell, t^bt_n^kQ_k\rangle$ for $\ell + k < 0$. Then Xv is a weight vector whose weight is strictly less than λ. At the same time Xv is a highest weight vector as X is central. Thus Xv generates a proper submodule but V is irreducible. Hence $Xv = 0$.

If $k + l > 0$ because of the weight reason (f) holds automatically. Thus (f) follows.

5.16. Claim 3. *Let $m \in \mathbb{Z}^n$, $m_n = 0$. Then*:

(1) $E_{qp}t^mQ_kv = 0$ *for $q > p$.*
(2) $E_{qp}t^mt_n^rQ_kv = 0$, $r < 0$, $q \neq p$.

(3) $(E_{pp}t^m t_n^r Q_k - E_{qq}t^m t_n^r Q_k)v = 0$, $r \leqslant 0$.
(4) $m + re_n \notin \operatorname{rad} f$, $(E_{ii}t^m t_n^r Q_k)v = 0$, $r \leqslant 0$.

Statement (1) follows from Claim 5.15(a) by induction on $q - p$. Statement (3) follows from (1), while Statement (4) follows from (2) with $q > p$. Statement (2) with $(q, p) = (1, d)$ follows from Claim 5.15(b) by induction on r.

To prove (2) with $q < p$ one has to use induction on $d - (p - q)$ and r to show that $E_{qp}t^m t_n^r Q_k v$ is a highest weight vector, yielding $E_{qp}t^m t_n^r Q_k v = 0$. To prove (2) with $q > p$ one has to use (1) and induction on r to show that $E_{qp}t^m t_n^r Q_k v$ is a highest weight vector, again yielding $E_{qp}t^m t_n^r Q_k v = 0$. We omit the details.

Let Q be the ideal of $\mathbb{C}_q$ generated by $Q_1(t_1^{N_1}), \ldots, Q_{n-1}(t_n^{N_n})$.

5.17. Claim. $\widetilde{Q}_c v = 0$.

From the fact that v is a highest weight vector and from Claim 3(1), (2) and (3), we have

$$E_{ij}t^m Q\, v = 0, \qquad (E_{ii} - E_{jj})t^m Q\, v = 0, \quad \forall i \neq j,\ m \in \mathbb{Z}^n.$$

What remains is to be proved that $(I \otimes [\mathbb{C}_q, Q])v = 0$.

We know that $[t^s, t^r Q_k] = 0$ if $s + r \in \operatorname{rad} f$. Suppose $s + r \notin \operatorname{rad} f$. Then from Claim 3(4) we know $E_{ii}t^{s+r}Q_k v = 0$. It follows that $I \otimes t^{s+r}Q_k v = 0$.

We will complete the proposition by considering the space

$$W = \{w \in V \mid \widetilde{Q}_c w = 0\}.$$

As $\widetilde{Q}_c$ is an ideal in $\widetilde{\tau}^n$, it is easy to check that W is a submodule of V, since V is irreducible and W has been proved to be non-zero by claim, we show that $W = V$, which completes the proof. □

5.18. Suppose one of the Q_k is a constant. Then the ideal $\langle Q \rangle = \mathbb{C}_q$ and hence by the above proposition the module V is trivial. Thus we can assume that each polynomial Q_k is not a constant. Suppose zero is a root of some polynomial Q_k. Then $t_k^{N_k}$ is a factor of Q_k and by multiplying Q_k by $t_k^{-N_k}$ we see that the ideal Q has not changed. Hence we can assume that each polynomial Q_k does not have zero as a root. We can further assume that the coefficient of the highest term of each Q_k is one.

Let $Q_j = \prod_{k=1}^{a_k}(t_j^{N_k} - a_{jk})^{\ell_{jk}}$ where $a_k > 0$, $\ell_{jk} > 0$, and a_{jk} for $1 \leqslant k \leqslant a_k$ are distinct and non-zero complex numbers.

Let $Q_j' = \prod_{k=1}^{a_k}(t_j^{N_k} - a_{jk})$. Let Q' be the ideal generated by $Q_1', \ldots, Q_{n-1}'$ inside $\mathbb{C}_q$. Clearly $Q \subseteq Q'$.

