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Ambitwistors, oscillators and massless fields on AdS_5

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ABSTRACT

Positive energy unitary irreducible representations of $SU(2,2)$ can be constructed with the aid of bosonic oscillators in (anti)fundamental representation of $SU(2)_L \times SU(2)_R$ that are closely related to Penrose twistors. Starting with the correspondence between the doubleton representations, homogeneous functions on projective twistor space and on-shell generalized Weyl curvature $SL(2,\mathbb{C})$ spinors and their low-spin counterparts, we study in the similar way the correspondence between the massless representations, homogeneous functions on ambitwistor space and, via the Penrose transform, with the gauge fields on Minkowski boundary of AdS_5. The possibilities of reconstructing massless fields on AdS_5 and some applications are also discussed.

1. Introduction

The problem of characterization of irreducible unitary representations of $SU(2,2)$ has rather long history in mathematical physics (see [1] and [2], where references to earlier literature can be found). In the mid 80-s the interest in positive energy unitary representations of corresponding supergroups $SU(2,2|N)$ [3,4] was stimulated mainly by the development of supersymmetry and supergravity, in particular the necessity to describe the spectrum of $D=10$ IIB supergravity compactified on $AdS_5 \times S^5$ that was shown [3,5] to be given by the infinite-tower of massless and massive representations of $SU(2,2|4)$. At the end of 90-s the interest in positive energy unitary representations of $SU(2,2|4)$ renewed (see, e.g., [6,7]) in the context of AdS_5/CFT_4 gauge/string duality, on both sides of which the $SU(2,2|4)$ supergroup constitutes finite-dimensional part of the infinite-dimensional symmetry related to integrable structure. More recently in the framework of vectorial AdS_5/CFT_4 duality it was shown [8] that the spectrum of $D=5$ Vasiliev-type higher-spin gauge theories,[1] dual to free $4d$ scalar, spinor or Maxwell theories, is described by the infinite set of the positive energy unitary representations of $SU(2,2)$ corresponding to massless fields.

Lowest-weight (positive energy) irreducible unitary representations of $SU(2,2)$ and $SU(2,2|N)$, as well as those of other (super)groups can be constructed using quantized oscillators carrying fundamental representation labels of the maximal compact subgroup [11,12]. In this approach (super)group generators are realized as bilinear combinations of oscillators and part of them, that contains only raising oscillators, is used to produce the whole representation by acting on the lowest-weight vector annihilated by the lowering oscillators. In the case of $SU(2,2)$ and $SU(2,2|N)$ such $SU(2) \times SU(2)$ oscillators are naturally combined into Penrose twistors and supertwistors [7,13]. (Super)twistor theory [14] in its turn has long been known to provide interesting alternative to traditional space–time description of $4d$ massless gauge fields that after the construction of the twistor-string models [15,16] have got significant attention and allowed to unveil remarkable features of Yang–Mills/gravity amplitudes written in the spinorial form.

Superalgebras $SU(2,2|N)$ can be extended to infinite-dimensional superalgebras [17,9,19] that admit realizations in terms of above mentioned quantized supertwistors (oscillators) that play the role of auxiliary variables in the construction of $4d$ conformal higher-spin theories [20] and $5d$ higher-spin theories [9,10,21]. The spectrum of constituent gauge fields fits into the representation of underlying higher-spin superalgebra and decomposes into an infinite sum of massless representations of $SU(2,2|N)$ with spins ranging from zero to infinity (see [22] and references therein). More recently it was shown that these superalgebras admit the realization in terms of deformed twistors as enveloping algebras of $SU(2,2|N)$ [23].

In view of the important role played by massless gauge fields in the bulk of AdS_5 and on its $D=4$ Minkowski boundary, in this pa-

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[1] Higher-spin theories involved include like already known ones [9,10], as well as those yet to be identified.

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per we examine the possibilities to reconstruct bulk (Fang-)Fronsdal fields [24,25],[2] starting from the corresponding positive-energy irreducible unitary representations of $SU(2,2)$ and using the isomorphism between oscillators and twistors and the properties of homogeneous functions on the (ambi)twistor space.

