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Spiral Flames

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Abstract—We describe computations of periodic and meandering spiral patterns in a reaction-diffusion model of flames.

1. INTRODUCTION

The formation of patterns and their spatiotemporal dynamics have been a subject of ongoing interest in many areas, including flames (see, e.g., [1–11]). Thus, there have been experimental observations as well as theoretical descriptions of cellular flames, pulsating flames, spinning flames, polyhedral flames, flames in the form of "rolls", solid flames, and others, executing a fascinating array of complex dynamics, including rotating, modulated rotating, hopping, and ratcheting cellular flames. In other experiments, Pearlman and Ronney [12,13] and Pearlman [14] observed flames in the form of a spiral propagating down a tube containing a premixed gas consisting of a lean mixture of butane (a heavy hydrocarbon) and oxygen highly diluted by helium. The mixture was designed to maximize the Lewis number Le (ratio of the thermal diffusivity of the mixture to the mass diffusivity of the deficient reaction component butane). The observation was not of a steady-state spiral flame. Rather, the spiral was observed for a finite time with subsequent transitions from the spiral state and returns to it, indicating the difficulty in controlling the flame behavior.

In an interesting paper, Scott *et al.* [15] employed a simplified 2D model of the experiment which is an extension of a model of Salnikov [16,17]. The model was thermodiffusive in nature, with heat loss effects included. Whereas, in the experiments, a fresh supply of fuel to sustain the reaction was available ahead of the propagating front, in the 2D model there is no third dimension and no propagation. Thus, to sustain the reaction in the model, a constant source of fuel was introduced. The authors of [15] focused their study on parameters such that the system exhibited a separation of time scales as in an excitable medium, which is known to possess spiral solutions. They then computed spiral solutions for these parameters. Moreover, they proposed that the spirals appear due to excitability. In support of their proposal, they noted that spirals appear not only for $\mathrm{Le} > 1$ but for $\mathrm{Le} \leq 1$ as well. However, the parameters considered are not typical of combustion.

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We consider the same system, focusing on parameters which are representative of combustion. The problem considered does not describe an excitable system. We nevertheless show that spiral structures exist and study their evolution as functions of the parameters. We describe both periodic and meandering spirals. We note that the spirals we determine are stable steady-state solutions of the model under consideration. The fact that spirals are shown to exist for parameters for which the system is not excitable demonstrates that excitability is not a necessary condition for their existence.

2. THE MATHEMATICAL MODEL

We consider the model of a flame employed in [15] for the temperature $\tilde{T}$ and deficient reaction component mass fraction $\tilde{Y}$ in 2D,

$$\begin{aligned}\tilde{\rho}\tilde{c}\tilde{T}_{\tilde{t}} &= \tilde{\kappa}\tilde{\nabla}^2\tilde{T} + \tilde{Q}\tilde{k}\tilde{Y}\exp\left(-\frac{\tilde{E}}{\tilde{R}\tilde{T}}\right) - \tilde{H}\left(\tilde{T}-\tilde{T}_0\right),\\ \tilde{Y}_{\tilde{t}} &= \tilde{D}\tilde{\nabla}^2\tilde{Y} - \tilde{k}\tilde{Y}\exp\left(-\frac{\tilde{E}}{\tilde{R}\tilde{T}}\right) + \tilde{M}.\end{aligned} \tag{1}$$

Here, $\tilde{\rho}, \tilde{c}, \tilde{\kappa}$ denote the density, specific heat, and thermal conductivity of the gas mixture, respectively, $\tilde{D}$ denotes the mass diffusivity of $\tilde{Y}, \tilde{Q}, \tilde{k}, \tilde{E}$ denote the heat release, preexponential factor and activation energy of the chemical reaction, respectively, $\tilde{R}$ is the gas constant, $\tilde{H}$ is a heat loss coefficient, $\tilde{T}_0$ is the unburned temperature, and the constant $\tilde{M}$ corresponds to the continuous supply of fuel, without which a flame could not be self-sustained. The model is considered in the domain $0 < \tilde{x}, \tilde{y} < \tilde{L}$ with no-flux boundary conditions.

In this model, there is no uniformly propagating planar flame solution, as exists in the experimental configuration under consideration. Recall that it is in the study of stability of an unconfined, adiabatic, uniformly propagating planar flame that Le $= 1$ serves to separate the monotonic instability (Le < 1) and the oscillatory instability (Le > 1). Thus, Le $= 1$ does not have the same significance in the present problem as it does in the propagating flame problem.