5.19. Proposition. *Let V be a non-trivial irreducible integrable highest weight module for $\widetilde{\tau}^n$ with finite dimensional weight spaces with respect to $\mathfrak{h}_{n-1}$. Then there exist polynomials Q_j' in $t_j^{N_j}$, $1 \leqslant j \leqslant n-1$ with distinct non-zero roots such that $\widetilde{Q}_c' V = 0$.*

Proof. We have already seen in Proposition 5.7 that there exist polynomials Q_j: $1 \leqslant j \leqslant n-1$ with non-zero roots such that $\widetilde{Q}_c V = 0$.

It is enough to prove Claim 5.15(a) and (b) for the polynomial Q_j' in the place of Q_j, because all the arguments given in the proof of Proposition 5.7 after Claim 2 follows from Claim 5.15(a) and (b). In particular we can conclude that $\widetilde{Q}_c' V = 0$ which proves the current proposition. Now we first to prove Claim 5.15(a) for the polynomial Q_j' in the place of Q_j, i.e.,

Claim 1. *$E_{ij}t^m Q_k' v = 0$, for $1 \leqslant i \neq j \leqslant d$, $1 \leqslant k \leqslant n-1$ and $m \in \mathbb{Z}^n$ with $m_n = 0$.*

We shall show $E_{ii+1}t^m Q_k' v = 0$ as an example. Consider the map

$$\pi : \widetilde{\tau}^n/\widetilde{Q}_c \to \widetilde{\tau}^n/\widetilde{Q}_c' \to 0.$$

The $\ker\pi = \bar{Q}_c'/\bar{Q}_c$ can be seen to be solvable. Indeed when we take commutators in $\bar{Q}_c'$ we see that the multiplicity of roots increase and after certain stage this belongs to $\bar{Q}_c$.

Let α be the root corresponding to E_{ii+1}. Let $\mathcal{G}_\alpha^+$ be the linear span of $E_{i,i+1}$ and $E_{i,i+1}t^m Q_k'$ for $m \in \mathbb{Z}^n$ with $m_n = 0$. Let $\mathcal{G}_\alpha^-$ be the linear span of $E_{i+1,i}t^{-m} Q_k'$ for $m \in \mathbb{Z}^n$ with $m_n = 0$. Let $\mathcal{G}_\alpha^0$ be the linear span of the following elements

$$(E_{i,i} - E_{i+1,i+1})t^m Q_k, \quad \forall m \in \Gamma \text{ with } m_n = 0;$$

$$E_{i,i}t^m Q_k, E_{i+1,i+1}t^m Q_k, \quad \forall m \in \mathbb{Z}^n \setminus \Gamma \text{ with } m_n = 0;$$

$$\langle t^m t_j^{-1}, t_j \rangle, \quad \forall m \in \Gamma \text{ with } m_n = 0,\ \forall 1 \leqslant j \leqslant n-1.$$

Set $\widetilde{\mathcal{G}}_\alpha^0 = \mathbb{C}(E_{i,i} - E_{i+1,i+1}) + \mathcal{G}_\alpha^0$, $\mathcal{G}_\alpha = \mathcal{G}_\alpha^- \oplus \mathcal{G}_\alpha^0 \oplus \mathcal{G}_\alpha^+$, $\widetilde{\mathcal{G}}_\alpha = \mathbb{C}(E_{i,i} - E_{i+1,i+1}) + \mathcal{G}_\alpha$, which can easily be seen to be nilpotent Lie algebras. We can also see that $\mathcal{G}_\alpha \subset [\widetilde{\mathcal{G}}_\alpha, \widetilde{\mathcal{G}}_\alpha]$.

We denote $U(\mathcal{G})$ the universal enveloping algebra of $\mathcal{G}$ for any Lie algebra $\mathcal{G}$. Let W be $\mathcal{G}_\alpha$ module generated by v. That is $W = U(\widetilde{\mathcal{G}}_\alpha)v$. We now prove that W is finite dimensional. First note that $U(\mathcal{G}_\alpha^+)v = \mathbb{C}v$ and $V_\lambda \supseteq U(\widetilde{\mathcal{G}}_\alpha^0)v$. Thus $W = U(\mathcal{G}_\alpha)v \subseteq (\mathcal{G}_\alpha^-)V_\lambda$. But each vector in $\mathcal{G}_\alpha^-$ acts locally nilpotent on V_λ as V is integrable and thus we conclude $U(\mathcal{G}_\alpha^-)V_\lambda$ is finite dimensional and hence W is finite dimensional (see Lemma 3.2(2)).

