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Computing a subgraph of the minimum weight triangulation

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Abstract

Given a set S of n points in the plane, it is shown that the $\sqrt{2}$-skeleton of S is a subgraph of the minimum weight triangulation of S. The β-skeletons are polynomially computable Euclidean graphs introduced by Kirkpatrick and Radke [8]. The $\sqrt{2}$-skeleton of S is the β-skeleton of S for $\beta = \sqrt{2}$.

Key words: Minimum weight triangulation; Euclidean graph

1. Introduction

Let S be a set of n points in the plane. A *triangulation* of S is a maximal straight line plane graph whose vertices are the points of S. Any triangulation of S partitions the convex hull of S into empty triangles. Triangulations have applications in numerical interpolation and in the finite element method.

A minimum weight triangulation (MWT) of S is a triangulation that minimizes the sum of the edge lengths. The problem of computing the minimum weight triangulation is included in Garey and Johnson's list of problems that are neither known to be NP-complete nor known to be solvable in polynomial time [5]. To date, the status of the minimum weight triangulation problem remains open.

The apparent difficulty of the problem suggests that approximation algorithms should be considered. Early, it was thought that the Delaunay triangulation and the Greedy triangulation may approximate the minimum weight triangulation. This is

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not true; the Delaunay triangulation can be as far as $\Omega(n)$ from the optimum [12, 7] and the Greedy triangulation can be as far as $\Omega(\sqrt{n})$ from the optimum [12, 10]. Using a different approach, Plaisted and Hong [14] were able to develop a heuristic triangulation that has length within $O(\log n)$ of the optimal. No polynomial time algorithm is known which constructs a triangulation guaranteed to have total length at most some constant time the optimum.

Another approach is to allow additional Steiner points to be added when constructing the triangulation. No exact algorithm is known for the minimum weight Steiner triangulation problem, but approximation algorithms have been developed. Clarkson [2] devised an algorithm which yields a Steiner triangulation with total length within a factor of $O(\log n)$ of the minimum possible. More recently, Eppstein presented an algorithm that constructs a Steiner triangulation which approximates the minimum weight Steiner triangulation within a constant factor [4]. Further, this triangulation also approximates the non-Steiner minimum weight triangulation within a constant factor. It turns out though that the minimum weight Steiner triangulation of n points can have total weight $\Omega(n)$ times smaller than the minimum weight triangulation of the points [4]. Thus allowing Steiner points makes the problem quite different.

When Steiner points are disallowed, one case that can be efficiently solved is that of a simple polygon. The minimum weight triangulation of a simple polygon can be computed using dynamic programming in $O(n^3)$ time [6, 9]. This fact provided the basis for a heuristic, developed by Lingas [11], for the general minimum weight triangulation problem. He first found the convex hull edges of S and added to these the edges of a minimum length planar forest connecting the convex hull with the remaining points of S. Finally, the resulting polygonal regions were triangulated using dynamic programming in $O(n^3)$ time. It is not known whether this heuristic approximates the minimum weight triangulation by better than an $O(n)$ factor.

The idea of first computing a subgraph of an optimal triangulation was used to a basis for an exact algorithm for computing the triangulation of S that minimizes the length of the longest edge [3]. Initially the edges of the convex hull and the relative neighbourhood graph of the point set S are included in the triangulation. The remaining polygonal regions are then optimally triangulated with dynamic programming in $O(n^3)$ time or an alternative $O(n^2)$ method. The key to this algorithm is that the relative neighbourhood graph provides a connected subgraph of the desired triangulation.

An exact method for a minimum weight triangulation algorithm could also proceed along these lines. First compute a subgraph of the minimum weight triangulation that is connected (actually it would be sufficient that the subgraph have a constant number of connected components [1]). The remaining edges could then be added by triangulating the resulting polygonal regions using dynamic programming. At present, very little is known about computing subgraphs of the minimum weight triangulation. Gilbert [6] showed that the shortest edge between two points in S can always be included in the minimum weight triangulation, and of course the convex hull is always included in every triangulation.