Positive energy irreducible unitary representations of $SU(2,2)$ can be labeled by positive (half-)integers (E, j_1, j_2), where the AdS_5 energy E is the eigenvalue of the $u(1)$ generator and $j_{1,2}$ are the representation labels of $SU(2)_{L(R)}$ factors of the maximal compact subgroup $SU(2)_L \times SU(2)_R \times U(1)$. Values of AdS_5 energy E are bounded from below and the form of the bound depends on the spin $s = j_1 + j_2$ and the representation [1]. The simplest doubleton representations $(s+1, s, 0)$ and $(s+1, 0, s)$, by tensoring which all other representations can be constructed and for which associated fields are localized on the $D = 4$ Minkowski boundary of AdS_5, saturate the bound

$$E \geq s + 1. \tag{1}$$

For massless fields on AdS_5 the bound has the following form

$$E \geq s + 2, \quad j_1, j_2 > 0. \tag{2}$$

Note that bounds (1), (2) provide the simplest instances of generic relation between the AdS_D energy of the irreducible unitary representation of $SO(2, D-1)$ and the labels of the corresponding $SO(D-1)$ representation that was derived in [27,28] from the requirement of the positive definiteness of scalar product in the Fock space of $SO(2, D-1)$ oscillators. Since these oscillators are the $SO(2, D-1)$ vectors, part of them satisfy 'wrong sign' commutation relations so that the norm of a state in such a Fock space can be positive, negative or null. Then the condition of the norm positivity leads to the above discussed energy bound in similarity with the derivation of the values of critical dimension and intercept in the old covariant quantization of (super)strings. On the contrary the oscillator approach of Refs. [11,12] applied to the description of positive energy unitary representations of $SU(2,2)$ relies on the introduction of (a number of copies of) bosonic oscillators transforming in the fundamental representation of the two $SU(2)_{L(R)}$ factors in the maximal compact subgroup of $SU(2,2)$ that obey positive-definite commutation relations. This approach thus resembles light-cone gauge (super)string quantization scheme.

In the next section taking doubletons as the simplest example we confront known (but generically considered independently) oscillator and twistor approaches to their description and then apply gained experience to AdS_5 massless fields starting from the corresponding representations. For them the oscillator description is also familiar starting from [3]. Twistor description naturally involves ambitwistors and we consider how the ambitwistor data applies to the construction of the (Fang-)Fronsdal fields.

2. Doubletons and twistors

Let us remind the relationship between the $SU(2)$ oscillators and Penrose twistors (see, e.g., [7,13]). The former correspond to diagonalization of the 'metric'

$$H = \begin{pmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{pmatrix} \tag{3}$$

used to contract indices of fundamental (twistor) and antifundamental (dual twistor) representations of $SU(2,2)$. The twistor is defined by its primary μ^α and secondary $\bar{u}_{\dot\alpha}$ $SL(2,\mathbb{C})$ spinor parts

$$Z^{\boldsymbol{\alpha}} = \begin{pmatrix} \mu^\alpha \\ \bar{u}_{\dot\alpha} \end{pmatrix} \tag{4}$$

and similarly the dual twistor $\bar{Z}_{\boldsymbol{\alpha}} = (u_\alpha, \bar{\mu}^{\dot\alpha})$. Since in transition to the oscillator basis only $SU(2)$ covariance is retained, following [18] we replace dotted indices of the spinor parts of the twistor and its dual by undotted ones in the opposite position that are identified with the $SU(2)$ spinor indices in accordance with the uniqueness of the $SU(2)$ spinor representation. Then the oscillator variables are defined by the linear combinations of the twistor components

$$a^\alpha = \frac{1}{\sqrt{2}}(-\mu^\alpha + \bar{u}^\alpha), \quad a_\alpha = \frac{1}{\sqrt{2}}(-\bar{\mu}_\alpha + u_\alpha) \tag{5}$$

and

$$b_\alpha = \frac{1}{\sqrt{2}}(\bar{\mu}_\alpha + u_\alpha), \quad b^\alpha = \frac{1}{\sqrt{2}}(\mu^\alpha + \bar{u}^\alpha) \tag{6}$$

that can be viewed as a kind of the Bogulyubov transform (for further discussion on that point see, e.g., [19]). Inverse relations express spinor parts of the twistor and its dual via the oscillators[3]

$$\mu^\alpha = \frac{1}{\sqrt{2}}(-a^\alpha + b^\alpha), \quad \bar{u}^\alpha = \frac{1}{\sqrt{2}}(a^\alpha + b^\alpha) \tag{7}$$

and

$$u_\alpha = \frac{1}{\sqrt{2}}(a_\alpha + b_\alpha), \quad \bar{\mu}_\alpha = \frac{1}{\sqrt{2}}(-a_\alpha + b_\alpha). \tag{8}$$