Denote the image of $\mathcal{G}_\alpha$ in $\widetilde{\tau}^n/\widetilde{Q}_c$ by $\bar{\mathcal{G}}_\alpha$ and we noticed earlier that $\bar{\mathcal{G}}_\alpha$ is nilpotent and W is a finite dimensional module for $\bar{\mathcal{G}}_\alpha$. Hence by Lie's theorem there exists a non-zero weight vector w in W such that $\bar{\mathcal{G}}_\alpha w = 0$. Consequently $\mathcal{G}_\alpha w = 0$.

From the definition of w we have a $y \in U(\mathcal{G}_\alpha)$ with $yv = w$. Since V is irreducible there exists $X \in U(\widetilde{\tau}^n)$ such that $Xw = v$. Write $X = X_- H X_+$ where $X_\pm \in U(\widetilde{\tau}^\pm)$ and $H \in U(\widetilde{\tau}^{0,n})$. First note that the weights of V are of the form $\lambda - \sum_{i=1}^{d-1} n_i\alpha_i$ where $\alpha_0, \alpha_1, \ldots, \alpha_{d-1}$ are simple roots of the affine system. Now from the definition, $W \subseteq U(\mathcal{G}_\alpha^-)V_\lambda$ and hence the weight of w is of the form $\lambda - s\alpha$, $s \geqslant 0$. Notice that $E_{ii+1} \otimes \mathbb{C}_q[t^{\pm 1}, \ldots, t_{n-1}^{\pm}]$ is a Lie algebra and $X_+ \in U(E_{ii+1} \otimes \mathbb{C}_q[t^{\pm 1}, \ldots, t_{n-1}^{\pm 1}])$. Otherwise $X_+ w = 0$ for weight reasons. Further X_- has to be scalar for weight reasons. From $Xw = v$, we may assume that $X \in U(E_{i,i+1} \otimes \mathbb{C}_q)$.

For any $(E_{i,i+1}t^{s_1}) \cdots (E_{i,i+1}t^{s_r}) \in U(E_{i,i+1} \otimes \mathbb{C}_q)$ where $s_1, \ldots, s_r \in \mathbb{Z}^n$ without nth components by induction on r we have $\mathcal{G}_\alpha(E_{i,i+1}t^{s_1}) \cdots (E_{i,i+1}t^{s_r})w = 0$ as $[\mathcal{G}_\alpha, E_{i,i+1}t^{s_i}] \subset \mathcal{G}_\alpha$. Thus we have proven that $\mathcal{G}_\alpha v = \mathcal{G}_\alpha XW = 0$. So Claim 1 holds.

In the same manner we can prove Claim 5.15(b) for the polynomial Q_j' in the place of Q_j, i.e.,

Claim 2. *$E_{ij}t^m Q_k' v = 0$, for $1 \leqslant i \neq j \leqslant d$, $1 \leqslant k \leqslant n-1$ and $m \in \mathbb{Z}^n$ with $m_n \neq 0$.*

This completes the proof of Proposition 5.19. □

6. Classification of irreducible integrable modules

In this section we will first prove that any irreducible finite dimensional module over $\tau(d,q)$ comes from the construction given in Section 4. We need to determine $\ker\widetilde{\varphi} \cap \tau(d,q)$ with the notation established in Section 4. We have already noted that $\ker\widetilde{\varphi} = M_d(\mathbb{C}_q \cap J)$ where J is the ideal generated by $P_j(t_j^{N_j})$.

6.1. Lemma. $\ker\widetilde{\varphi} \cap \tau(d,q) = \widetilde{J}$.

Proof. Note that for $i \neq j$, $E_{ij} \otimes J \subseteq \ker\widetilde{\varphi}$. Thus $\widetilde{J} \subseteq \ker\widetilde{\varphi}$ as $\widetilde{J}$ is the ideal generated by $\bigoplus_{i\neq j} E_{ij} \otimes J$. It remains to be proved that

6.2. $(H \otimes J) \cap \tau(d,q) \subseteq \widetilde{J}$, where H is the diagonal matrices.