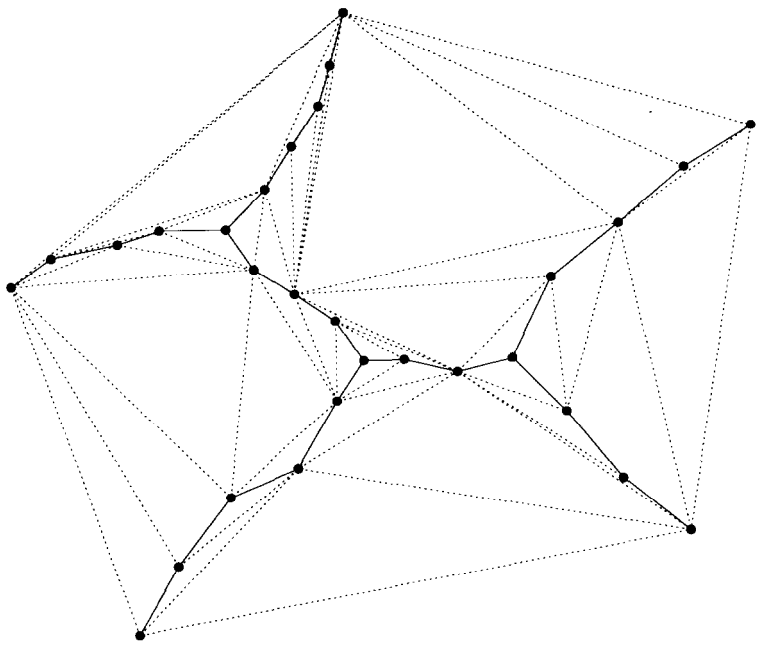

Fig. 1. A set of planar points with its $\sqrt{2}$-Skeleton given in solid lines and the rest of the minimum weight triangulation given in dotted lines.

In this paper we show that a polynomially computable Euclidean graph, the disk based $\sqrt{2}$-skeleton is always a subgraph of the minimum weight triangulation. The $\sqrt{2}$-skeleton is the β-skeleton of Kirkpatrick and Radke [8] for $\beta = \sqrt{2}$. See Fig. 1 for an example $\sqrt{2}$-skeleton of a set of planar points. Note that, for this point set, since the $\sqrt{2}$-skeleton is connected, the complete minimum weight triangulation can be found in polynomial time. Unfortunately, in general the $\sqrt{2}$-skeleton is not sufficiently connected to (by itself) provide the basis for a polynomial time minimum weight triangulation algorithm.

2. The $\sqrt{2}$-skeleton

In the minimum weight triangulation a point is joined by edges to other points that are 'near by' as much as possible. This intuition suggests that a subgraph of the minimum weight triangulation may be found among the various 'neighbourly' Euclidean graphs that have been studied. Such graphs join two points by an edge when a certain neighbourhood is empty. The relative neighbourhood graph (RNG) and the Gabriel graph (GG) are among the most studied empty neighbourhood graphs. For two points x and y, the neighbourhood for the relative neighbourhood graph is the intersection of the two disks centered at x and y with radius d(x, y),

