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Curves of genus g on an abelian variety of dimension g

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ABSTRACT

In this paper we prove a general theorem concerning the number of translation classes of curves of genus g belonging to a fixed cohomology class in a polarized abelian variety of dimension g. For $g = 2$ we recover results of Göttsche and Bryan-Leung. For $g = 3$ we deduce explicit numbers for these classes.

1. INTRODUCTION

Let (A, L) be a polarized abelian variety of dimension g and of type $(d_1, \dots, d_g)$ over the field $\mathbb{C}$ of complex numbers. Two curves, C_1 and C_2 in A are called *equivalent*, if there is an $x \in A$ such that $C_1 = t_x C_2$. Here $t_x : A \to A$ denotes the translation by x. We call a class of equivalent curves in A a *translation class* of curves. Equivalent curves in A define the same cohomology class in $H^{2g-2}(A, \mathbb{Z})$. Let $r \in \mathbb{Q}, r > 0$ such that the cohomology class $r \bigwedge^{g-1} c_1(L)$ is contained in $H^{2g-2}(A, \mathbb{Z})$. It is easy to see that the number of translation classes of irreducible reduced curves of genus g in the class $r \bigwedge^{g-1} c_1(L)$ is finite. The problem is to compute this number.

In the case of an abelian surface this problem is equivalent to computing the number of curves of genus 2 in a linear system $|L|$. In fact, if L is of type (d_1, d_2), then $|L|$ contains exactly $d_1^2 \cdot d_2^2$ curves which are equivalent to a given curve of genus 2. The number of curves of genus 2 in $|L|$ has been computed in the special case of a simple abelian surface of type $(1, d)$ by Göttsche and Debarre in

[G] and [D] and finally by Bryan and Leung in [BrL], applying intersection cohomology methods.

In this paper we prove a general theorem (Theorem 2.1) expressing the number of equivalence classes of curves of genus g in the class $r \bigwedge^{g-1} c_1(L) \in H^{2g-2}(A, \mathbb{Z})$ in terms of maximal isotropic subgroups of Jacobian type of a finite symplectic space associated to L. For the definitions see Section 2. In dimension $g \geq 4$ it is difficult to decide whether a given maximal isotropic subgroup is of Jacobian type. However any maximal isotropic subgroup of a simple abelian variety of dimension 2 and 3 is of Jacobian type. For these abelian varieties the theorem can be applied to actually compute the number of translation classes of curves in a fixed cohomology class. For abelian surfaces we obtain the results of Göttsche, Debarre and Bryan-Leung in a slightly more general form. The main application of the theorem is dimension 3, where we compute the number of translation classes of curves of genus 3 for any minimal cohomology class in $H^4(A, \mathbb{Z})$ and any simple abelian threefold. It would be interesting to find generating functions for these numbers.

We would like to thank the referee for suggesting some improvements in the proof of Proposition 4.4.

2. STATEMENTS OF THE RESULTS

First we recall some preliminaries. By a curve we will always mean a complete reduced irreducible curve. By its genus we understand its geometric genus, i.e. the genus of its normalization. A polarization on an abelian variety A is by definition the first Chern class of an ample line bundle L on A. By abuse of notation we denote the polarization by L instead of $c_1(L)$ and consider it as a line bundle. For any polarization L on A the group $K(L)$ is defined to be the kernel of the isogeny

$$\phi_L : A \to \hat{A}, \ x \mapsto t_x^* L \otimes L^{-1}.$$

If L is of type $(d_1, \ldots, d_g)$, then $K(L) \simeq (\mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_g\mathbb{Z})^2$. The group $K(L)$ admits a nondegenerate (multiplicative) alternating form $e^L : K(L) \times K(L) \to \mathbb{C}^*$, so that one can speak of totally isotropic subgroups of $K(L)$ which we will shortly call *isotropic subgroups*. According to [CAV], Corollary 6.3.5 there is a canonical bijection between the sets of

(i) maximal isotropic subgroups of $K(L)$ and

(ii) isogenies $f : (A, L) \to (B, M)$ of polarized abelian varieties (i.e. with f^*M algebraic equivalent to L) onto a principally polarized abelian variety (B, M).

We call a maximal isotropic subgroup *of Jacobian type* if the associated principally polarized abelian variety (B, M) is the Jacobian of a smooth projective curve.

For any polarization L on A there is a unique *dual polarization* $\hat{L}$ on the dual abelian variety $\hat{A}$, characterized by the following 2 equivalent properties (see [BL]).