From weight reasons and from the definition of $\tau(d,q)$ we have

$$(H \otimes J) \cap \tau(d,q) = (H \otimes J) \cap \left((\dot{\mathfrak{h}} \otimes J) \oplus \left(I \otimes [\mathbb{C}_q, \mathbb{C}_q]\right)\right).$$

But $\dot{\mathfrak{h}} \otimes J \subseteq \widetilde{J}$ and hence to see 6.2 we need to prove that

$$(H \otimes J) \cap \left(I \otimes [\mathbb{C}_q, \mathbb{C}_q]\right) \subseteq \widetilde{J}.$$

But by Lemma 5.2, $\widetilde{J}$ contain $I \otimes [\mathbb{C}_q, J]$. Thus to prove 6.2 it is sufficient to prove

$$J \cap [\mathbb{C}_q, \mathbb{C}_q] = [\mathbb{C}_q, J].$$

Clearly $[\mathbb{C}_q, J] \subseteq J \cap [\mathbb{C}_q, \mathbb{C}_q]$. Let $t^m p_j(t_j^{N_j}) \in J \cap [\mathbb{C}_q, \mathbb{C}_q]$. Then $m \notin \operatorname{rad} f = \Gamma$. So $t^m = \lambda[t^r, t^s]$ for some non-zero scaler λ and for some $r, s \in \mathbb{Z}^n$ with $r + s = m$. Now $t^m p_j(t_j^{N_j}) = \lambda[t^r p_j(t_j^{N_j}), t^s] \in [J, \mathbb{C}_q]$. $\square$

6.3. Consider $\tau(d,q)$ with $n \geqslant 2$. Let $\tau^+ = \bigoplus_{i<j, m\in\mathbb{Z}^n} \mathbb{C}E_{ij}t^m$, $\tau^- = \bigoplus_{i<j, m\in\mathbb{Z}^n} \mathbb{C}E_{ji}t^m$,

$$\tau^0 = \left(I \otimes [\mathbb{C}_q, \mathbb{C}_q]\right) \bigoplus_{m\in\mathbb{Z}^n} \mathbb{C}(E_{ii} - E_{jj})t^m.$$

Then clearly $\tau^+ \oplus \tau^0 \oplus \tau^- = \tau(d,q)$.

6.4. Definition. A weight module for $\tau(d,q)$ is called *a highest weight module* if there exists a weight vector v such that $\tau^+ v = 0$ and v generates the module.

6.5. Let V be a finite dimensional irreducible module for τ with $n \geqslant 2$. Now we continue to use the notation in Section 4. Since it is a finite dimensional module (considered over $\mathfrak{sl}_d(\mathbb{C})$), it has to be a weight module and there exists a highest weight vector and hence V is a highest weight module. From a similar argument as in the proof of Proposition 5.19 there exist polynomials $Q_1, \ldots, Q_n$ in $t_1^{N_1}, \ldots, t_n^{N_n}$ with non-zero distinct roots such that $\widetilde{Q}V = 0$, where Q is the ideal of $\mathbb{C}_q$ generated by $Q_1, \ldots, Q_n$ in $t_1^{N_1}, \ldots, t_n^{N_n}$. Thus V is a module for

$$\tau/\widetilde{Q} = \bigoplus_{(K\text{-copies})} \mathfrak{sl}_{dN}(\mathbb{C}),$$

where $N = N_1 N_2 \cdots N_n$, $K = M_1 M_2 \cdots M_n$ and each M_i is the number of distinct roots of Q_i. See Section 4 for further details. Thus we have proven

6.6. Theorem. *Any finite dimensional irreducible module for $\tau(d,q)$ has to come from the construction in Section* 4.

Now we discuss the problem of classifying irreducible integrable highest weight modules V for $\widetilde{\tau}^n$ with finite dimensional weight spaces. We have already seen in Proposition 5.19 that there exist polynomials $Q_1, \ldots, Q_{n-1}$ with distinct non-zero roots for each Q_i such that $\widetilde{Q}_c V = 0$.