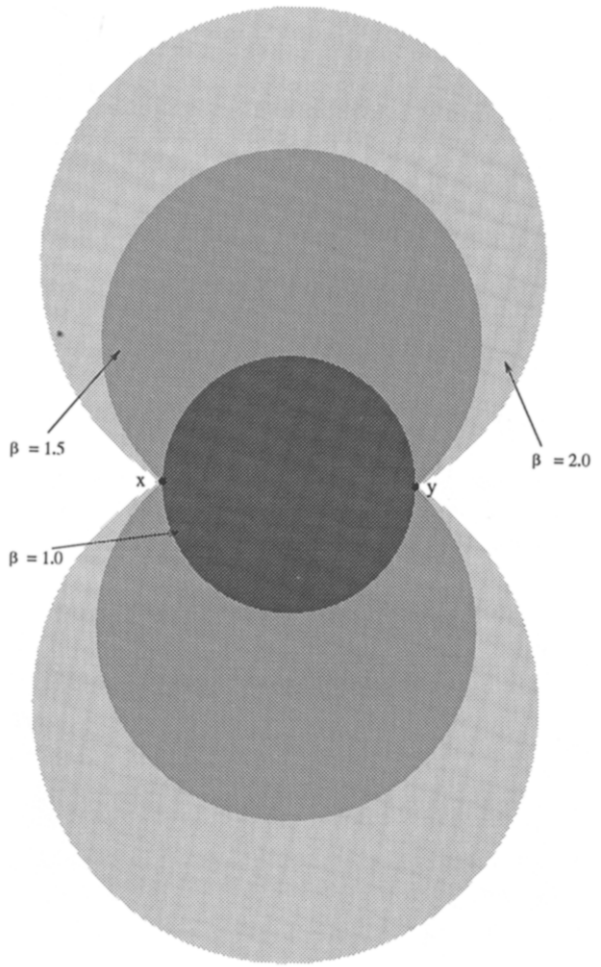

Fig. 2. β-Skeletons.

whereas the neighbourhood for the Gabriel graph is the smallest disk through x and y, which has radius d(x, y)/2. The fact that the Euclidean minimum spanning tree (EMST) is not a subgraph of the minimum weight triangulation [13], and the fact that MST $\subseteq$ RNG $\subseteq$ GG implies that the relative neighbourhood graph and the Gabriel graph are not the desired subgraphs of the minimum weight triangulation.

Another class of empty neighbourhood Euclidean graphs are the β-skeletons defined by Kirkpatrick and Radke [8]. These come in two variants, one based on lune like neighbourhoods and the other based on neighbourhoods constructed from the unions or intersections of disks. We make use of the disk-based variant here. For $\beta > 1$, the forbidden neighbourhood for points x and y is defined to be the union of the two disks of radius $\beta\,\mathrm{d}(x, y)/2$ that pass through both x and y. See Fig. 2. These

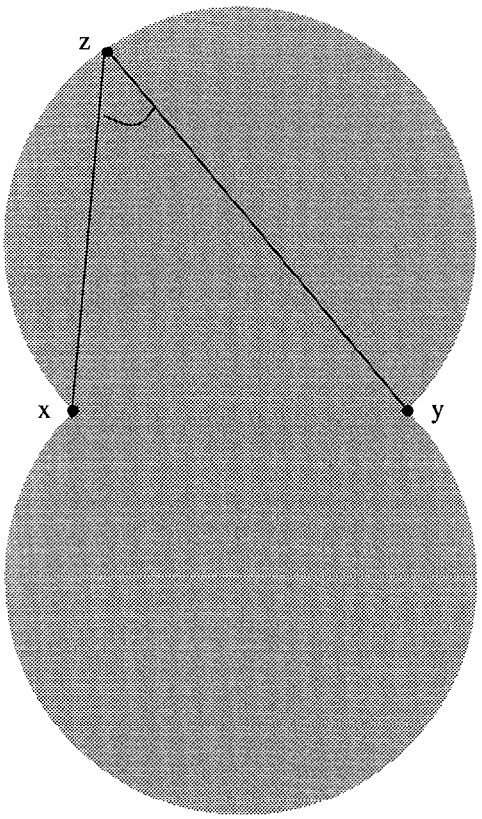

Fig. 3. $\sqrt{2}$-Skeleton forbidden zone.

β-skeletons have been incorporated in the parameterized γ-neighbourhood graph scheme of Veltkamp [15] as $\gamma(1 - 1/\beta, 1 - 1/\beta)$ graphs.

The following elementary lemma provides a useful alternative way of viewing the forbidden neighbourhoods defining β-skeletons.

Lemma 1. *Given a set S of points in the plane. Points x and y are adjacent in the disk-based β-skeleton ($\beta > 1$) if and only if there does not exist a point $z \in S$ such that angle $xzy \geqslant \arcsin(1/\beta)$.*

Figure 3 shows the forbidden neighbourhood zone for the $\sqrt{2}$-skeleton. Notice that if z is a point on the boundary of the neighbourhood that angle $xzy = \pi/4$.

For the remainder of this section we shall set β to $\sqrt{2}$ and study the properties of the $\sqrt{2}$-skeleton. If x and y are points in S we use xy to denote the line segment with endpoints x and y and $|xy|$ to denote its length. The following lemma compares the length of segments adjacent to the vertices of a $\sqrt{2}$-skeleton edge to the length of a line segment that intersects the $\sqrt{2}$-skeleton edge.