In quantum theory introduced above oscillators can be shown to satisfy commutation relations

$$[\hat{a}_\alpha, \hat{a}^\beta] = \delta^\beta_\alpha, \quad [\hat{b}^\alpha, \hat{b}_\beta] = \delta^\alpha_\beta \tag{9}$$

that allows interpret $\hat{a}_\alpha$ and $\hat{b}^\alpha$ as annihilation operators and their conjugates $\hat{a}^\alpha = (\hat{a}_\alpha)^\dagger$, $\hat{b}_\alpha = (\hat{b}^\alpha)^\dagger$ as creation operators acting on the unitary vacuum $|0\rangle$. Important invariant – the twistor norm then transforms into the difference between the occupation numbers of b- and a-oscillators

$$\bar{Z}Z \to -\hat{N}_a + \hat{N}_b, \quad \hat{N}_a = \hat{a}^\alpha \hat{a}_\alpha, \quad \hat{N}_b = \hat{b}_\alpha \hat{b}^\alpha \tag{10}$$

and $su(2,2)$ algebra relations can be realized by the bilinears of quantized a- and b-oscillators.

Positive energy unitary representations one can build using just one copy of a- and b-oscillators (9) are called doubletons [3,29]. They correspond to $4d$ massless fields 'living' on the Minkowski boundary of AdS_5. The lowest-weight vectors corresponding to definite doubleton representations are constructed out of the product of creation $\hat{a}^\alpha$ or $\hat{b}_\alpha$ oscillators[4]

$$|\mathrm{lwv}\rangle = \hat{a}^{\alpha(2s_L)}|0\rangle \quad \text{or} \quad \hat{b}_{\alpha(2s_R)}|0\rangle \tag{11}$$

acting on the oscillator vacuum annihilated by the $\hat{a}_\alpha$ and $\hat{b}^\alpha$ operators. The whole representation in the basis corresponding to the maximal compact subalgebra $SU(2)_L \times SU(2)_R \times U(1)$ is constructed by applying to (11) the raising operators $\hat{L}_+{}^\alpha{}_\beta = \hat{a}^\alpha \hat{b}_\beta$. They commute with $-\hat{N}_a + \hat{N}_b$, thus in any representation fixed

[2] The issue of deriving non-linear extensions of the Fronsdal equations for bosonic fields on AdS_4 starting from the Vasiliev equations was addressed recently in [26].

[3] The oscillators with upper/lower indices are complex conjugate of each other but bars like in the case of $SL(2,\mathbb{C})$ spinors with dotted indices are not placed conventionally. This does not cause a confusion since changing the position of indices is not required in constructing $SU(2,2)$ positive energy unitary representations.

[4] Throughout the paper we adhere to the notation that a number in round brackets following an index stands for the group of indices equal to that number that are symmetrized with unit weight. Accordingly a number in square brackets following an index denotes the group of indices antisymmetrized with unit weight.

Table 1
Description of $D=4$ massless gauge fields by homogeneous functions on $\mathbb{PT}^\bullet$ and $\mathbb{PT}_\bullet$.

Irrep	Helicity	Hom. degree on $\mathbb{PT}^\bullet$	$D=4$ field	Hom. degree on $\mathbb{PT}_\bullet$	$D=4$ field
$(1-s,-s,0)$	$s<0$	$-2-2s$	$\Gamma_{\alpha\dot\alpha(-2s-1)}(x)$	$-2+2s$	$W_{\alpha(-2s)}(x)$
$(3,2,0)$	-2	2	$\Gamma_{\alpha\dot\alpha(3)}(x)$	-6	$W_{\alpha(4)}(x)$
$(5/2,3/2,0)$	$-3/2$	1	$\Lambda_{\alpha\dot\alpha(2)}(x)$	-5	$\Psi_{\alpha(3)}(x)$
$(2,1,0)$	-1	0	$A_{\alpha\dot\alpha}(x)$	-4	$f_{\alpha(2)}(x)$
$(3/2,1/2,0)$	$-1/2$	-1	$\lambda_\alpha(x)$	-3	$\lambda_\alpha(x)$
$(2,0,0)$	0	-2	$\varphi(x)$	-2	$\varphi(x)$
$(3/2,0,1/2)$	$1/2$	-3	$\lambda_{\dot\alpha}(x)$	-1	$\lambda_{\dot\alpha}(x)$
$(2,0,1)$	1	-4	$f_{\dot\alpha(2)}(x)$	0	$A_{\dot\alpha\alpha}(x)$
$(5/2,0,3/2)$	$3/2$	-5	$\Psi_{\dot\alpha(3)}(x)$	1	$\Lambda_{\dot\alpha\alpha(2)}(x)$
$(3,0,2)$	2	-6	$W_{\dot\alpha(4)}(x)$	2	$\Gamma_{\dot\alpha\alpha(3)}(x)$
$(s+1,0,s)$	$s>0$	$-2-2s$	$W_{\dot\alpha(2s)}(x)$	$-2+2s$	$\Gamma_{\dot\alpha\alpha(2s-1)}(x)$

is the integer $-2s_L$ or $+2s_R$. The $SU(2)_L\times SU(2)_R$ labels $j_{1,2}$ are given by half the eigenvalues of $\hat N_a$ and $\hat N_b$ on the lowest-weight vectors and the energy equals $E=j_1+j_2+1=s_{L(R)}+1$.

