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The Radical of a Vertex Operator Algebra Associated to a Module

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The radical of a vertex operator algebra associated to a module is defined and computed.

1. INTRODUCTION

The study of the radical $J(V)$ for a vertex operator algebra V was initiated in [DLMM], where we defined the radical $J(V)$ and determined $J(V)$ in the case V is of CFT type (see Section 3 for the definition of a CFT-type vertex operator algebra). Let M be an admissible V-module (see [DLM2] and below for the definition). The M-radical $J_M(V)$ of V consists of vectors $v \in V$ such that $o(v) = 0$ on M where $o(v) = v_{\mathrm{wt}v-1}$ if v is homogeneous and $o(u+v) = o(u) + o(v)$. In the case that $M = V$, $J_V(V)$ is exactly the $J(V)$ which was determined to be $(L(0)+L(-1))V + J(V)_1$ in [DLMM] where $J(V)_1$ is the weight-one subspace of $J(V)$. It turns out that a similar result is true for $J_M(V)$. We show in this paper that $J_M(V) = (L(0)+L(-1))V + J_M(V)_{(0,1)}$ where $J_M(V)_{(0,1)}$ is the intersection of $J_M(V)$ with $V_0 \oplus V_1$. Although the method for determining $J_M(V)$ is similar to that for determining $J_V(V)$ in [DLMM], the argument here is more complicated. The reason is that V has a vacuum $\mathbf{1}$ but M does not, in general. If V is also rational and C_2-finite (see Section 3 for these definitions) and satisfies $L(1)V_1 = 0$, it is proved in [DM2] that $o(u)$ is not zero

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on any admissible module M for nonzero u in V_1. So if V satisfies all these conditions, we, in fact, prove that $J_M(V)$ is exactly $(L(-1)+L(0))V$. We expect that the concept of M-radical $J_M(V)$ of V will play a very important role in the theory of vertex operator algebra.

The second main result in this paper is a criterion for irreducibility of an admissible V-module M (see Proposition 4.3). The result says that M is irreducible if and only if each homogeneous subspace is an irreducible $\widehat{V}(0)$-module or $S_M(V)$-module (see Section 4 for the definition of $\widehat{V}(0)$ and $S_M(V)$). We also formulate this result in terms of the theory of associative algebras $A_n(V)$ developed in [DLM4]. This result is important in the study of dual pairs associated to a vertex operator algebra and an automorphism group (cf. [DLM1]).

Both results are extended to twisted modules. In particular, we also define the radical $J_V(M)$ for an admissible g-twisted V-module for an automorphism g of V of finite order and determine $J_V(M)$ precisely. A similar criterion of irreducibility of M is obtained, too, in terms of a certain Lie algebra $S_M(V^0)$ (see Section 5) and an associative algebra $A_{g,n}(V)$ [DLM5].

2. PRELIMINARY

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (see [B] and [FLM]). We shall use the commuting formal variables z, z_0, z_1, z_2. We shall also use the delta function $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ whose elementary properties can be found in [FLM].

First recall from [FLM, Z, DLM2] the definitions of the weak module, the admissible module, and the ordinary module for a vertex operator algebra V. A *weak module* M for V is a vector space equipped with a linear map

$$\begin{aligned} &V \to (\operatorname{End} M)[[z^{-1}, z]] \\ &v \mapsto Y_M(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} M) \quad \text{for } v \in V \end{aligned}$$

satisfying the conditions for $u, v \in V, w \in M$,

$$v_n w = 0 \quad \text{for} \quad n \in \mathbb{Z} \quad \text{sufficiently large;}$$

$$Y_M(\mathbf{1}, z) = 1;$$

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1) Y_M(v, z_2) \\ &\quad - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v, z_2) Y_M(u, z_1) \\ &\qquad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2). \end{aligned} \tag{2.1}$$

Here and below $(z_i - z_j)^n$ for $n \in \mathbb{C}$ is to be expanded in nonnegative powers of the second variable z_j.

This completes the definition. We denote this weak module by (M, Y_M) (or briefly by M).

An *ordinary V-module* is a weak V-module which carries a $\mathbb{C}$-grading

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda,$$

such that $\dim M_\lambda$ is finite and $M_{\lambda+n} = 0$ for fixed λ and $n \in \mathbb{Z}$ small enough. Moreover, one requires that M_λ is the λ-eigenspace for $L(0)$,

$$L(0)w = \lambda w = (\mathrm{wt} w)w, \qquad w \in M_\lambda$$

where $L(0)$ is the component operator of $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.

An *admissible V-module* is a weak V-module M which carries a $\mathbb{Z}_+$-grading

$$M = \bigoplus_{n\in\mathbb{Z}_+} M(n)$$

($\mathbb{Z}_+$ is the set all nonnegative integers) such that if $r, m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, and $a \in V_r$, then

$$a_m M(n) \subseteq M(r + n - m - 1).$$

Note that any ordinary module is an admissible module.