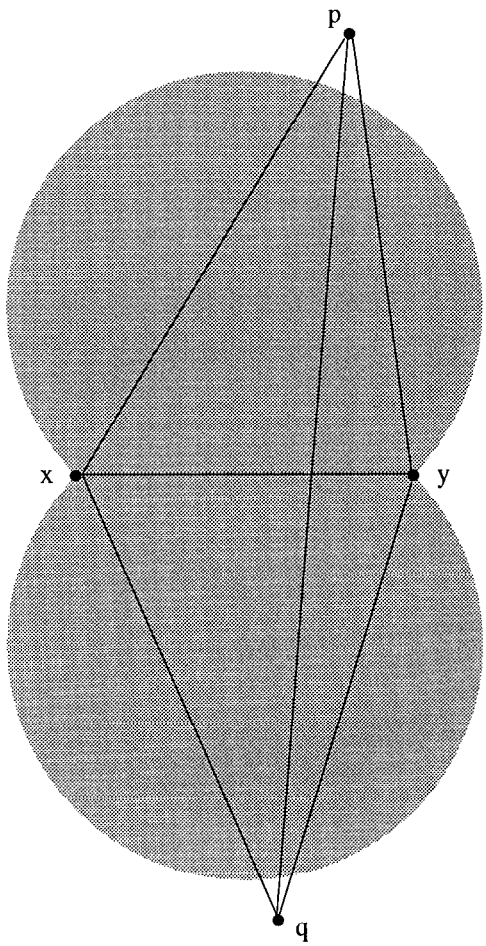

Fig. 4. Length Lemma.

Lemma 2 [Length Lemma]. *Let x and y be the endpoints of an edge in the $\sqrt{2}$-skeleton of a set S of points in the plane. Let p and q be two points in S such that the line segment pq intersects the segment xy. Then $|pq|$ is greater than $|xy|$, $|xp|$, $|xq|$, $|yp|$, and $|yq|$.*

Proof. Refer to Fig. 4. Consider the triangle xpq. Since the angles xpq and pqx are less than $\pi/4$, we have that $|pq| > |xp|$ and $|pq| > |xq|$. Likewise by considering triangle pqy we can show that $|pq| > |yp|$ and $|pq| > |yq|$. Now consider triangle xpy. Since angle xpy is less than $\pi/4$, either $|xp| > |xy|$ or $|yp| > |xy|$. Since both $|xp|$ and $|yp|$ are less than $|pq|$ we can conclude that $|xy| < |pq|$ as well. □

We can also bound the length of more remote segments by the length of segments that cross $\sqrt{2}$-skeleton edges.

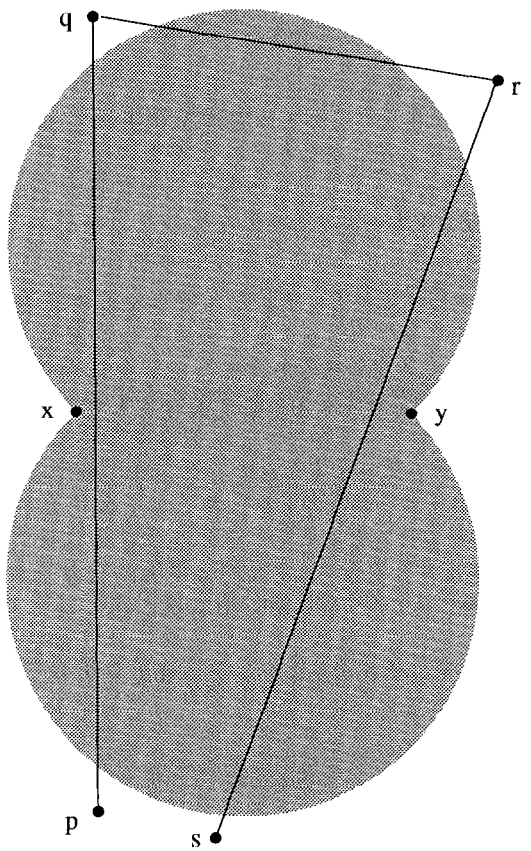

Fig. 5. Remote Length Lemma.