(i) $\phi_L^* \hat{L} \approx L^{d_1 d_g}$

(ii) $\phi_{\hat{L}} \phi_L = d_1 d_g \mathrm{id}_A$.

Here '$\approx$' denotes algebraic equivalence. The polarization $\hat{L}$ is of type $\left(d_1, \frac{d_1 d_g}{d_{g-1}}, \ldots, \frac{d_1 d_g}{d_2}, d_g\right)$, if L is of type $(d_1, \ldots, d_g)$. Moreover, one has biduality: $\hat{\hat{L}} \approx L$. The main result of the paper is the following theorem:

Theorem 2.1. *Let (A, L) be a polarized abelian variety of dimension $g \geq 2$ and type $(d_1, \ldots, d_g)$ and $r \in \mathbb{Q}, r > 0$, such that $r \bigwedge^{g-1} c_1(L) \in H^{2g-2}(A, \mathbb{Z})$. There is a canonical bijection between the sets of*

(1) *translation classes of curves of genus g in the class $r \bigwedge^{g-1} c_1(L)$.*

(2) *maximal isotropic subgroups in $K(\hat{L}^{r(g-1)! d_2 \cdots d_{g-1}})$ of Jacobian type.*

Here for $g = 2$ the empty product $d_2 \cdots d_{g-1}$ is considered to be equal to 1.

Note that, since a general abelian variety of dimension ≥ 4 is not isogenous to a Jacobian, for such an (A, L) the cardinalities of the sets (1) and (2) of the theorem are both equal to 0. In particular, a generic polarized abelian variety of dimension $g \geq 4$ does not contain any curve of genus g. On the other hand it is difficult to decide whether a given polarization in dimension ≥ 4 is of Jacobian type or not. Consequently the main interest of the theorem is for abelian varieties of dimensions 2 and 3.

It is well known (see e.g. [CAV], Corollary 11.8.2) that any simple principally polarized abelian variety of dimension 2 or 3 is the Jacobian of a nonsingular curve. Hence for $g = 2$ and 3 any maximal isotropic subgroup of a simple abelian variety is of Jacobian type. So in these cases we can be more precise.

Corollary 2.2. *Let (A, L) be a simple abelian surface of type (d_1, d_2). The number of curves of genus 2 in the linear system $|L|$ is $d_1^2 d_2^2 \cdot |\{$maximal isotropic subgroup of $K(L)\}|$.*

Note that if L is of type $(1, d)$, the number of maximal isotropic subgroups is $\sigma(d) = \sum_{m|d} m$ (see Proposition 4.3).

Proof. It remains to show that the number of curves in $|L|$ equivalent to a given curve $C \in |L|$ of genus 2 is $d_1^2 d_2^2$. But if C' is equivalent to C then $C' = t_x^* C$ for some $x \in A$. This implies $x \in K(L)$. Since $K(L)$ is of order $d_1^2 d_2^2$, it suffices to show that $t_x^* C \neq C$ for any $x \in A, x \neq 0$. But if $t_x^* C = C$, then t_x^* has to map the finite set of singularities of C into itself. This is impossible for $x \neq 0$, if we replace C by a suitable translate. In fact, replacing C by $t_{x_0}^* C$ for a general x_0, we obtain the identity $t_{x-x_0}^*(t_{x_0}^* C) = t_{x_0}^* C$ with $x - x_0$ not a division point and such a translation t_{x-x_0} cannot stabilize a finite set of points. □

Denote by $K(d_1, \ldots, d_g)$ the group $(\mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_g\mathbb{Z})^2$ together with the standard symplectic form $e : K(d_1, \ldots, d_g)^2 \to \mathbb{C}^*$ defined as follows. If $f_1, \ldots, f_{2g}$ denotes the standard generators of $K(d_1, \ldots, d_g)$, then

$$e(f_\mu, f_\nu) = \begin{cases} \exp\left(-\frac{2\pi i}{d_\nu}\right) & \text{if} \quad \mu = g + \nu \\ \exp\left(\frac{2\pi i}{d_\nu}\right) & \text{if} \quad \nu = g + \mu \\ 1 & \text{otherwise.} \end{cases}$$

Corollary 2.3. *Let (A, L) be a simple abelian threefold of type (d_1, d_2, d_3). The number of translation classes of curves of genus 3 in the class $r \bigwedge^2 c_1(L)$ for some $r \in \mathbb{Q}, r > 0$ is equal to the number of maximal isotropic subgroups of $K(2rd_1d_2, 2rd_1d_3, 2rd_2d_3)$.*

In particular the number of equivalence classes of curves of genus 3 in $\frac{1}{2d_1d_2} \bigwedge^2 c_1(L)$ is equal to the number of maximal isotropic subgroups of $K\left(\frac{d_3}{d_2}, \frac{d_3}{d_1}\right)$. If L is of type (d, d, d), then there is exactly one class of curves of genus 3 in the class $\frac{1}{2d^2} \bigwedge^2 c_1(L)$.