A vertex operator algebra V is called *rational* if any admissible module is a direct sum of irreducible admissible modules. It was proved in [DLM3] that if V is rational then there are only finitely many inequivalent irreducible admissible modules and each irreducible admissible module is an ordinary module.

The following proposition can be found in [L2] and [DM1].

PROPOSITION 2.1. *Any irreducible weak V-module M is spanned by $\{u_n w | u \in V, n \in \mathbb{Z}\}$, where $w \in W$ is any fixed nonzero vector.*

Let M be a weak V-module. We define the M-radical of V to be

$$J_M(V) = \{v \in V | o(v)|_M = 0\}, \tag{2.2}$$

where $o(v) = v_{\mathrm{wt} v - 1}$ for homogeneous $v \in V$ and $o(u + v) = o(u) + o(v)$. If $M = V$ this is precisely the definition of radical of V given in [DLMM]. ($J_V(V)$ was denoted by $J(V)$ in [DLMM].) If $M = \bigoplus_{n\geq 0} M(n)$ is an admissible module then $o(v)M(n) \subset M(n)$ for all $n \in \mathbb{Z}$.

Recall from [DLMM] that V is of CFT type if V is simple and $V = \bigoplus_{n\geq 0} V_n$ with V_0 one-dimensional. It was proved in [DLMM] that if V is of CFT type then $J_V(V)$ is equal to $(L(0) + L(-1))V + J_V(V)_1$, where $J_V(V)_1 = V_1 \cap J(V)$. Here we prove a similar result for $J_M(V)$ for any admissible module M with the same assumption on V.

3. DETERMINATION OF $J_M(V)$

We need several lemmas.

LEMMA 3.1. *Let V be a simple vertex operator algebra and M a weak V-module. Let $u \in V$ such that the vertex operator $Y_M(u, z)$ on M involves only either finitely many positive powers or finitely many negative powers of z, then $u \in V_0$.*

Proof. The proofs in the two cases are similar. We only deal with the case that $Y_M(u, z)$ involves only finitely many positive powers of z. We first prove that

$$Y_M(u, z_1)Y_M(v, z_2) = Y_M(v, z_2)Y_M(u, z_1);$$

for all $v \in V$. By (7.24) of [DL] (also see [FLM]) there exists a nonnegative integer n such that

$$(z_1 - z_2)^n Y_M(u, z_1)Y_M(v, z_2) = (z_1 - z_2)^n Y_M(v, z_2)Y_M(u, z_1). \tag{3.3}$$

Since each factor in (3.3) involves only finitely many positive powers of z_1 we multiply (3.3) by $(z_1 - z_2)^{-n}$ to obtain $Y_M(u, z_1)Y_M(v, z_2) = Y_M(v, z_2)Y_M(u, z_1)$.

From $[Y_M(\omega, z_1), Y_M(u, z_2)] = 0$ we see that

$$0 = [L(-1), Y_M(u, z)] = Y_M(L(-1)u, z).$$

From the Jacobi identity (2.1) we have the associator formula (see Chap. 8 of [FLM]): for $a, b \in V$ and $w \in M$ there exists a nonnegative integer n, which depends on a and w only, such that

$$(z_0 + z_2)^n Y_M(Y(a, z_0)b, z_2)w = (z_0 + z_2)^n Y_M(a, z_0 + z_2)Y_M(b, z_2)w.$$

So if $b = L(-1)u$ then $Y_M(b, z_2) = 0$ on M and

$$(z_0 + z_2)^n Y_M(Y(a, z_0)b, z_2)w = 0$$

or

$$Y_M(Y(a, z_0)b, z_2)w = 0.$$

This shows that $Y_M(a_m b, z) = 0$ on M for any $a \in V$ and $m \in \mathbb{Z}$. Assume that $b \neq 0$. Since V is simple then the span of $a_m b$ for $a \in V$ and $m \in \mathbb{Z}$ is the whole V by Proposition 2.1. As a result we have $Y_M(v, z) = 0$ for every $v \in V$. This is a contradiction as $Y_M(\mathbf{1}, z) = id_M$. Thus $b = L(-1)u = 0$. Since $L(-1)\colon\ V_n \to V_{n-\mathbf{1}}$ is injective if $n \neq 0$ (cf. [L1] and [DLiM] we immediately have that $u \in V_0$, as required. ∎

LEMMA 3.2. *Let V be a vertex operator algebra of CFT type. Let $0 \neq v \in V_n$ with $n \geq 2$ such that $L(1)v = 0$. Then $v \notin J_M(V)$.*