In the twistor picture doubleton representations are described by homogeneous functions $f(Z)$ on the twistor space $\mathbb{PT}^\bullet$ or homogeneous functions $\tilde f(\bar Z)$ on the dual twistor space $\mathbb{PT}_\bullet$. Such a description is based on the quantized twistors

$$[\hat Z^{\boldsymbol\alpha},\hat{\bar Z}_{\boldsymbol\beta}]=\delta^{\boldsymbol\alpha}_{\boldsymbol\beta} \tag{12}$$

realization as the multiplication and differentiation operators

$$\hat Z^{\boldsymbol\alpha}\to Z^{\boldsymbol\alpha},\quad \hat{\bar Z}_{\boldsymbol\alpha}\to-\frac{\partial}{\partial Z^{\boldsymbol\alpha}} \tag{13}$$

or vice versa

$$\hat Z^{\boldsymbol\alpha}\to\frac{\partial}{\partial\bar Z_{\boldsymbol\alpha}},\quad \hat{\bar Z}_{\boldsymbol\alpha}\to\bar Z_{\boldsymbol\alpha}. \tag{14}$$

Operator realization (13) is adapted to the action on the twistor space functions and (14) – on the dual twistor space functions.

In the twistor approach doubleton representations built upon the lowest-weight vectors (11) are described by the homogeneous functions on the twistor space with the homogeneity degrees $2s_L-2$ or $-2s_R-2$ as follows from (10). The twistor helicity operator

$$\hat s=\frac14(\hat{\bar Z}\hat Z+\hat Z\hat{\bar Z}) \tag{15}$$

in the realization (13) acquires the form

$$\hat s=-\frac12 Z\frac{\partial}{\partial Z}-1 \tag{16}$$

so that the function $f_{(2s_L-2)}(Z)$ homogeneous of degree $2s_L-2>-2$ corresponds to the field with negative helicity $-s_L$ that describes left-polarized massless particles, whereas the function $f_{(-2s_R-2)}(Z)$ homogeneous of degree $-2s_R-2<-2$ corresponds to the field with positive helicity $+s_R$ and right-polarized particles [14]. In the case of positive helicity fields reconstruction of the on-shell curvatures (linearized Weyl curvature $SL(2,\mathbb C)$ spinors) on the $D=4$ Minkowski space–time proceeds using the contour integral representation

$$\bar W_{\dot\alpha(2s_R)}(x)=\int\bar u_{\dot\beta}d\bar u^{\dot\beta}\,\bar u_{\dot\alpha_1}\cdots\bar u_{\dot\alpha_{2s_R}}\times f_{(-2s_R-2)}(i\bar u_{\dot\lambda}x^{\dot\lambda\lambda},\bar u_{\dot\lambda}):\quad \partial^{\dot\alpha\alpha}\bar W_{\dot\alpha(2s_R)}(x)=0, \tag{17}$$

where the incidence relations $\mu^\alpha=i\bar u_{\dot\alpha}\tilde x^{\dot\alpha\alpha}$ that express primary spinor part of the twistor via the coordinates $\tilde x^{\dot\alpha\alpha}=x^a\tilde\sigma_a^{\dot\alpha\alpha}$ of the (complexified conformally compactified) $D=4$ Minkowski space–time are assumed to hold. For the negative helicity fields cohomological arguments suggest a description in terms of the spinor form $\Gamma_{\alpha\dot\alpha(2s_L-1)}(x)$ of linearized Christoffel-type connections [30] modulo the gauge transformations. Accordingly Penrose transform of the dual-twistor-space function for negative helicity yields generalized Weyl curvature $SL(2,\mathbb C)$ spinor of opposite chirality $W_{\alpha(2s_L)}(x)$, while for positive helicity it produces Christoffel-type connection $\Gamma_{\dot\alpha\alpha(2s_R-1)}(x)$ (see Table 1). General relation between the homogeneity degrees of functions on the twistor space $h_{\mathbb{PT}^\bullet}$ and on the dual twistor space $h_{\mathbb{PT}_\bullet}$ that correspond to the field of helicity s is

$$h_{\mathbb{PT}_\bullet}=-h_{\mathbb{PT}^\bullet}-4. \tag{18}$$

The correspondence between respective cohomology groups of homogeneous functions is known as the twistor transform.