3. PROOF OF THE THEOREM

We need some preliminaries. Let (A, L) be a polarized abelian variety of dimension g and C a curve of genus g in A generating A. Then we have the following diagram

$$(1) \qquad \begin{array}{ccc} \tilde{C} & \hookrightarrow & J = J(\tilde{C}) \\ \nu \downarrow & & \downarrow \hat{f} \\ C & \hookrightarrow & A \end{array}$$

where ν denotes the normalization map, J the Jacobian of $\tilde{C}$, and f the homomorphism induced by the composed map $\tilde{C} \xrightarrow{\nu} C \hookrightarrow A$. Note that f is an isogeny, since $\tilde{C}$ is of genus g and C generates A as an abelian variety. Let M denote a line bundle on J defining the canonical principal polarization on J.

Lemma 3.1. *For any $e \in \mathbb{N}$ the following statements are equivalent:*

(i) $f^*L \equiv M^{e \deg L}$.

(ii) $[C] = \frac{e}{(g-1)!} \bigwedge^{g-1} c_1(L)$ *in* $H^{2g-2}(A, \mathbb{Z})$.

Here $\equiv$ denotes numerical equivalence and the *degree of L* is defined as $\deg L = d_1 \ldots d_g$ if L is of type $(d_1, \ldots, d_g)$.

Proof. Note first that (i) is equivalent to $\hat{f} \phi_L f = e \deg L \cdot 1_J$, i.e. to the commutativity of the diagram

$$
(2) \qquad \begin{array}{ccc} A & \xrightarrow{\phi_L} & \hat{A} \\ \hat{f}\uparrow & & \downarrow\hat{f} \\ J & \xrightarrow{(e\deg L)_J} & J, \end{array}
$$

where we identify $J = \hat{J}$ under the isomorphism $\phi_M : J \to \hat{J}$. We claim that (2) is equivalent to

$$
(3) \qquad f\hat{f}\phi_L = e\deg L \cdot 1_J.
$$

For the proof note that being an isogeny $f : J \to A$ is an isomorphim in $Hom_{\mathbb{Q}}(J, A) = Hom(J, A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence (2) is equivalent to

$$
\hat{f}\phi_L = e\deg L \cdot f^{-1}.
$$

which in turn is equivalent to (3).

Recall from [CAV] Section 5.4 that for any cycles V and W of complementary dimensions one defines an endomorphism $\delta(V, W)$ of A by

$$
\delta(V, W)(x) = S(V \cdot (t_x^* W - W))
$$

where $S(V \cdot (t_x^* W - W)) = \sum_{i=1}^n r_i x_i \in A$ if $V \cdot (t_x^* W - W) = \sum_{i=1}^n r_i x_i$ as a zero-cycle. Moreover $\delta(V, W)$ depends only on the algebraic equivalence classes of V and W. In particular $\delta(C, L)$ is a well defined endomorphism of A.

Now assume (3) holds. Then

$$
\begin{aligned}
\delta(C, L) &= -f\hat{f}\phi_L && \text{by [CAV], Proposition 11.6.1} \\
&= -e\deg L \cdot 1_A && \text{by (3)} \\
&= \delta\left(\frac{e}{(g-1)!}\bigwedge^{g-1} c_1(L), L\right) && \text{by [CAV], Proposition 5.4.7}
\end{aligned}
$$

Since L is an ample line bundle this implies using [CAV], Theorem 11.6.4 that C is numerically equivalent to $\frac{e}{(g-1)!}\bigwedge^{g-1} c_1(L)$ which is equivalent to (ii), since numerical and homological equivalence coincide on an abelian variety.