Lemma 3 [Remote Length Lemma]. *Let x and y be the endpoints of an edge in the $\sqrt{2}$-skeleton of a set S of points in the plane. Let p, q, r, and s be four other distinct points of S such that pq intersects xy, rs intersects xy, pq does not intersect rs and p and s lie on the same side of the line through xy. Then either $|pq| > |qr|$ or $|rs| > |qr|$.*

Proof. Refer to Fig. 5. If $|qr| \leqslant |xr|$, then since the length lemma implies that $|xr| < |rs|$, we have that $|qr| < |rs|$ as required. The remaining case has $|qr| > |xr|$. Now angle pqr is less than angle xqr, which in turn is less than $xq'r$, where q' is the point on segment qr at distance $|xr|$ from r. Points q' and x are on the circle with centre r and radius $|rx|$, thus we may conclude that angle $xq'r$ is less than $\pi/2$. Therefore angle pqr is also less than $\pi/2$. By lemma 1 angle rpq is less than $\pi/4$ and by considering triangle pqr we conclude that angle prq must be greater than $\pi/4$. These angle relationships imply that $|qr| < |pq|$ as required. $\square$

Before we come to our main theorem we include a useful lemma from elementary geometry.

Lemma 4 [Triangle Lemma]. *Let xy be a line segment contained in the closure of triangle abc, then $|xy| \leqslant \max\{|ab|, |ac|, |bc|\}$.*

We are now able to prove the main result.

Theorem 1. *Let S be a set of points in the plane, then $\sqrt{2}$-skeleton$(S) \subseteq$ MWT(S).*

Proof. Assume to the contrary, that there exists an edge xy in the $\sqrt{2}$-skeleton of S that is not contained in the MWT(S). Since MWT(S) is maximal, there exists a set of MWT(S) edges that intersect xy. Let $C = \{c_1, \ldots, c_r\}$ be the set of such edges indexed in nondecreasing order of length. If the edges in C are removed from MWT(S) an empty polygonal region P results. We shall show that region P can be retriangulated by a set of edges that contain xy whose total length is less than that of C. We will build up the alternative triangulation T' in a series of r stages so that after the ith stage the partial triangulation T' will have the following properties.

Property A. *The length of any edge added to the alternative triangulation is less than the length of c_i, and thus also c_j, $j = i \cdots r$.*

Property B. *The alternative partial triangulation T' forms a triangulation of a polygon P' whose vertex set $V(T')$ is a subset of the vertex set of P.*

Property C. *If $e = (u, v)$ is an edge in C and $u \in V(T')$ then the portion of e between u and its intersection with xy lies within P'.*

We begin the construction of the alternative triangulation by including the edge xy. Next consider $c_1 = (t_1, b_1)$, the shortest edge in MWT(S) that intersects xy. If edges xt_1, yt_1, xb_1, yb_1 exist as chords in P, we add them to the alternative triangulation and conclude stage one. The added edges form a triangulation of subpolygon $P' = xt_1yb_1$ thus satisfying Property B. The length lemma implies that all added edges have length less than c_1 thus satisfying Property A. Segment xy cuts each chord of C into two pieces. Property C is satisfied as any such piece of a chord in C containing endpoint t_1 lies within triangle xt_1y and any piece of a chord in C containing endpoint b_1 lies within triangle yb_1x.

In general edges xt_1, yt_1, xb_1, and yb_1 may not exist as chords of P. This would happen when P contains points interior to the quadrangle xt_1yb_1, as in Fig. 6. Consider the case where P contains points interior to triangle xt_1y. The case where P may also contain points interior to xb_1y can be treated similarly. All points in P are endpoints of edges of MWT(S) that intersect xy, thus the points of P interior to triangle xt_1y can be ordered by the order in which their adjacent crossing MWT edges