3. Massless field representations and ambitwistors

Description of the positive energy unitary irreducible representations associated with massless fields on AdS_5 necessitates introduction of two copies of a- and b-oscillators [3,29]

$$[\hat a_\alpha(p),\hat a^\beta(r)]=\delta_{pr}\delta^\beta_\alpha,\quad [\hat b^\alpha(p),\hat b_\beta(r)]=\delta_{pr}\delta^\alpha_\beta,\quad p,r=1,2. \tag{19}$$

In the twistor approach this corresponds to dealing with two Penrose twistors and their duals. Associate in accordance with (7), (8) to the first set of oscillators the twistor $\hat Z^{\boldsymbol\alpha}$ and its dual $\hat{\bar Z}_{\boldsymbol\alpha}$, and analogously to the second set of oscillators – another pair of twistors $\hat W^{\boldsymbol\alpha}=(\hat\upsilon^\alpha,\hat{\bar\upsilon}_{\dot\alpha})$ and $\hat{\bar W}_{\boldsymbol\alpha}=(\hat\upsilon_\alpha,\hat{\bar\upsilon}^{\dot\alpha})$. Similarly to the doubleton case, in any representation fixed is the difference between the occupation numbers of a- and b-oscillators of the first and the second sets

$$(-\hat N_a(1)+\hat N_b(1))|\mathrm{lwv}\rangle=-2s_1|\mathrm{lwv}\rangle,$$
$$(-\hat N_a(2)+\hat N_b(2))|\mathrm{lwv}\rangle=2s_2|\mathrm{lwv}\rangle. \tag{20}$$

For the representations that correspond to massless fields on AdS_5 $s_{1,2}$ are positive (half-)integers modulo relabeling the oscillators. So one is led to consider homogeneous functions on the ambitwistor space $\mathbb A$:

$$Z\tfrac{\partial}{\partial Z}F_{(2s_1-2|2s_2-2)}(Z,\bar W)=(2s_1-2)F_{(2s_1-2|2s_2-2)}(Z,\bar W),$$
$$\bar W\tfrac{\partial}{\partial\bar W}F_{(2s_1-2|2s_2-2)}(Z,\bar W)=(2s_2-2)F_{(2s_1-2|2s_2-2)}(Z,\bar W). \tag{21}$$

We have chosen $Z^{\boldsymbol{\alpha}}$ and $\bar{W}_{\boldsymbol{\alpha}}$ to parametrize $\mathbb{A}$, while the operators corresponding to $\bar{Z}_{\boldsymbol{\alpha}}$ and $W^{\boldsymbol{\alpha}}$ have been traded for the derivatives of $Z^{\boldsymbol{\alpha}}$ and $\bar{W}_{\boldsymbol{\alpha}}$ (cf. (13), (14)). The condition $\bar{W}_{\boldsymbol{\alpha}}Z^{\boldsymbol{\alpha}}=0$ on the ambitwistor space coordinates is imposed via the δ-function

$$F_{(2s_1-2|2s_2-2)}(Z,\bar{W})=\delta(\bar{W}Z)f_{(2s_1-1|2s_2-1)}(Z,\bar{W}). \tag{22}$$

In the oscillator approach it translates into the constraint

$$(\hat{b}_\alpha(2)\hat{b}^\alpha(1)-\hat{a}_\alpha(2)\hat{a}^\alpha(1))|\,\mathrm{lwv}\rangle=0. \tag{23}$$

For homogeneous functions $f_{(s-1|s-1)}(Z,\bar{W})\equiv f_{(s-1)}(Z,\bar{W})$ on $\mathbb{A}$ with s non-negative integer details of the Penrose transform can be found, e.g., in [31] or in Ref. [32]. On-shell of the incidence relations

$$\mu^\alpha=i\bar{u}_{\dot\alpha}\tilde{x}^{\dot\alpha\alpha},\quad \bar{\nu}^{\dot\alpha}=-i\tilde{x}^{\dot\alpha\alpha}v_\alpha \tag{24}$$

the function $f_{(s-1)}$ satisfies

$$v^\alpha\bar{u}^{\dot\alpha}\partial_{\alpha\dot\alpha}f_{(s-1)}(\mu,\bar{u},\bar{\nu},v)=0. \tag{25}$$