Conversely (ii) implies

$$
\delta(C, L) = \frac{e}{(g-1)!}\delta(\bigwedge^{g-1} c_1(L), L)
$$

according to a theorem of Matsusaka (see [M]). But as above $\delta(C, L) = -f\hat{f}\phi_L$ and $\delta(\bigwedge^{g-1} c_1(L), L) = -(g-1)!\deg L$. So this implies (3), which completes the proof of Lemma 3.1. □

Lemma 3.2. *Let (A, L) be a polarized abelian variety of type $(d_1, \ldots, d_g)$ and (B, M) a principally polarized abelian variety. If $g : (A, L) \to (B, M)$ is an isogeny of polarized abelian varieties, i.e. $g^* M \approx L$, then $\hat{g}^* \hat{L} \approx \hat{M}^{d_1 d_g}$.*

Proof. The polarization M being principal, we may identify $B = \hat{B}$ such that

$\phi_M = 1_B$. Then $g^*M \approx L$ implies $\phi_{g^*M} = \phi_L$ and the following diagram commutes

$$\begin{array}{ccccc} A & \xrightarrow{\phi_L} & \hat{A} & \xrightarrow{\phi_{\hat{L}}} & A \\ g\downarrow & & \uparrow\hat{g} & & \downarrow g \\ B & \xrightarrow{1_B} & B & \xrightarrow{\phi_{\hat{g}^*\hat{L}}} & B. \end{array}$$

By definition of $\hat{L}$ we have

$$\phi_{\hat{L}} \circ \phi_L = d_1 d_g 1_A.$$

Hence

$$\phi_{\hat{g}*\hat{L}} \circ g = d_1 d_g g.$$

Since g is an isogeny, this gives

$$\phi_{\hat{g}^*\hat{L}} = d_1 d_g \cdot 1_B.$$

On the other hand

$$\phi_{\hat{M}^{d_1 d_g}} = d_1 d_g \phi_{\hat{M}} = d_1 d_g \cdot 1_B.$$

Hence $\phi_{\hat{g}*\hat{L}} = \phi_{\hat{M}^{d_1 d_g}}$ which implies the assertion. □

Proof of Theorem 2.1. Let C be a curve of genus g in the class $r \bigwedge^{g-1} c_1(L)$ in $H^{2g-2}(A, \mathbb{Z})$. Denote for abbreviation $\beta = r(g-1)! \deg L$. According to Lemma 3.1

$$f^*L \approx M^\beta$$

and the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\phi_L} & \hat{A} \\ f\uparrow & & \downarrow\hat{f} \\ J & \xrightarrow{\beta_J} & J. \end{array}$$

Hence

$$f^*\phi_L^*\hat{f}^*M = \beta^*M \approx M^{\beta^2} \approx f^*L^\beta.$$

This implies

$$\phi_L^*\hat{f}^*M \approx L^\beta.$$

On the other hand by definition of $\hat{L}$ we have $\phi_L^*\hat{L} \approx L^{d_1 d_g}$ and thus

$$\phi_L^*\left(\hat{L}^{\frac{\beta}{d_1 d_g}}\right) \approx L^\beta.$$

It follows

$$\phi_L^*\hat{f}^*M \approx \phi_L^*\hat{L}^{\frac{\beta}{d_1 d_g}}.$$

This implies $\hat{f}^*M \approx \hat{L}^{r(g-1)!d_2\cdots d_{g-1}}$. Hence the isogeny $\hat{f}$ corresponds to a maximal isotropic subgroup in $K(\hat{L}^{r(g-1)!d_2\cdots d_g})$ of Jacobian type. It is clear that equivalent curves lead to the same maximal isotropic subgroup.

Conversely let K be a maximal isotropic subgroup of $K(\hat{L}^{r(g-1)!d_2\cdots d_{g-1}})$ of Jacobian type. Then there is an isogeny

$$\hat{f} : \hat{A} \to \hat{J} = \hat{A}/K$$

such that

$$\hat{f}^*\hat{M} \approx \hat{L}^{r(g-1)!d_2\cdots d_{g-1}}.$$

Let M denote the dual polarization of $\hat{M}$ on $J = \hat{\hat{J}}$. $(J, M) \simeq (\hat{J}, \hat{M})$ is the Jacobian of a smooth curve $\tilde{C}$ of genus g. Let $f : J \to A$ denote the dual isogeny of $\hat{f}$. Then $C := f(\tilde{C})$ is a curve of genus g on A and $f|\tilde{C} : \tilde{C} \to C$ is birational, since C generates the abelian variety A.