Proof. Assume that $v \in J_M(V)$. Then $o(v) = v_{n-1} = 0$ on M. Using the relation $[L(-1), v_m] = -mv_{m-1}$ we see that $v_k = 0$ for $0 \leq k \leq n-1$. Thus for any $u \in V$ we have

$$0 = [v_i, u_{-i}] = \sum_{t=0}^{i} \binom{i}{t} (v_t u)_{-t}$$

for $i = 0, \ldots, n-1$. This shows that

$$(v_i u)_{-i} = 0$$

for $i = 0, \ldots, n-1$. Using the relation $[L(-1), a_m] = -ma_{m-1}$ repeatedly for a $a \in V$ gives

$$(v_i u)_k = 0$$

for $i = 1, \ldots, n-1$ and $k \leq -i$. Thus $Y_M(v_i u, z)$ involves only finitely many positive powers of z. It follows from Lemma 3.1 that $v_i u \in V_0$ for $i = 1, \ldots, n-1$. If u is homogeneous of weight $s \geq 0$ then the weight $v_{n-1}u$ again is s. Thus if $s > 0$ then $v_{n-1}u = 0$. If $s = 0$ then u is a multiple of $\mathbf{1}$ and again $v_{n-1}u = 0$. Thus $v \in J_V(V)$.

On the other hand, $J_V(V) = J_V(V)_{\mathbf{1}} + (L(-1) + L(0))V$ (Theorem 1 of [DLMM]). It is clear that $v \notin J_V(V)$. This is a contradiction. ∎

LEMMA 3.3. *Let V be a vertex operator algebra of CFT type and M a weak V-module. Let $v \in V_1$ such that $o(v)$ is a constant on M. Then $v \in J_V(V)$. Moreover, if M is irreducible then $o(v)$ is a constant on M if and only if $v \in J_V(V)$.*

Proof. For any $u \in V$ and $n \in \mathbb{Z}$ we have

$$0 = [o(v), Y_M(u, z)] = [v_0, Y_M(u, z)] = Y_M(v_0 u, z).$$

As in the proof of Lemma 3.1 we conclude that $v_0 u = 0$ for all $u \in V$. That is, $v \in J_V(V)$.

If $v \in J_V(V)$ then again we have $[o(v), Y_M(u, z)] = Y_M(v_0 u, z) = 0$. If M is irreducible then M has countable dimension. Let $\mathrm{Hom}_V(M, M)$ denote the set of all V-homomorphisms from M to itself. Then $\mathrm{Hom}_V(M, M)$ is a division ring over $\mathbb{C}$. Let $w \in M$ be any nonzero vector. Then $f \mapsto f(w)$ gives a bijection from $\mathrm{Hom}_V(M, M)$ to $\mathrm{Hom}_V(M, M)w$ which has countable dimension. Thus $\mathrm{Hom}_V(M, M)$ has countable dimension. Since any division ring over $\mathbb{C}$ with countable dimension is $\mathbb{C}$ itself (cf. [DLM3]) we conclude that $\mathrm{Hom}_V(M, M) = \mathbb{C}$. We now see immediately $o(v)$ is a constant on M. ∎

We can now determine the radical $J_M(V)$ precisely.

THEOREM 3.4. *Suppose that V is a vertex operator algebra of CFT type. Then for any admissible V-module M we have*

$$J_M(V) = (L(0) + L(-1))V + J_M(V)_{(0,1)}$$

where $J_M(V)_{(0,1)} = (V_0 + V_1) \cap J_M(V)$. Moreover, if $a = a^0 + a^1 \in J_M(V)_{(0,1)}$ with $a^i \in V_i$ then $a^1 \in J_V(V)$. That is, the image of the projection of $J_M(V)_{(0,1)}$ into V_1 is contained in $J_V(V)$.

Proof. The proof of this theorem is similar to that of Theorem 1 of [DLMM]. The conclusion $(L(0)+L(-1))V + J_M(V)_{(0,1)} \subset J_M(V)$ is clear.

First we recall a result from [DLiM] (Corollary 3.2). As a module for $sl(2, \mathbb{C}) = \langle L(-1), L(0), L(1)\rangle$, V is a direct sum of highest weight modules $X(\mu)$ with highest weights μ $(\mu > 0)$, the trivial module, and the projective cover $P(1)$ of $X(1)$. Thus for any $x \in J_M(V)$ we can write

$$x = \sum_{n=0}^{m} L(-1)^n u^n,$$

where each u^n either is in V_1 or satisfies $L(1)u^n = 0$. We assume that $u^m \neq 0$. We prove by induction on m that x lies in $J_M(V)_{(0,1)} + (L(0) + L(-1))V$.