$f_{(s-1)}$ is cohomologically trivial since $H^1(\mathbb{CP}^1\times\mathbb{CP}^1,\mathcal{O}(s-1))=0$ implying that it can be globally defined as a polynomial of the respective degree in $\bar{u}^{\dot\alpha}$ and v^α, homogeneous coordinates on $\mathbb{CP}^1\times\mathbb{CP}^1$,

$$v^\alpha\bar{u}^{\dot\alpha}\partial_{\alpha\dot\alpha}f_{(s-1)}\Rightarrow v^{\alpha(s)}\bar{u}^{\dot\alpha(s)}\widetilde{b}_{\alpha(s)\dot\alpha(s)}(x), \tag{26}$$

where the symmetric multispinor field $\widetilde{b}_{\alpha(s)\dot\alpha(s)}$ is defined modulo the gauge symmetry

$$\delta\widetilde{b}_{\alpha(s)\dot\alpha(s)}(x)=\partial_{\alpha(1)\dot\alpha(1)}\widetilde{\xi}_{\alpha(s-1)\dot\alpha(s-1)}(x) \tag{27}$$

with the symmetric multispinor parameter $\widetilde{\xi}_{\alpha(s-1)\dot\alpha(s-1)}$. In 4d vector notation it corresponds to symmetric traceless rank-s tensor field $\widetilde{b}_{a(s)}(x)$[5] and the gauge parameter is given by the symmetric traceless rank-$(s-1)$ tensor field $\widetilde{\xi}_{a(s-1)}(x)$ so that

$$\delta\widetilde{b}_{a(s)}(x)=\partial_{a(1)}\widetilde{\xi}_{a(s-1)}(x)-\frac{1}{s}\eta_{a(2)}\partial^c\widetilde{\xi}_{ca(s-2)}(x). \tag{28}$$

The discussion of the last paragraph corresponds to the simplest case of totally symmetric massless bosonic fields. In the oscillator description the solution to the constraints

$$(-\hat{N}_a(1)+\hat{N}_b(1))|\,\mathrm{lwv}\rangle=-s|\,\mathrm{lwv}\rangle,$$
$$(-\hat{N}_a(2)+\hat{N}_b(2))|\,\mathrm{lwv}\rangle=s|\,\mathrm{lwv}\rangle,\quad s\ge 0 \tag{29}$$

and (23) is given by the lowest-weight vector (cf. [29])

$$|\,\mathrm{lwv}\rangle=\hat{a}(1)^{\alpha(s)}\hat{b}(2)_{\beta(s)}|\,0\rangle. \tag{30}$$

Respective representation labels are $j_1=j_2=s/2$ and $E=s+2$.

Generalizing the above consideration to the functions $f_{(s-1|s)}$ and $f_{(s|s-1)}$ on $\mathbb{A}$ allows to obtain also fermionic fields $\widetilde{\psi}_{\alpha(s+1)\dot\alpha(s)}(x)$ and $\widetilde{\chi}_{\alpha(s)\dot\alpha(s+1)}(x)$ defined modulo the gauge transformations

$$\delta\widetilde{\psi}_{\alpha(s+1)\dot\alpha(s)}(x)=\partial_{\alpha(1)\dot\alpha(1)}\widetilde{\varepsilon}_{\alpha(s)\dot\alpha(s-1)}(x) \tag{31}$$

and

$$\delta\widetilde{\chi}_{\alpha(s)\dot\alpha(s+1)}(x)=\partial_{\alpha(1)\dot\alpha(1)}\widetilde{\epsilon}_{\alpha(s-1)\dot\alpha(s)}(x). \tag{32}$$

In vector form these fields are given by the totally symmetric σ-traceless tensor-spinors $\widetilde{\psi}_{a(s)\alpha}(x)$ and $\widetilde{\chi}_{a(s)\dot\alpha}(x)$, for which the gauge variations read

$$\delta\widetilde{\psi}_{a(s)}(x)=\partial_{a(1)}\widetilde{\varepsilon}_{a(s-1)}(x)+\frac{1}{2(s+1)}\sigma_{a(1)}\tilde{\sigma}^b\partial_b\widetilde{\varepsilon}_{a(s-1)}(x)$$
$$-\frac{1}{s+1}\eta_{a(2)}\partial^b\widetilde{\varepsilon}_{ba(s-2)}(x) \tag{33}$$

and

$$\delta\widetilde{\chi}_{a(s)}(x)=\partial_{a(1)}\widetilde{\epsilon}_{a(s-1)}(x)+\frac{1}{2(s+1)}\tilde{\sigma}_{a(1)}\sigma^b\partial_b\widetilde{\epsilon}_{a(s-1)}(x)$$
$$-\frac{1}{s+1}\eta_{a(2)}\partial^b\widetilde{\epsilon}_{ba(s-2)}(x). \tag{34}$$

Associated lowest-weight vectors in the oscillator approach

$$\hat{a}^{\alpha(s)}(1)\hat{b}_{\beta(s+1)}(2)|\,0\rangle,\quad \hat{a}^{\alpha(s+1)}(1)\hat{b}_{\beta(s)}(2)|\,0\rangle \tag{35}$$

correspond to representations with $j_1=s/2$, $j_2=(s+1)/2$ and $j_1=(s+1)/2$, $j_2=s/2$ that both have $E=s+5/2$.