We have to show that C is in the class $r \bigwedge^{g-1} c_1(L) \in H^{2g-2}(A, \mathbb{Z})$. Note first that $\hat{L}^{r(g-1)!d_2\cdots d_{g-1}}$ is of type

$$r(g-1)!\left(\frac{\deg L}{d_g}, \frac{\deg L}{d_{g-1}}, \ldots, \frac{\deg L}{d_1}\right).$$

In particular we have

$$\deg f = r^g((g-1)!)^g(\deg L)^{g-1}.$$

So Lemma 3.2 applied to $g = \hat{f}$ yields

$$f^*L^{r(g-1)!d_2\cdots d_{g-1}} = \hat{\hat{f}}^*((\hat{L}^{r(g-1)!d_2\cdots d_{g-1}})^\wedge) \approx \hat{M}^{\frac{((r(g-1)!\deg L)^2}{d_1 d_g}}$$

and thus

$$f^*L \approx \hat{M}^{r(g-1)!\deg L}.$$

On the other hand $f|\tilde{C} : \tilde{C} \to C$ is birational, which gives

$$\begin{aligned}
f^*[C] &= \sum_{x \in K} t_x^*[\tilde{C}] \\
&\equiv \deg f \cdot [\tilde{C}] \\
&= r^g((g-1)!)^{g-1}(\deg L)^{g-1} \cdot \bigwedge^{g-1} c_1(\hat{M}) \\
&= r \bigwedge^{g-1} (r(g-1)! \deg L \cdot c_1(\hat{M})) \\
&= r \bigwedge^{g-1} (f^*c_1(L)) \\
&= f^*(r \bigwedge^{g-1} c_1(L)).
\end{aligned}$$

But for an isogeny f^* is injective on cohomology, implying

$$[C] = r \bigwedge^{g-1} c_1(L).$$

It is easy to see that both maps are inverse to each other. □

4. MAXIMAL ISOTROPIC SUBGROUPS

Suppose $d_1, ..., d_g$ are positive integers with $d_i | d_{i+1}$ for $i = 1, ..., g-1$ and let $K(d_1, ..., d_g) = (\mathbb{Z}/d_1\mathbb{Z} \times ... \times \mathbb{Z}/d_g\mathbb{Z})^2$ denote the finite group with the standard symplectic form $e(\,.\,,\,.\,)$ of Section 2. Recall that a subgroup $H \subset K(d_1, ..., d_g)$ is called *(totally) isotropic* if $e(h, h') = 1$ for all $h, h' \in H$. An isotropic subgroup of $K(d_1, ..., d_g)$ is *maximal isotropic* if and only if it is of order $d := d_1 \dots d_g$. Let

$$\nu(d_1, ..., d_g)$$

denote the number of maximal isotropic subgroups of $K(d_1, ..., d_g)$. In this section we compute the number $\nu(d_1, ..., d_g)$ in some cases. First note that from the definition of the symplectic form $e(\cdot, \cdot)$ we immediately obtain

Proposition 4.1. *Let $d_1', ..., d_g'$ be another set of positive integers with $d_i' | d_{i+1}'$ for $i = 1, ..., g$. If $(d_g, d_g') = 1$ then there is a symplectic isomorphism*

$$K(d_1 d_1', ..., d_g d_g') \simeq K(d_1, ..., d_g) \times K(d_1', ..., d_g').$$

In particular $\nu(d_1 d_1', ..., d_g d_g') = \nu(d_1, ..., d_g) \cdot \nu(d_1', ..., d_g')$.

Proposition 4.2. $\nu(p, ..., p) = \prod_{i=1}^{g} (p^i + 1)$ *for any prime number p.*

Proof. According to Witt's theorem the group $Sp(2g, p)$ acts transitively on the set of maximal isotropic subgroups of $K(p, \cdots, p)$. Hence

$$\nu(p, \cdots, p) = \frac{|Sp(2g, p)|}{|\mathrm{Fix} K_0|}$$

where K_0 denotes the isotropic subgroup generated by $f_1, \dots, f_g$, if $f_1, \dots, f_{2g}$ denotes a symplectic basis of $K(p, \cdots, p)$. An element $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in Sp(2g, p)$ is contained in K_0 if and only if $\alpha = 0$, $\beta = -{}^t\gamma^{-1}$ and $\gamma\, {}^t\delta = \delta\, {}^t\gamma$. Hence

$$K_0 \simeq GL(g, p) \rtimes \mathrm{Sym}_g(p)$$

where $\mathrm{Sym}_g(p)$ denotes the additive group of symmetric $g \times g$-matrices over $\mathbb{F}_p$. This implies

$$\nu(p,\cdots,p) = \frac{|Sp(2g,p)|}{|GL(n,p)|\cdot p^{\frac{g(g+1)}{2}}}$$

$$= \frac{p^{g^2}\prod_{i=1}^{g}(p^{2i}-1)}{\prod_{i=0}^{g}(p^g - p^{i-1})p^{\frac{g(g+1)}{2}}}$$

(by [H], II, Hilfssatz 6.2 and Satz 9.13).

$$= \prod_{i=1}^{g}(p^i+1). \quad \square$$

Proposition 4.3. $\nu(1,...,1,d) = \sigma(d) = \sum_{n|d} n$ *for any positive integer d.*