Let $\mathbb{C}_{n-1}$ be the subalgebra of $\mathbb{C}_q$ generated by the first $n-1$ variables $t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}$; $\tau_{n-1} = \mathfrak{sl}_d(\mathbb{C}_{n-1})$;

$$\tau_n^+ = \sum_{\substack{\alpha\in\dot{\Delta}\cup\{0\}\\ m\in\mathbb{Z}^n,\, m_n>0}} \widetilde{\tau}_{\delta_m+\alpha};$$

$$\tau_n^- = \sum_{\substack{\alpha\in\dot{\Delta}\cup\{0\}\\ m\in\mathbb{Z}^n,\, m_n<0}} \widetilde{\tau}_{\delta_m+\alpha};$$

$$\widetilde{\tau}_n^0 = \tau_{n-1} \oplus \sum_{\substack{m\in\mathbb{Z}^n,\, m_n=0\\ 1\leqslant i\leqslant n}} \mathbb{C}\langle t^m t_i, t_i^{-1}\rangle \oplus \mathbb{C}d_n.$$

Then $\widetilde{\tau}^n = \tau_n^- \oplus \tau_n^+ \oplus \widetilde{\tau}_n^0$. Let V_λ be the highest weight space of V. Since V is irreducible, V_λ is an irreducible $\widetilde{\tau}^{0,n}$-module (see 5.4 for definitions).

6.7. Lemma. *Let $W = U(\widetilde{\tau}_n^0)V_\lambda$. Then W is an irreducible integrable highest weight module over $\widetilde{\tau}_n^0$ and also over τ_{n-1} with finite dimensional weight spaces.*

Proof. From properties of V we know that W is an integrable highest weight module over $\widetilde{\tau}_n^0$ with finite dimensional weight spaces. We need only show that W is irreducible.

Let $w \in W \setminus \{0\}$ and $v \in V_\lambda \setminus \{0\}$ then there exists $X \in U(\widetilde{\tau})$ such that $Xw = v$. Let $X = X_- X_0 X_+$ for $X_\pm \in U(\widetilde{\tau}^\pm)$ and $X_0 \in U(\widetilde{\tau}_n^0)$. Then $X_+ w = 0$ unless X_+ is a scalar in which case X_- has to be scaler for weight reasons. Thus we can assume $X = X_0$. This proves W is $\widetilde{\tau}_n^0$ irreducible. Now d_n acts by a scaler on the whole module W and also the center acts by scalar. Note that the center that comes from the brackets of τ_{n-1} is zero on the module. Then $\tau_{n-1}|_W$ is a subalgebra. Thus W remains irreducible over τ_{n-1}. □

6.8. Proposition. *The subspace W, as defined in Lemma* 6.7*, is finite dimensional.*

Proof. Note that $\tau_{n-1}/(\widetilde{Q}_c \cap \tau_{n-1})$ is a finite dimensional Lie algebra and W is an integrable module over this Lie algebra. Let $\mathfrak{n}^-$ denote the strictly lower triangular matrices of $\mathfrak{gl}_d$. Then any element in $\mathfrak{n}^- \otimes (\mathbb{C}_q/\langle Q_1, Q_2, \dots, Q_{n-1}\rangle)$ acts nilpotently on any element in V. Since V_λ and $\mathfrak{n}^- \otimes (\mathbb{C}_q/\langle Q_1, Q_2, \dots, Q_{n-1}\rangle)$ are finite dimensional and $W = U(\mathfrak{n}^- \otimes (\mathbb{C}_q/\langle Q_1, Q_2, \dots, Q_{n-1}\rangle))V_\lambda$, using PBW theorem we deduce that W is finite dimensional. □

6.9. We will now indicate how to recover $\widetilde{\tau}^n$-module V from a finite dimensional irreducible $\widetilde{\tau}_n^0$-module W. Let $\widetilde{\tau}_n^+ W = 0$. Consider the induced module

$$M(W) = U(\widetilde{\tau} \oplus \mathbb{C}d_n) \otimes W.$$

Since W is irreducible it follows by the standard arguments that $M(W)$ has a unique irreducible quotient say $\mathcal{V}(W)$. By universal property it follows that $\mathcal{V}(W) \cong V$ as $\widetilde{\tau} \oplus \mathbb{C}d_n$ module as both modules has the same "top".

We have not answered the question that if we start with an arbitrary finite dimensional irreducible module W for τ_{n-1}, whether the module $\mathcal{V}(W)$ is integrable with finite dimensional weight spaces. We believe this is true when we choose appropriate central action. This problem we take up in the next paper.