Generic ambitwistor functions $f_{(2s_1-1|2s_2-1)}(Z,\bar{W})$ with $s_1,s_2>0$, $|s_1-s_2|>1/2$ give rise to multispinor bosonic fields $\widetilde{b}_{\alpha(2s_2)\dot\alpha(2s_1)}(x)$ if s_1+s_2 is integer, or, if s_1+s_2 is half-integer, fermionic fields $\widetilde{\psi}_{\alpha(2s_2)\dot\alpha(2s_1)}(x)$ ($s_2>s_1$) or $\widetilde{\chi}_{\alpha(2s_2)\dot\alpha(2s_1)}(x)$ ($s_1>s_2$). They are defined modulo the gauge freedom

$$\delta\widetilde{b}_{\alpha(2s_2)\dot\alpha(2s_1)}(x)=\partial_{\alpha(1)\dot\alpha(1)}\widetilde{\xi}_{\alpha(2s_2-1)\dot\alpha(2s_1-1)} \tag{36}$$

and analogous transformations for

$\widetilde{\psi}_{\alpha(2s_2)\dot\alpha(2s_1)}(x)$ and $\widetilde{\chi}_{\alpha(2s_2)\dot\alpha(2s_1)}(x)$.

Respective lowest-weight vectors are

$$a^{\alpha(2s_1)}(1)b_{\beta(2s_2)}(2)|\,0\rangle \tag{37}$$

and representation labels are $(s+2,s_1,s_2)$ with $s=s_1+s_2$. Together with the conjugate representations $(s+2,s_2,s_1)$ they make up completely traceless tensor fields $\widetilde{b}_{a(s_1+s_2)b(|s_1-s_2|)}(x)$ and tensor-bispinor fields $\widetilde{\Psi}_{a(s_1+s_2-1/2)b(|s_1-s_2|-1/2)}(x)$. These are mixed-symmetry fields associated with two-row Young tableaux. Above considered representations are the only massless ones. Other $SU(2,2)$ representations that can be constructed using the two sets of a- and b-oscillators [29], in particular those arising in the limit $s_1=0$ or $s_2=0$ of (37), correspond to the so called massive self-dual fields [33].

Symmetric traceless bosonic fields (28) and σ-traceless fermionic fields (33) and (34) can naturally be identified with bosonic and fermionic $D=4$ shadow fields [34,35], and treated as boundary values of AdS_5 totally symmetric massless gauge fields corresponding to the non-normalizable solutions of the Dirichlet problem for the Fronsdal equations. AdS/CFT adapted description of massless mixed-symmetry is much more involved.[6] These results are summarized in Table 2, where $a',b'=0,...,3,5$ are $D=5$ tangent-space vector indices.

The direct reconstruction of AdS_5 fields requires modification of the Penrose incidence relations (24) to accommodate the contribution of the fifth space–time coordinate.[7] It is possible to unify secondary spinor parts of ambitwistors v^α and $\bar{u}_{\dot\alpha}$ into four-component $D=5$ spinor

$$U_{\underline{\alpha}}=\begin{pmatrix} v_\alpha \\ \bar{u}^{\dot\alpha}\end{pmatrix} \tag{38}$$

[5] Tilde is used to indicate tracelessness of a tensor w.r.t. Minkowski metric.

[6] To date available is light-cone gauge formulation of AdS_5 massless mixed-symmetry fields [33], $SO(1,D-1)$-covariant formulation for three-cell hook-type Young diagram field [36], as well as the ambient-space one [37].

[7] For the space–time of dimension $D=6$, for which the space–time coordinate matrices span the space of 4×4 antisymmetric matrices, natural generalization of Penrose twistors and on-shell contour integral relations does exist [38,39].

Table 2
$SU(2,2)$ massless representations, respective ambitwistor functions and space–time fields.