Proof. Note that $K(1,...,1,d) = K(d)$ and any subgroup of order d of $K(d)$ is maximal isotropic. It is well-known that $(\mathbb{Z}/d\mathbb{Z})^2$ contains exactly $\sigma(d)$ subgroups of order d. $\square$

In the remaining cases it is a little more difficult to compute the number $\nu(d_1,...,d_g)$, mainly because it may happen that there are maximal isotropic subgroups of different types. We do this only for $K(d_1,d_2)$ which turns out to be sufficient in order to determine the number of equivalence classes of curves of genus 3 in any primitive cohomology class of an abelian threefold.

In order to compute $\nu(d,d)$ it suffices according to Proposition 4.1 to compute $\nu(p^n,p^n)$ for every prime power p^n. The different types of maximal isotropic subgroups of $K(p^n,p^n)$ are listed in the following table together with a typical example. For the examples recall the standard generators of $K(p^n,p^n)$ with $e(f_1,f_3) = e(f_2,f_4) = exp(\frac{2\pi i}{p^n})$, $e(f_3,f_1) = e(f_4,f_2) = exp(-\frac{2\pi i}{p^n})$ and $e(f_\nu,f_\mu) = 1$ otherwise. In the following table (and only there) we denote for abbreviation $\mathbb{Z}_{p^m} := \mathbb{Z}/p^m\mathbb{Z}$ for all m.

type	M isomorphic to	restrictions	example
1	$\mathbb{Z}_{p^n}^2$		$\langle f_1,f_2\rangle$
2_k	$\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^{n-k}}\times\mathbb{Z}_{p^k}$	$0<k<n-k<n$	$\langle f_1,p^kf_2,p^{n-k}f_4\rangle$
3	$\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^k}\times\mathbb{Z}_{p^k}$	$2k=n$	$\langle f_1,p^kf_2,p^kf_4\rangle$
$4_{k,l}$	$\mathbb{Z}_{p^{n-l}}\times\mathbb{Z}_{p^{n-k}}\times\mathbb{Z}_{p^k}\times\mathbb{Z}_{p^l}$	$0<l<k<n-k$	$\langle p^lf_1,p^kf_2,p^{n-k}f_4,p^{n-l}f_3\rangle$
5_l	$\mathbb{Z}_{p^{n-l}}\times\mathbb{Z}_{p^k}\times\mathbb{Z}_{p^k}\times\mathbb{Z}_{p^l}$	$0<l<k<n, 2k=n$	$\langle p^lf_1,p^kf_2,p^kf_4,p^{n-l}f_3\rangle$
6_k	$\mathbb{Z}_{p^{n-k}}\times\mathbb{Z}_{p^{n-k}}\times\mathbb{Z}_{p^k}\times\mathbb{Z}_{p^k}$	$0<k<n-k<n$	$\langle p^kf_1,p^kf_2,p^{n-k}f_3,p^{n-k}f_4\rangle$
7	$\mathbb{Z}_{p^k}\times\mathbb{Z}_{p^k}\times\mathbb{Z}_{p^k}\times\mathbb{Z}_{p^k}$	$2k=n$	$\langle p^kf_1,p^kf_2,p^kf_3,p^kf_4\rangle$

Note that there are some obvious restrictions: For example type $4_{k,l}$ only occurs for $n \geq 5$. For types 3, 5_k and 7 the number n is necessarily even.

The following proposition computes the number $\nu(p^n,p^n)(-)$ of maximal isotropic subgroups of $K(p^n,p^n)$ of type $-$ from the previous table.

Proposition 4.4. (1) $\nu(p^n,p^n)(1) = p^{3n-3}(p^2+1)(p+1)$,
(2) $\nu(p^n,p^n)(2_k) = p^{3n-2k-4}(p^2+1)(p+1)^2$,
(3) $\nu(p^n,p^n)(3) = p^{2n-3}(p^2+1)(p+1)$,
(4) $\nu(p^n,p^n)(4_{k,l}) = p^{3n-4l-2k-4}(p^2+1)(p+1)^2$,
(5) $\nu(p^n,p^n)(5_l) = p^{2n-4l-3}(p^2+1)(p+1)$,
(6) $\nu(p^n,p^n)(6_k) = p^{3n-6k-3}(p^2+1)(p+1)$
(7) $\nu(p^n,p^n)(7) = 1$.