3. THE HOMOGENEOUS SOLUTION AND ITS STABILITY

The model admits a homogeneous stationary solution $\tilde{T}_H = \tilde{T}_0 + \tilde{Q}\tilde{M}/\tilde{H}$, $\tilde{Y}_H = \tilde{M}\exp(N)/\tilde{k}$, $(N = \tilde{E}/\tilde{R}\tilde{T}_H)$, which will serve as the basic state. We nondimensionalize by introducing $\theta = (\tilde{T}-\tilde{T}_0)/(\tilde{T}_H-\tilde{T}_0)$, $Y = \tilde{Y}/\tilde{Y}_H$. The temporal and spatial variables are nondimensionalized by $\exp(N)/\tilde{k}$ and $\sqrt{\tilde{\kappa}\exp(N)/(\tilde{k}\tilde{\rho}\tilde{c})}$, respectively, so that (1) becomes

$$\begin{aligned}\theta_t &= \nabla^2\theta + h\left(Y\exp\left(\frac{N(1-\sigma))(\theta-1)}{\sigma+(1-\sigma)\theta}\right) - \theta\right),\\ Y_t &= \frac{1}{\text{Le}}\nabla^2 Y - Y\exp\left(\frac{N(1-\sigma))(\theta-1)}{\sigma+(1-\sigma)\theta}\right) + 1,\end{aligned} \tag{2}$$

in the domain $0 < x$, $y < L$ subject to no-flux boundary conditions. Here, Le $= \tilde{\kappa}/(\tilde{D}\tilde{\rho}\tilde{c})$ is the Lewis number, $h = \tilde{H}\exp(N)/(\tilde{k}\tilde{\rho}\tilde{c})$, $\sigma = \tilde{T}_0/\tilde{T}_H$. The basic state is given by $\theta = Y = 1$.

We next consider the stability of this state. We consider perturbations of the form $\cos[\pi(k_1x + k_2y)/L]$. The dispersion relation for the growth rate ω of the perturbations is

$$\left[h + k^2\left(1 - \frac{h}{\text{Le}}(Z-1)\right) + \frac{k^4}{\text{Le}}\right] + \left[1 - h(Z-1) + k^2\left(1 + \frac{1}{\text{Le}}\right)\right]\omega + \omega^2 = 0, \tag{3}$$

where $k^2 = \pi^2(k_1^2 + k_2^2)/L^2$ and $Z = N(1-\sigma)$ is the Zeldovich number.

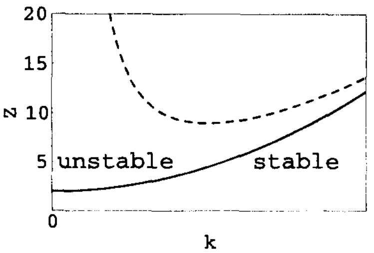

Figure 1. Neutral stability boundary.

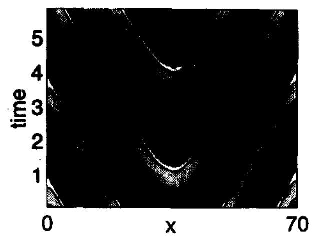

Figure 2. Space time plot of the spiral wave (mass fraction).

Monotonic ($\mathrm{Re}\,\omega = \mathrm{Im}\,\omega = 0$—solid curve) and oscillatory ($\mathrm{Re}\,\omega = 0, \mathrm{Im}\,\omega \neq 0$—dashed curve) stability boundaries are shown as Z vs. k in Figure 1 for Le $= 4$ and $h = 1$. We see that the dominant instability is the oscillatory instability since its curve lies below the monotonic instability curve. We note that the plot exhibited for Le $= 4$ is typical for Le $>$ Le$_c = (\sqrt{1+h} - \sqrt{h})^2$. Below, Le$_c$ the dominant instability becomes the monotonic instability. Thus, here it is Le $=$ Le$_c$, rather than Le $= 1$, which separates the two instabilities. Note that Le$_c$ decreases monotonically with h, from Le$_c = 1$ at $h = 0$. In particular, for $h = 3487$ as in [15], Le$_c = 0.00014$. The minimum of the oscillatory neutral instability curve corresponds to $k = 0$ and is given by $h(Z - 1) = 1$. Increasing Z or h is destabilizing. In addition, decreasing Le enhances the monotonic instability, whereas increasing Le enhances the oscillatory instability only if $h > h_c$.

We note that $h \gg 1$ corresponds to the regime considered in [15].