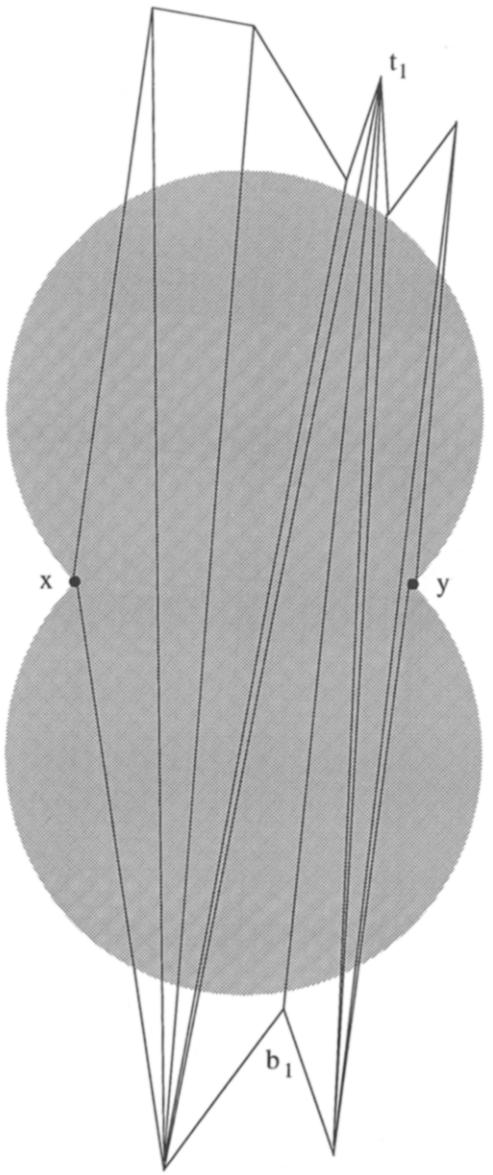

Fig. 6. Minimum weight triangulation of P.

cross xy. This is the same ordering in which these vertices appear around the boundary of P from x to y. For an example see Fig. 7. Let $P_{xt_1y} = \{p_1 \cdots p_q\}$ be the ordered set of points of S that lie within the closure of triangle xt_1y including $x = p_1$, $t_1 = p_k$ and $y = p_q$.

Claim 1. *Either p_ip_{i+1}, $i = 1 \cdots q-1$ is an original edge of P or p_ip_{i+1} is a chord within P.*

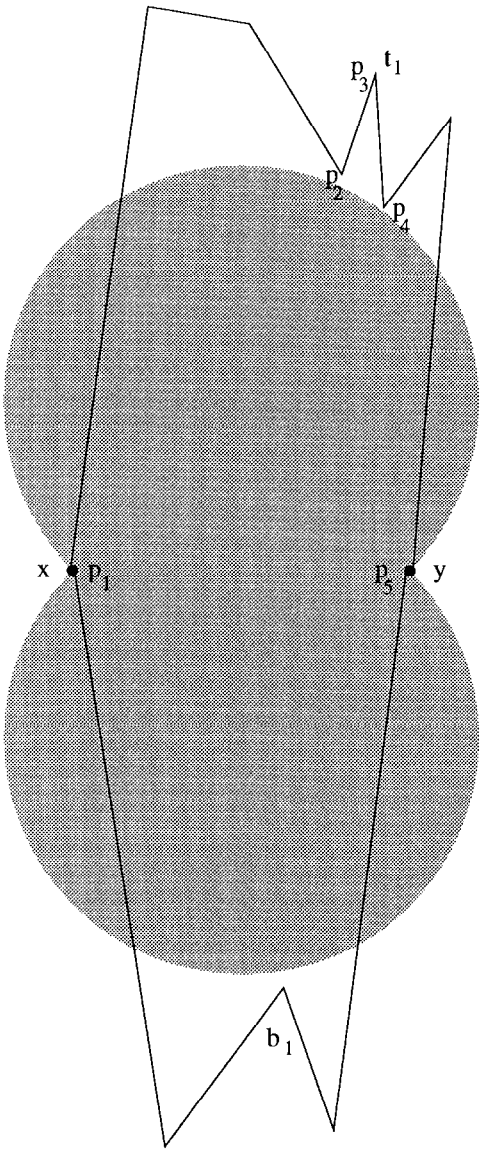

Fig. 7. Ordering the points of P that lie within triangle xt_1y.