Suppose first that $m = 0$. Then $x = u^0$. Write $x = \sum_{i\geq 0} x^i$ where $x^i \in V_i$ and $L(1)x^i = 0$ if $i \neq 1$. Since $o(x) = 0$ on M we have

$$0 = [L(1), o(x)] = \sum_{i\geq 0}[L(1), o(x^i)] = \sum_{i\geq 2} 2(i - (i-1)/2 - 1)x^i_i$$

on M where we have used the fact that $[L(1), o(x^i)] = 0$ for $i \leq 1$ (see Lemma 2.5 of [DLMM]). That is,

$$\sum_{i\geq 2}(i-1)x^i_i = 0.$$

Thus

$$\begin{aligned} 0 &= \sum_{i\geq 2}(i-1)[L(1), [L(-1), x^i_i]] \\ &= \sum_{i\geq 2}(i-1)[L(1), -ix^i_{i-1}] = -\sum_{i\geq 2}(i-1)^2 i x^i_i \end{aligned}$$

on M. Continuing in this way we get

$$\sum_{i\geq 2}(i-1)^k i^{k-1} x^i_i = 0$$

for all $k \geq 1$. It follows that each $x^i_i = 0$ for all $i \geq 2$. Using the relation $[L(-1), u_n] = -nu_{n-1}$ shows inductively that $x^i_j = 0$ for $j = 0, \dots, i$.

Thus $x^i \in J_M(V)$. If $x^i \neq 0$ then x^i is not in $J_M(V)$ by Lemma 3.2. This is a contradiction. Thus $x^i = 0$ for all $i \geq 2$.

So $x = x^0 + x^1$. Since x^0 is a multiple of $\mathbf{1}$ we see that $o(x^{\mathbf{1}}) = -o(x^0)$ is a constant on M. By Lemma 3.3 $x^{\mathbf{1}} \in J_V(V)$. This proves the result for $m = 0$.

For $m > 0$ set $a = L(-1)^{m-1}u^m$ and $b = \sum_{n=0}^{m-1} L(-1)^n u^n$. Thus $x = L(-1)a + b$. From $(L(0) + L(-1))a \in J_M(V)$ we have

$$0 = o(x) = o(L(-1)a) + o(b) = -o(L(0)a) + o(b) = o(b - L(0)a).$$

Note that $L(0)a = (m-1)L(-1)^{m-1}u^m + L(-1)^{m-1}L(0)u^m$ so that

$$b - L(0)a = \sum_{n=0}^{m-2} L(-1)^n u^n - L(-1)^{m-1}((m-1)u^m + L(0)u^m - u^{m-1})$$

lies in $J_M(V)$. Since either $L(0)u^m \in V_1$ or $L(1)L(0)u^m = 0$, we conclude by induction that $b - L(0)a$ lies in $J_M(V)_{(0,1)} + (L(0) + L(-1))V$. But then the same is true for $x = b - L(0)a + (L(0) + L(-1))a$. This completes the proof of the theorem. ∎

We now sharpen Theorem 3.4 if V satisfies additional conditions. Recall from [DLM6] and [Z] that V is called C_2-finite if dim $V/C_2(V)$ is finite dimensional where $C_2(V)$ is linearly spanned by $u_{-2}v$ for all $u, v \in V$.

THEOREM 3.5. *Let V be a simple, rational, C_2-finite vertex operator algebra of CFT type such that $L(1)V_1 = 0$. Then for any admissible V-module M we have $J_M(V) = (L(-1) + L(0))V$. In particular, $J(V) = (L(-1) + L(0))V$.*

Proof. It follows from a result in [DM2] which says that $o(u)$ is not zero on any admissible module for $u \in V_1$. ∎

4. A CRITERION FOR IRREDUCIBILITY

In this section we give a criterion for irreducibility of an admissible module for an arbitrary vertex operator algebra V which we do not assume to be simple. We consider the quotient space

$$\widehat{V} = \mathbb{C}[t, t^{-1}] \otimes V/D\mathbb{C}[t, t^{-1}] \otimes V,$$

where $D = \frac{d}{dt} \otimes 1 + 1 \otimes L(-1)$. Denote by $v(n)$ the image of $t^n \otimes v$ in $\widehat{V}$ for $v \in V$ and $n \in \mathbb{Z}$. Then $\widehat{V}$ is $\mathbb{Z}$-graded by defining the degree of $v(n)$ to be wt$v - n - 1$ if v is homogeneous. Denote the homogeneous subspace of degree n by $\widehat{V}(n)$. The space $\widehat{V}$ is, in fact, a $\mathbb{Z}$-graded Lie algebra with bracket

$$[a(m), b(n)] = \sum_{i=0}^{\infty} \binom{m}{i} a_i b(m+n-i)$$

for $a, b \in V$ (see [B, L2 and DLM3]). Note that $\widehat{V}(0)$ is a subalgebra of $\widehat{V}$ and is isomorphic to $V/(L(-1)+L(0))V$ whose Lie bracket is given by

$$[a, b] = \sum_{n=0}^{\mathrm{wt}a-1} \binom{\mathrm{wt}a - 1}{n} a_n b$$

for homogeneous $a, b \in V$.