6.10. Let us streamline the results we have established. We started with an irreducible integrable module V over $\widehat{\tau}(d,q)$ with finite dimensional weight spaces and with non-trivial action of the center C. First we proved in Theorem 3.5 that V has to be a highest weight module after suitable change of co-ordinates. Then we produced a quotient module V/W for the Lie algebra $\widetilde{\tau}^n = \widetilde{\tau}(d,q) \oplus \mathbb{C}d_n$. In Proposition 5.6 we proved that V/W has an irreducible module $\bar{V}$ over $\widetilde{\tau}^n$, which is an integrable highest weight module with the same highest weight and with finite dimensional weight spaces. Then in Proposition 6.8 we proved that $\bar{V}$ has a finite dimensional irreducible submodule $\widetilde{W}$ over τ_{n-1} and all finite dimensional irreducible module over τ_{n-1} have been explicitly constructed in Theorem 6.6. We have indicated how to get the module $\bar{V}$ for $\widetilde{\tau}(d,q) \oplus \mathbb{C}d_n$ from such a finite dimensional module $\widetilde{W}$ in 6.9. So what remains to be done is to recover original module V for $\widehat{\tau}(d,q)$ from an irreducible integrable highest weight module $\bar{V}$ for $\widetilde{\tau}(d,q) \oplus \mathbb{C}d_n$.

6.11. Let V be an irreducible integrable module for $\widehat{\tau}(d,q)$ with finite dimensional weight module and with non-trivial action of the center C. Then from Theorem 3.5 we can assume that V is a highest weight module. From Proposition 2.4 we have non-zero central operators $Z_1, \ldots, Z_{n-1}$ on V with degree $(k_1, 0, \ldots, 0), \ldots, (0, 0, \ldots, k_{n-1}, 0)$. Let $\bar{V}$ be the irreducible quotient for the Lie algebra $\widetilde{\tau}(d,q) \oplus \mathbb{C}d_n$ (Proposition 5.6), and such modules were classified in 6.9. We will now define $\widehat{\tau}(d,q)$-module structure on $\bar{V} \otimes A_{n-1}$ where $A_{n-1} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is a Laurent polynomial ring in $n-1$ commuting variables. We will prove that $\bar{V} \otimes A_{n-1}$ is a completely reducible $\widehat{\tau}(d,q)$-module and all components are isomorphic up to a grade shift. We will further prove that one of the components of $\bar{V} \otimes A_{n-1}$ is isomorphic to V as $\widehat{\tau}(d,q)$-module. This completes the classification problem.

Now we define the $\widehat{\tau}(d,q)$-module structure on $\bar{V} \otimes A_{n-1}$. For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ define $m' = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$. We write $m = (m', m_n)$. Fix $\alpha \in \mathbb{C}^{n-1}$. For any homogeneous element X in $\widehat{\tau}(d,q)$ of degree m, define

$$X \cdot \big(w \otimes t^{r'}\big) = (Xw) \otimes t^{r'+m'}, \qquad d_n\big(w \otimes t^{r'}\big) = (d_n w) \otimes t^{r'},$$
$$d_i\big(w \otimes t^{r'}\big) = (r_i + \alpha_i) w \otimes t^{r'}, \quad \forall r' \in \mathbb{Z}^{n-1},$$

for any $w \in \bar{V}$. It is easy to check that the above action defines a module for $\widetilde{\tau}(d,q)$. Recall that the central operators Z_i act as one on $\bar{V}$. We may choose the maximal submodule of V to contain $v - Z_i v$, $v \in V$, $1 \leqslant i \leqslant n-1$. Thus $Z_i(v \otimes t^{r'}) = v \otimes t^{r'+k_i e_i}$.

6.12. Theorem.

(i) *$\bar{V} \otimes A_{n-1}$ is completely reducible as $\widehat{\tau}(d,q)$-module and all components are isomorphic up to graded shift.*
(ii) *One of the component of $\bar{V} \otimes A_{n-1}$ is isomorphic to V as $\widehat{\tau}(d,q)$-module for a suitable α.*

Proof. The proof of the theorem is identical to the proof of Proposition 3.8 and Theorem 3.9 of [E4]. There we have to assume the quantum tours $\mathbb{C}_q$ to be commutative but the proofs are valid for any $\mathbb{C}_q$. □

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