Proof of Claim 1. If p_ip_{i+1} is not an original edge of P, then the portion of P between p_i and p_{i+1} lies outside triangle xt_1y. Let z_i be the intersection point of a MWT edge adjacent to p_i with xy that is closest to y. See Fig. 8. The proof then follows from the fact that the quadrangle $p_iz_iz_{i+1}p_{i+1}$ is empty. □

It follows from the claim that $p_1 \cdots p_q$ is an empty subpolygon of P. We include in the alternative triangulation T' a triangulation of the above mentioned subpolygon.

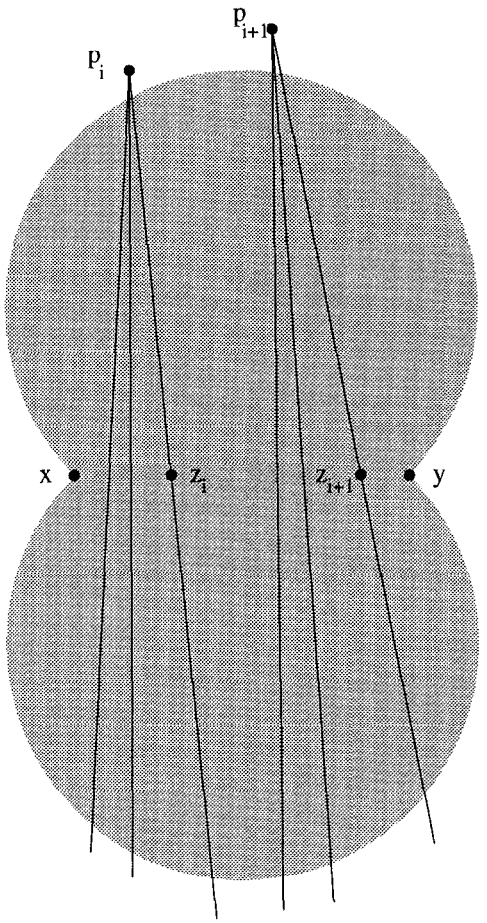

Fig. 8. Quadrangle $p_i z_i z_{i+1} p_{i+1}$ contains no points of P.

This addition satisfies Property B. Claim 1 and the ordering of P_{xt_1y} imply that the subpolygon $z_{i-1}p_{i-1}p_iz_i$ contains no points of P and lies within the polygon $p_1 \cdots p_q$. Thus Property C also holds for the newly constructed triangulation.

By the Triangle Lemma all the new edges will have length less than or equal to the longest among xy, xt_1, and t_1y and by the length lemma will thus be shorter than c_1. Therefore Property A also holds.

We now proceed by induction on the number of stages. Assume that after the (i-1)th stage an alternative partial triangulation T' of P has been constructed that satisfies Properties A, B and C. Consider $c_i = (t_i, b_i)$ the ith shortest edge in MWT(S) that intersects xy. If c_i lies within P', then we do not add to the alternative triangulation and Properties A, B and C trivially continue to hold. Otherwise c_i must cross the boundary of P′ in either one or two places, as t_i or b_i or both may lie outside P'.

Without loss of generality assume it is t_i that lies outside of P' such that $t_i z_i$ intersects boundary edge pq of P'. If b_i also lies outside of P' it can be handled similarly.

As in the basis case let $P_{pt_iq} = \{p_1 \cdots p_s\}$ be the points of P that lie within the closure of triangle pt_iq, ordered by the order in which their adjacent MWT edges cross xy. We again have that $p_1, \ldots p_s$ is an empty subpolygon of P. We include in the alternative triangulation an arbitrary triangulation of this subpolygon thus satisfying Property B. Recall that z_i is the intersection of a MWT edge adjacent to p_i with xy that is closest to y. Again the ordering of P_{pt_iq} also implies that the polygon $z_{j-1}p_{j-1}p_jz_j$, $j = 2, \ldots, q-1$, contains no points of P and lies within the subpolygon $z_1p_1p_2$, $\ldots, p_sz_s$ which in turn lies in the extended P'. Thus Property C holds for the extended alternate triangulation.