Let M be an admissible V-module. Then the map from $\widehat{V}$ to End M by sending $v(m)$ to v_m is a Lie algebra homomorphism (cf. [L2] and [DLM3]). In particular, the restriction of this map to $\widehat{V}(0)$ gives a Lie algebra homomorphism from $V/(L(-1)+L(0))V$ to End M. The kernel of this map is exactly the M-radical $J_M(V)$. Set

$$S_M(V) = V/J_M(V).$$

Then $S_M(V)$ is quotient Lie algebra of $V/(L(-1)+L(0))V$ by Theorem 3.4 and acts on M faithfully.

LEMMA 4.1. *Let V be a finite dimensional vertex operator algebra. Then $V = V_0$ is a commutative associative algebra such that $Y(a, z)b = ab$ for $a, b \in V$.*

Proof. Since $L(-1)$ is injective on $\sum_{n>0} V_n$ (see [L1] and [DLiM]), we observe that $\sum_{n>0} V_n = 0$. In particular, $\omega = 0$ and $L(0) = 0$. This shows that $V = V_0$.

It is clear now that $a_n = 0$ for $a \in V_0$ and $n \neq -1$. This implies that $Y(a, z)b = a_{-1}b$. The reader can verify that $ab = a_{-1}b$ defines a commutative associative algebra structure on V_0 (see [B] and [L2]). ∎

LEMMA 4.2. *Let V be a vertex operator algebra and $M = \bigoplus_{n\geq 0} M(n)$ an admissible V-module with $M(0) \neq 0$. Then M is not equal to $\bigoplus_{n=0}^{k} M(n)$ for any $k \geq 0$ unless V is finite dimensional.*

Proof. If $M = \bigoplus_{n=0}^{k} M(n)$ such that $M(k) \neq 0$. Take a nonzero $u \in M(k)$. Then from the definition of the admissible module $L(-1)u \in M(k+1) = 0$. Thus u is a vacuum-like vector and the submodule W of M generated by u is isomorphic to the adjoint module V [L1]. Since $L(-2)u \in M(n+2) = 0$ we see that $\omega = 0$ and $V = V_0$ is finite dimensional. ∎

Now we use Proposition 2.1 and Lemma 4.2 to give a criterion for irreducibility of an admissible module.

PROPOSITION 4.3. *Let V be a vertex operator algebra with $\omega \neq 0$. An admissible V-module $M = \bigoplus_{n\geq 0} M(n)$ with $M(0) \neq 0$ is irreducible if and only if each $M(n)$ is an irreducible $\widehat{V}(0)$-module or each $M(n)$ is an irreducible $S_M(V)$-module.*

Proof. We have already mentioned that M is a module for $\widehat{V}$ under the action $v(n) \mapsto v_n$ where $Y_M(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ are vertex operators on M for $v \in V$. First we assume that M is irreducible. By Proposition 2.1 $M = \widehat{V} \cdot w$ for any nonzero vector w of M. Now take $w \in M(n)$. Then $M(k) = \widehat{V}(k-n) \cdot w$. In particular, $M(n) = \widehat{V}(0) \cdot w$. Thus $M(n)$ is an irreducible $\widehat{V}(0)$-module.

Conversely, suppose each $M(n)$ is an irreducible $\widehat{V}(0)$-module. From the proof of Lemma 1.2.1 of [Z] we see that $L(0)$ acts on each $M(n)$ as a scalar. Let W be any nonzero submodule of M. Then

$$W = \bigoplus_{n\geq 0} W(n),$$

where $W(n) = M(n) \cap W$. From Lemma 4.2 and the injectivity of $L(-1)$ on $M(n)$ for all large n (cf. [L1] and [DLiM]) we see that $W(n) \neq 0$ for all large n. Note that each $W(n)$ is a submodule of $M(n)$ for $\widehat{V}(0)$. So $W(n) = M(n)$ for all large n as $M(n)$ is an irreducible $\widehat{V}(0)$-module. If $W \neq M$ then the quotient M/W is an admissible V-module with only finitely many homogeneous subspaces. This is a contradiction by Lemma 4.2 unless V is finite dimensional. Thus $W = M$ and M is irreducible. ∎

Remark 4.4 In the case $V = V_0$ is a commutative associative algebra, the assertion in Proposition 4.3 is false. For example, if we take $V = \mathbb{C}$, then $M = \sum_{n\geq 0} M(n)$ is a V-module with each $M(n) = V$. Clearly, each $M(n)$ is an irreducible $S_M(V)$-module, but M is not irreducible under V.