Proof. The number $\nu(p^n,p^n)(-)$ of maximal isotropic subgroups of $K(p^n,p^n)$ of type $-$ can always be computed as

$$\nu(p^n,p^n)(-) = \frac{N(-)}{D(-)}$$

where $N(-)$ denotes the number of ordered bases of maximal isotropic subgroups of $K(p^n,p^n)$ of type $-$ and $D(-)$ denotes the number of ordered bases of the corresponding abelian group.

There is a more elegant, though conceptually more involved proof of (1): Let $\tilde{e}_n : (\mathbb{Z}/p^n\mathbb{Z})^4 \times (\mathbb{Z}/p^n\mathbb{Z})^4 \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$ denote the additive version of the multiplicative alternating form $e(\cdot,\cdot) : (\mathbb{Z}/p^n\mathbb{Z})^4 \times (\mathbb{Z}/p^n\mathbb{Z})^4 \longrightarrow \mathbb{C}^*$ of above. The alternating forms $\tilde{e}_n$ form a projective system and define an alternating form $\tilde{e} : (\mathbb{Z}_p)^4 \times (\mathbb{Z}_p)^4 \longrightarrow \mathbb{Z}_p$ (Here $\mathbb{Z}_p$ denotes the ring of p-adic integers). The symplectic group $Sp_4(\mathbb{Z}_p)$ acts transitively on the set of maximal isotropic subgroups of $(\mathbb{Z}_p)^4$. Let S denote the stabilizer of a fixed maximal isotropic subgroup. The set of all maximal isotropic subgroups of $(\mathbb{Z}_p)^4$ can be identified with the smooth scheme $M := Sp_4(\mathbb{Z}_p)/S$ over $\mathbb{Z}_p$. The maximal isotropic subgroups of $(\mathbb{Z}/p^n\mathbb{Z})^4$ can be considered as the $\mathbb{Z}/p^n\mathbb{Z}$-valued points of the scheme M. Since for any smooth scheme N of relative dimension d over $\mathbb{Z}_p$ the $\mathbb{Z}/p^n\mathbb{Z}$-valued points and the $\mathbb{Z}/p\mathbb{Z}$-valued points of N are related by

$$\#N(\mathbb{Z}/p^n\mathbb{Z}) = p^{d(n-1)} \cdot \#N(\mathbb{Z}/p\mathbb{Z})$$

and since M is of relative dimension 3 over $\mathbb{Z}_p$, Proposition 4.2 implies assertion (1).

Assertions (4), (5) and (6) can be deduced from (1), (2) and (3) using the following remark: If $W \subset K(p^n,p^n)$ is any isotropic subgroup then the maximal isotropic subgroups containing W are in natural bijection with the maximal isotropic subgroups of $W^\perp/W$. Thus we find: $\nu(p^n,p^n)(4_{k,l}) = \nu(p^{n-2l},p^{n-2l})(2_{k-l})$, $\nu(p^n,p^n)(5_l) = \nu(p^{n-2l},p^{n-2l})(3)$ and $\nu(p^n,p^n)(6_k) = \nu(p^{n-2k},p^{n-2k})(1)$. □

It is easy to make a similar computation for $\nu(p^m,p^n)$ with $m < n$. However there are considerably more types of maximal isotropic subgroups to distinguish. We omit the corresponding tables and formulas. The following table gives the number $\nu(d_1, d_2)$ for small d_1, d_2, which are computed using Propositions 4.1, 4.3, 4.4 and the corresponding formulas for $m < n$.

d	$\nu(1,d)$	$\nu(d,d)$	(d_1,d_2)	$\nu(d_1,d_2)$
2	3	15	(2,4)	51
3	4	40	(2,6)	60
4	7	151	(2,8)	114
5	6	156	(2,10)	90
6	12	600	(2,12)	204
7	8	400	(3,6)	120
8	15	1335	(3,9)	184
9	13	1201	(3,12)	280
10	18	2340	(4,8)	363
11	12	1464	(4,12)	604
12	28	6040	(5,10)	468
16	31	10191	(6,12)	2040