Irrep (E, j_1, j_2)	f on $\mathbb{A}$	$SL(2,\mathbb{C})$ multispinors	AdS_5 field
$(s+2, s/2, s/2)$	$f_{(s-1)}$	$\widetilde{b}_{\alpha(s)\dot\alpha(s)}$	$\widetilde{B}_{a'(s)}$
$(s+5/2, s/2, (s+1)/2) \oplus (s+5/2, (s+1)/2, s/2)$	$f_{(s-1\|s)} \oplus f_{(s\|s-1)}$	$\widetilde{\psi}_{\alpha(s+1)\dot\alpha(s)} \oplus \widetilde{\chi}_{\alpha(s)\dot\alpha(s+1)}$	$\widetilde{\Psi}_{a'(s)}$
$(s_1+s_2+2, s_1, s_2) \oplus (s_1+s_2+2, s_2, s_1)$ $s_{1,2} > 0$, $\|s_1 - s_2\| > 1/2$	$f_{(2s_1-1\|2s_2-1)} \oplus f_{(2s_2-1\|2s_1-1)}$	s_1+s_2 integer: $\widetilde{b}_{\alpha(2s_2)\dot\alpha(2s_1)} \oplus \widetilde{b}_{\alpha(2s_1)\dot\alpha(2s_2)}$ s_1+s_2 half-integer: $\widetilde{\psi}_{\alpha(2\max\{s_1,s_2\})\dot\alpha(2\min\{s_1,s_2\})} \oplus \widetilde{\chi}_{\alpha(2\min\{s_1,s_2\})\dot\alpha(2\max\{s_1,s_2\})}$	$\widetilde{B}_{a'(s_1+s_2)b'(\|s_1-s_2\|)}$ $\widetilde{\Psi}_{a'(s_1+s_2-1/2)b'(\|s_1-s_2\|-1/2)}$

and combine it with the Dirac conjugate into $SU(2)$-symplectic Majorana spinor. Such an $SU(2)$-symplectic Majorana spinor can be treated as the 4×2 rectangular block of the $D=5$ spinor Lorentz-harmonic matrix. Then by generalizing the construction of [40], an integral of homogeneous functions that belong to definite $SU(2)$ irreducible representations over the spinor harmonics produces the spinor form of $D=5$ Yang–Mills, Rarita–Schwinger field strengths, linearized Weyl tensor and its higher-spin counterparts that satisfy Dirac-type equations [41]. Alternatively $D=5$ symmetric traceless gauge fields can be obtained via the Penrose transform for $D=5$ ambitwistors [32].

4. Conclusions

In this note we discussed the correspondence between the positive energy (lowest-weight) unitary irreducible representations of $SU(2,2)$ and the space–time fields on AdS_5 taking as an example doubleton and massless representations that saturate the bounds $E=s+1$ and $E=s+2$ respectively. Isomorphism between bosonic oscillators, that can be used to construct positive energy unitary irreducible representations of $SU(2,2)$, and Penrose twistors allows to establish one-to-one correspondence between the doubleton representations and homogeneous functions of the single argument on projective twistor space (or dual projective twistor space) that via the Penrose transform yield on-shell linearized Weyl curvature spinors and their low-spin counterparts in 4 dimensions. We sought for possible extension of this twistor description to the case of massless representations. Since to construct massless representations it is necessary to use twice more oscillators compared to the doubletons, their natural twistor counterparts are ambitwistors. We have shown that Penrose transform for homogeneous functions on ambitwistor space yields shadow fields on $D=4$ Minkowski space–time that admit interpretation as boundary values of the non-normalizable solutions to the Dirichlet problem for (Fang-)Fronsdal equations for the corresponding AdS_5 massless gauge fields. This establishes one-to-one correspondence between homogeneous ambitwistor functions and $SU(2,2)$ massless representations. The fact that one arrives at the boundary values of AdS_5 massless gauge fields rather than the bulk fields themselves is encoded in the form of the incidence relations that take into account only the contributions of $D=4$ Minkowski space coordinates. Direct obtention of the AdS_5 massless gauge fields requires extension of Penrose incidence relations to account for the contribution of the fifth space–time coordinate.

Natural generalization of the results reported in this note is to introduce supersymmetry and also consider massive representations pertinent to the adjoint version of the AdS_5/CFT_4 correspondence. Potentially interesting applications can be also in twistor-string theory. Ambitwistor string models have already been used to reproduce $D=4$ Yang–Mills and Einstein gravity tree amplitudes in [42]. It is tempting to speculate that their appropriate ramifications can produce tree amplitudes of $D=5$ gauge theories.

While finalizing this paper we learned about Ref. [43] that to some extent overlaps with our results and clarifies some points that we discuss here.

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