The result discussed in Proposition 4.3 can also be formulated in terms of the theory of the associative algebra $A_n(V)$ developed in [DLM4].

Let $O_n(V)$ be the linear span of all $u \circ_n v$ and $L(-1)u + L(0)u$, where for homogeneous $u \in V$ and $v \in V$,

$$u \circ_n v = \operatorname{Res}_z Y(u, z)v \frac{(1+z)^{\mathrm{wt}\,u+n}}{z^{2n+2}}.$$

Define the linear space $A_n(V)$ to be the quotient $V/O_n(V)$. We also define a second product $*_n$ on V for u and v as above,

$$u *_n v = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \operatorname{Res}_z Y(u, z) \frac{(1+z)^{\mathrm{wt}\,u+n}}{z^{n+m+1}} v.$$

Extend linearly to obtain a bilinear product on V which coincides with that of Zhu [Z] if $n = 0$. The following theorem was proved in [DLM4]; in the case $n = 0$ it was proved previously in [Z].

THEOREM 4.5. *Let* $M = \sum_{n\geq 0} M(n)$ *be an admissible* V*-module with* $M(0) \neq 0$. *Then*

(i) *The product* $*_n$ *induces an associative algebra structure on* $A_n(V)$ *with the identity* $\mathbf{1} + O_n(V)$. *Moreover,* $\omega + O_n(V)$ *is a central element of* $A_n(V)$.

(ii) *The identity map on* V *induces an onto algebra homomorphism from* $A_n(V)$ *to* $A_m(V)$ *for* $0 \leq m \leq n$.

(iii) *The map* $u \mapsto o(u)$ *gives a representation of* $A_n(V)$ *on* $M(i)$ *for* $0 \leq i \leq n$.

Moreover, V *is rational if and only if* $A_n(V)$ *are finite-dimensional semisimple algebras for all* n.

Note that both the actions of $A_n(V)$ and $S_M(V)$ on $\sum_{0\leq m\leq n} M(m)$ are given by $v \mapsto o(v)$. Combining Proposition 4.3 and Theorem 4.5 immediately gives

THEOREM 4.6. *Assume that the Virasoro element* ω *of* V *is nonzero. Then an admissible* V*-module* $M = \bigoplus_{n\geq 0} M(n)$ *with* $M(0) \neq 0$ *is irreducible if and only if each* $M(n)$ *is an irreducible* $A_n(V)$*-module.*

5. TWISTED CASE

This section is an analogue of Section 4 for a twisted module M. We will omit a lot of details and refer the reader to the previous sections when it is clear how the corresponding proofs and arguments given before carry out in this case.

First we give definitions of various twisted modules following [FLM] and [DLM3]. Let g be an automorphism of V of order T. Then we have the eigenspace decomposition $V = \sum_{k=0}^{T-1} V^k$ where $V^k = \{v \in V | gv = e^{-2\pi ik/T} v\}$. Then V^0 is a vertex operator subalgebra of V with the same Virasoro vector.

A weak g-twisted V-module M is a vector space equipped with a linear map

$$\begin{aligned} V &\to (\mathrm{End}\, M)\{z\} \\ v &\mapsto Y_M(v,z) = \sum_{n\in\mathbb{Q}} v_n z^{-n-1} \qquad (v_n \in \mathrm{End}\, M) \end{aligned}$$

such that for all $0 \leq r \leq T-1$, $u \in V^r$, $v \in V$, and $w \in M$,

$$Y_M(u,z) = \sum_{n\in \frac{r}{T}+\mathbb{Z}} u_n z^{-n-1}$$

$$u_l w = 0 \qquad \text{for } l \gg 0,$$

$$Y_M(\mathbf{1}, z) = id_M;$$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y_M(v, z_2)Y_M(u, z_1)$$
$$= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-r/T}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_M(Y(u, z_0)v, z_2).$$

It is clear that if $g = 1$ this reduces the definition of the weak module in Section 3.

An *ordinary g-twisted V-module* is a weak g-twisted V-module M with a $\mathbb{C}$-grading induced by the eigenvalues of $L(0)$;

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda,$$

where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$, $\dim M_\lambda$ is finite, and, for fixed λ, $M_{n/T+\lambda} = 0$ for all small enough integers n.

An admissible g-twisted V-module is a weak $\frac{1}{T}\mathbb{Z}$-graded g-twisted V-module M

$$M = \bigoplus_{n=0}^{\infty} M(n/T)$$

such that $M(0) \neq 0$ and that $v_m M(n/T) \subseteq M(n/T + wt v - m - 1)$ for homogeneous $v \in V$. Clearly, an ordinary g-twisted V-module is an admissible g-twisted V-module.

Remark 5.1 From the definition we see that any weak (admissible, ordinary) g-twisted V-module is a weak (admissible, ordinary) V^0-module.