It remains to show that Property A holds for the new alternative triangulation. In order to do this we label all vertices of $V(T')$, which are adjacent to MWT edges in $\{c_1, \ldots, c_i\}$ as 'planned' and denote the planned vertices in $V(T')$ as $\mathrm{PV}(T')$. Now let P'_t be the portion of P' cut off by xy that contains point t_i. (i.e. the top portion)

Claim 2. *Any vertex other than x or y, that lies on the convex hull of P'_t is planned.*

Proof. Assume to the contrary that some vertex w, other than x or y, that lies on the convex hull of P'_t is not planned. When w was added to $V(T')$, either it was t_j for some $c_j, j < i$ (and thus planned) or it was a vertex of P lying inside triangle abc where b was t_j for some $c_j, j < i$, and a and c were previously existing vertices of P'. In the later case w could not lie on the convex hull of P'_t. □

We are now able to show that the newly extended alternative triangulation continues to satisfy Property A. By the Triangle Lemma all the newly added edges will be no longer than the longest of pt_i, t_iq and pq. Thus it remains to show that these three segments are all shorter than c_i. By the inductive assumption pq is shorter than c_{i-1} and thus also shorter than c_i. Now consider pt_i. If p is planned than the remote length lemma implies that pt_i is shorter than either c_j, for some $j < i$ which is adjacent to p or c_i which is adjacent to t_i and since $|c_j| < |c_i|$ for $j < i$ we have that $|pt_i| < |c_i|$. If p is not planned consider two cases. If p lies inside triangle xt_iy then the Triangle Lemma and the Length Lemma imply that $|pt_i| < |c_i|$. Otherwise p lies on the polygonal chain P_{xt_i} of P, between x and t_i. By definition the convex hull of the vertices of P_{xt_i} is convex and thus the segment pt_i lies within a triangle t_iab, where a and b are convex hull vertices and thus also planned vertices. By the Triangle Lemma, pt_i is not longer than the longest of t_ia, ab and bt_i. Since a, b and t_i are all planned, by the remote length lemma (or by the length lemma if a or b is x) we have that the segments at_i, ab and bt_i are all shorter than c_i. Therefore pt_i is shorter than c_i as required. That t_iq is shorter than c_i can be shown similarly, and thus Property A has been established.

After the rth and final stage $V(T') = V(P)$ and Properties B and C ensure that T' is a triangulation of P. Since Property A was true after every stage, we have that the total

length of the edges added in T' is less than the sum of the lengths of the edges in C. This contradicts the minimality of the MWT, thus ruling out the possibility that there exists an edge of the $\sqrt{2}$-skeleton of S that is not contained in the MWT(S).

3. Discussion

One question that is immediately suggested by the previous section is "Why $\sqrt{2}$?". What about other values for β. The proof in the previous section also works for values of β greater than $\sqrt{2}$, but these graphs, in general, have fewer edges thus contributing less to a potential MWT algorithm. When $\beta = 1$, the β-skeleton is the Gabriel graph which we know is not a subgraph of the MWT. What about $1 < \beta < \sqrt{2}$? There are two questions that can be asked here. When $\beta = 1$, the β-skeleton is connected, how high can β go so that the β-skeleton remains connected? When $\beta = \sqrt{2}$, the β-skeleton forms a subgraph of the MWT, how low can β go so that the β-skeleton remains a subgraph of the MWT?

For $\beta = \sqrt{2}$ the β-skeleton is not sufficiently connected to provide the basis for a polynomial time MWT algorithm. Also unfortunately, for any β is greater than 1 the β-skeleton is not, in general, connected. Consider a point set with one point in the center of a circle on which the remaining points are equally spaced. If the size of the point set is large enough the center point will be isolated.

When $\beta = 1/\sin(\pi/3)$, I conjecture that the β-skeleton is a subgraph of the MWT. The only part of the proof from the previous section that fails is the remote length lemma. If $\beta < 1/\sin(\pi/3)$, then the length lemma fails as well and a four point example shows that the β-skeleton is not a subgraph of the MWT.

It should be noted that the $\sqrt{2}$-skeleton can be computed in $O(n \log n)$ time [8].

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