5. CURVES IN A MINIMAL COHOMOLOGY CLASS

Let (A, L) be a polarized abelian variety of type $(d_1, \ldots, d_g)$. Let $c_1(L)$ denote the cohomology class of the line bundle L. The class $\bigwedge^{g-1} c_1(L) \in H^{2g-2}(A, \mathbb{Z})$ is divisible by $(g-1)!d_1 \cdots d_{g-1}$. The class $\frac{1}{(g-1)!d_1 \cdots d_{g-1}} \bigwedge^{g-1} c_1(L)$ is not divisible in $H^{2g-2}(A, \mathbb{Z})$ and is called the *minimal cohomology class (of dimension* 1*)* in (A, L) (This follows easily from the fact that for a suitable choice of a real basis $x_1, \ldots, x_{2g}$ of the tangent space of A at 0 we have $c_1(L) = -\sum_{i=1}^{g} d_i dx_i \wedge dx_{g+i}$ (see [CAV], Lemma 3.6.4)). Denote by $N_{min}(d_1, \ldots, d_g)$ the number of translation classes of curves of genus g in the minimal cohomology class of (A, L). If L' is a d_1-th root of L, then L' defines a polarization of type $(1, \frac{d_2}{d_1}, \ldots, \frac{d_g}{d_1})$ and the minimal cohomology classes in (A, L) and (A, L') coincide. In particular we have $N_{min}(d_1, \ldots, d_g) = N_{min}(1, \frac{d_2}{d_1}, \ldots, \frac{d_g}{d_1})$. Hence we may always assume that (A, L) is of type $(1, d_2, \ldots, d_g)$.

In the case of an abelian surface (A, L) of type $(1, d)$ the minimal cohomology class is the polarization $c_1(L)$ itself. In this case the number $N_{min}(1, d)$ has been computed by Göttsche [G], Debarre [D] and Bryan - Leung [BL]. Theorem 2.1 and Proposition 4.3 yield

$$N_{min}(d_1, d_2) = \sigma(\frac{d_2}{d_1})$$

Note that the results of Section 4 also give the number of translation classes of curves of genus 2 in non minimal cohomology classes of abelian surfaces.

Now let (A, L) be a simple abelian threefold of type $(1, d_2, d_3)$. Recall from Section 4 that $\nu(m, n)$ denotes the number of maximal isotropic subgroups of $K(m, n)$, hence it equals the number of maximal isotropic subgroups of $K(1, m, n)$. So Corollary 2.3 yields

$$N_{min}(1, d_2, d_3) = \nu(\frac{d_3}{d_2}, d_3) \tag{4}$$

and we obtain from Proposition 4.3:

Proposition 5.1. *For any simple abelian threefold (A, L) of type $(1, d, d)$ we have*

$$N_{min}(1, d, d) = \sigma(d).$$

Similarly we have

Proposition 5.2. *Let (A, L) be a simple abelian threefold of type $(1, 1, d)$ and $d = \prod_{i=1}^r p_i^{n_i}$ the prime decomposition. Then*

$$N_{min}(1, 1, d) = \prod_{i=1}^{r} \nu(p_i^{n_i}, p_i^{n_i})$$

and for any prime number p we have

$$\nu(p^{2m+1}, p^{2m+1}) = (p^2+1)(p+1)\left[p^{6m} + \frac{p^{4m-1}(p^{2m}-1)}{p-1} + \frac{p^{6m}-1}{p^6-1}\right] + \\ + \frac{p+1}{p-1}\left[p^{4m-1}\frac{(p^{2m-2}-1)}{p^2-1} - p^3\frac{p^{6m-6}-1}{p^6-1}\right]$$

for all $m \geq 2$ and

$$\nu(p^{2m}, p^{2m}) = (p^2+1)(p+1)\left[p^{6m-3} + p^{4m-2}\frac{p^{2m-2}-1}{p-1} + p^{4m-3} + p\frac{p^{4m-4}-1}{p^4-1} \right. \\ \left. + p^3\frac{p^{6m-6}-1}{p^6-1}\right] + \frac{p+1}{p-1}\left[p^{4m-2}\frac{p^{2m-4}-1}{p^2-1} - p^6\frac{p^{6m-12}-1}{p^6-1}\right] + 1$$

for all $m \geq 3$.

For smaller values of m the formulas are similar, but simpler.

Proof. By identity (4) we have $N_{min}(1, 1, d) = \nu(d, d)$ and by applying Proposition 4.1 we obtain $\nu(d, d) = \prod_{i=1}^r \nu(p_i^{n_i}, p_i^{n_i})$. Thus we get

$$N_{min}(1, 1, d) = \prod_{i=1}^{r} \nu(p_i^{n_i}, p_i^{n_i}).$$

The expression for $\nu(p^{2m-1}, p^{2m-1})$ and $\nu(p^{2m}, p^{2m})$ is obtained by adding the appropriate terms in Proposition 4.4. □

Similar formulas can be given for $N_{min}(1, d_2, d_3)$ for $d_2 < d_3$. We omit them since they look more complicated. But note that the table on page 9 gives $N_{min}(1, d_2, d_3)$ in some cases. For example $N_{min}(1, 2, 4) = 51$ etc.

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