Let M be an admissible g-twisted V-module. For homogeneous $v \in V$ we denote $o(v) = v_{wt v - 1}$ on M and extend it linearly to whole V, as before. Then it is immediate from the definition that $o(v) = 0$ for $v \in V^1 \oplus \cdots \oplus V^{T-1}$. Since M is an admissible V^0-module we consider the M-radical of V^0 given in (2.2). By Theorem 3.4 we have

THEOREM 5.2. *Suppose that V is a vertex operator algebra of CFT type. Then for any admissible g-twisted V-module M we have*

$$J_M(V^0) = (L(0) + L(-1))V^0 + J_M(V^0)_{(0,1)}.$$

Moreover, if $a = a^0 + a^1 \in J_M(V^0)_{(0,1)}$ with $a^i \in V_i^0$ then $a^1 \in J_{V^0}(V^0)$.

Proposition 4.3 still holds in this case.

PROPOSITION 5.3. *Let V be a simple vertex operator algebra with $\omega \neq 0$. An admissible g-twisted V-module $M = \bigoplus_{n \geq 0} M(n/T)$ with $M(0) \neq 0$ is irreducible if and only if each $M(n/T)$ is an irreducible $S_M(V^0)$-module.*

Proof. If M is irreducible then one can show that the analogue of Proposition 2.1 is true. That is, $M = \{u_n w | u \in V, n \in \frac{1}{T}\mathbb{Z}\}$ for any nonzero $w \in M$. Thus for any nonzero $w \in M(n/T)$ we have $\{o(v)w | v \in V^0\} = M(n/T)$. So $M(n/T)$ is an irreducible $S_M(V^0)$-module.

Note that for each $k = 0, \dots, T-1$, $M^k = \bigoplus_{n \geq 0} M(n + k/T)$ is an admissible V^0-module. If all $M(n/T)$ are irreducible $S_M(V^0)$-modules then M^k is an irreducible admissible V^0-module for $k = 0, \dots, T-1$ by Proposition 4.3. Using the associativity of vertex operators on M we show that if $Y_M(v, z)w = 0$ for some nonzero $v \in V$ and $w \in M$ then $Y_M(u, z) = 0$ for all $u \in V$ (cf. Proposition 11.9) of [DL]: here we use the assumption that V is simple). Since $\sum_{n=0}^{\infty} M(n)$ is nonzero ($M(0) \neq 0$ by assumption) we see from the associativity of vertex operators on M that $\{u_n \sum_{n=0}^{\infty} M(n) | u \in V^k, n \in \mathbb{Z}\}$ is a nonzero V^0-submodule of M^k. Thus

$$\left\{ u_n \sum_{n=0}^{\infty} M(n) | u \in V^k, n \in \mathbb{Z} \right\} = M^k.$$

In particular, M^k is nonzero for all k. Clearly, M^k and M^i for $i \neq k$ are inequivalent V^0-modules.

Let $0 \neq w \in M^k$. Then $\{u_n w | u \in V^i, n \in \mathbb{Q}\}$ is an admissible V^0-submodule of M^{i+k} (where $i+k$ is understood modulo T) and thus must be equal to M^{i+k} for all i. That is, $\{u_n w | u \in V, n \in \mathbb{Q}\} = M$ and M is an irreducible g-twisted V-module. ∎

As in the untwisted case, we can also formulate Proposition 5.3 in terms of the theory of the associative algebra $A_{g,n}(V)$ developed in [DLM3] and [DLM5].

Let V and g be as before. Fix $n = l + \frac{i}{T} \in \frac{1}{T}\mathbb{Z}$, with l a nonnegative integer and $0 \leq i \leq T-1$. For $0 \leq r \leq T-1$ we define $\delta_i(r) = 1$ if $i \geq r$ and $\delta_i(r) = 0$ if $i < r$. We also set $\delta_i(T) = 1$. Let $O_{g,n}(V)$ be the linear span of all $u \circ_{g,n} v$ and $L(-1)u + L(0)u$ where for homogeneous $u \in V^r$ and $v \in V$,

$$u \circ_{g,n} v = \operatorname{Res}_z Y(u, z)v \frac{(1+z)^{\mathrm{wt}u - 1 + \delta_i(r) f + l + r/T}}{z^{2l + \delta_i(r) + \delta_i(T-r)}}.$$

Define the linear space $A_{g,n}(V)$ to be the quotient $V/O_{g,n}(V)$. Then $A_{g,n}(V)$ is the untwisted associative algebra $A_n(V)$ as defined in Section 4 if $g = 1$ and is $A_g(V)$ in [DLM3] if $n = 0$. We also define a second product $*_{g,n}$ on V for u and v as above;

$$u *_{g,n} v = \sum_{m=0}^{l} (-1)^m \binom{m+1}{l} \operatorname{Res}_z Y(u, z) \frac{(1+z)^{\mathrm{wt}u + 1}}{z^{l+m+1}} v$$

if $r = 0$ and $u *_{g,n} v = 0$ if $r > 0$. Extend linearly to obtain a bilinear product on V.

Recall from [DLM3] that V is called *g-rational* if any admissible g-twisted V-module is completely reducible. The following theorem was given in [DLM5].

THEOREM 5.4. *Let $M = \sum_{n=0}^{\infty} M(n/T)$ be an admissible g-twisted V-module with $M(0) \neq 0$. We have*:

(i) *The product $*_{g,n}$ induces an associative algebra structure on $A_{g,n}(V)$ with the identity $\mathbf{1} + O_{g,n}(V)$. Moreover, $\omega + O_{g,n}(V)$ is a central element of $A_{g,n}(V)$.*

(ii) *The identity map on V induces an onto algebra homomorphism from $A_{g,n}(V)$ to $A_{g,m}(V)$ for $m, n \in \frac{1}{T}\mathbb{Z}$ and $0 \leq m \leq n$.*

(iii) *The map $u \mapsto o(u)$ gives a representation of $A_{g,n}(V)$ on $M(i)$ for $i \in \frac{1}{T}\mathbb{Z}$ and $0 \leq i \leq n$. Moreover, V is g-rational if and only if $A_{g,n}(V)$ are finite-dimensional semisimple algebras for all n.*

Clearly, both the actions of $A_{g,n}(V)$ and $S_M(V^0)$ on $\sum_{0 \leq m \leq n} M(m)$ are induced by $v \mapsto o(v)$. Combining Proposition 5.3 and Theorem 5.4 gives an analogue of Theorem 4.6

THEOREM 5.5. *Assume that the Virasoro element ω of V is nonzero. Then an admissible g-twisted V-module $M = \bigoplus_{n \geq 0} M(n)$ with $M(0) \neq 0$ is irreducible if and only if each $M(n)$ is an irreducible $A_{g,n}(V)$-module.*

REFERENCES

[B] R. E. Borcherds, Vertex algebras, Kac–Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.

[DL] C. Dong and J. Lepowsky, "Generalized Vertex Algebras and Relative Vertex Operators," Progress in Mathematics, Vol. 112, Birkhäuser, Boston, 1993.

[DLM1] C. Dong, H. Li, and G. Mason, Compact automorphism groups of vertex operator algebras, *Int. Math. Res. Notices* **18** (1996), 913–921.

[DLM2] C. Dong, H. Li, and G. Mason, Regularity of rational vertex operator algebras, *Adv. Math.* **132** (1997), 148–166.

[DLM3] C. Dong, H. Li, and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571–600.

[DLM4] C. Dong, H. Li, and G. Mason, Vertex operator algebras and associative algebras, *J. Algebra* **206** (1998), 67–96

[DLM5] C. Dong, H. Li, and G. Mason, Twisted representations of vertex operator algebras and associative algebras, *Int. Math. Res. Notices* **8** (1998), 389–397.

[DLM6] C. Dong, H. Li, and G. Mason, Modular invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* **214** (2000), 1–56.

[DLMM] C. Dong, H. Li, G. Mason, and P. Montague, The radical of a vertex operator algebra, *in* "Proceedings of the Conference on the Monster and Lie algebras at The Ohio State University, May 1996" (J. Ferrar and K. Harada, Eds.), pp. 17–25, de Gruyter, Berlin/New York, 1998.

[DLiM] C. Dong, Z. Lin, and G. Mason, On vertex operator algebras as sl_2-modules, *in* "Groups, Difference Sets, and the Monster, Proceedings of a Special Research Quarter at the Ohio State University, Spring 1993" (K. T. Arasu, J. F. Dillon, K. Harada, S. Sehgal, and R. Solomon, Eds.), 349–362, de Gruyter, Berlin/New York, 1996.

[DM1] C. Dong and G. Mason, On quantum Galois theory, *Duke Math J.* **86** (1997), 305–321.

[DM2] C. Dong and G. Mason, Reductive Lie algebras and vertex operator algebras, preprint.

[FLM] I. B. Frenkel, J. Lepowsky, and A. Meurman, "Vertex Operator Algebras and the Monster," Pure and Applied Mathematics, Vol. 134, Academic Press, New York, 1988.

[L1] H. Li, Symmetric invariant bilinear forms on vertex operator algebras, *J. Pure Appl. Algebra* **96** (1994), 279–297.

[L2] H. Li, "Representation Theory and Tensor Product Theory for Vertex Operator Algebras," Ph.D. thesis, Rutgers University, 1994.

[Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J Amer. Math. Soc.* **9** (1996), 237–302.