4. SPIRAL FLAMES

In this section, we present the results of numerical computations of flames in the form of spiral waves. In [15], a spiral solution of (1) was computed (using a different nondimensionalization) for $Z \simeq 0.9$, a value significantly below those typical for combustion. In our computations, we fixed $N = 4.5$ and $\sigma = 0.2$ so that $Z = 3.6$, which is more representative of typical values for combustion. Indeed, this value is close to the stability boundary for an unconfined uniformly propagating planar flame when Le $= 4$ [5]. The system length was fixed at 69.86. The parameters Le, h were varied in the ranges Le $\in [1, 5.5]$ and $h \in [0.3, 30]$. In [15], the parameters Le, h were taken as Le $= 1$ and $h = 3487$. We note that in the experiments where spiral flames were observed, the Lewis number was estimated to be in the range Le $\in [3.88, 4.315]$ [14].

We employ an explicit finite difference scheme which is first order in time and second order in space. For all computations, we used $\Delta t = 0.00024$ and $\Delta x = \Delta y = 0.14$, which were sufficient to resolve the spiral solutions. We solve the initial boundary value problem, marching forward in time until a steady state is achieved. Thus, the solutions we compute are necessarily stable.

Figure 2 shows a space-time plot of a cross-section ($y = L/2$) of the mass fraction for a spiral (Le $= 4, h = 3$) which rotates outward from the spiral center. The space-time plot for the temperature is similar. The motion, including that of the spiral tip is periodic in time. The spiral was created from initial conditions corresponding to the homogeneous solution in the lower half of the domain and a one-dimensional pulse, localized in x at $x = L/2$, in the upper half of the domain. The pulse begins to move and evolves to the spiral.

Spirals corresponding to increasing values of h are shown in Figure 3. As h increases, the motion of the spiral tip becomes more complex and simple periodic spirals evolve to meandering spirals, as is typical in other systems (see, e.g., [18,19]). The spiral tip is defined by the intersection of the contour lines $\theta = 2.0$, $Y = 0.2$. We note that the curves describing the motion of the meandering spiral tips in Figure 3 are not generally closed. We also see that as h increases, the number of spiral arms in a given region of space increases. As a result, more resolution is required, so we restrict our computations to $h \leq 30$, though h may become quite large before extinction occurs.

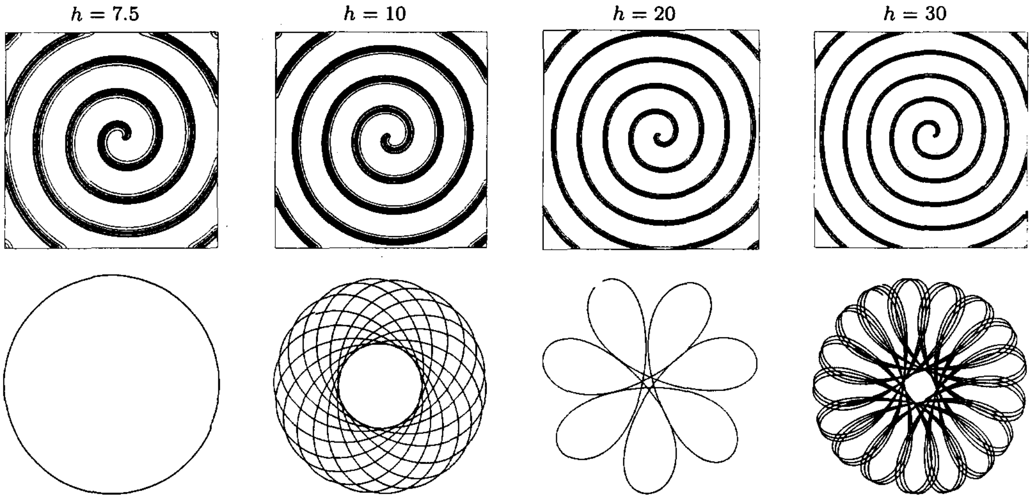

Figure 3. Temperature contours of spirals and their tip motions.

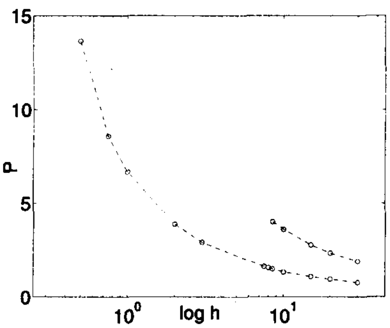

Figure 4. Period (P) of tip motion.

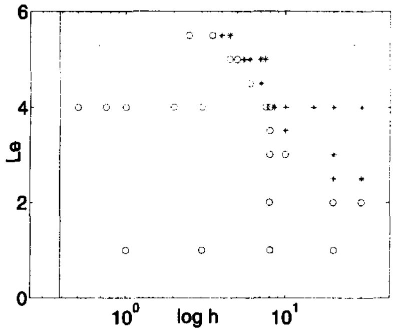

Figure 5. Phase portrait.

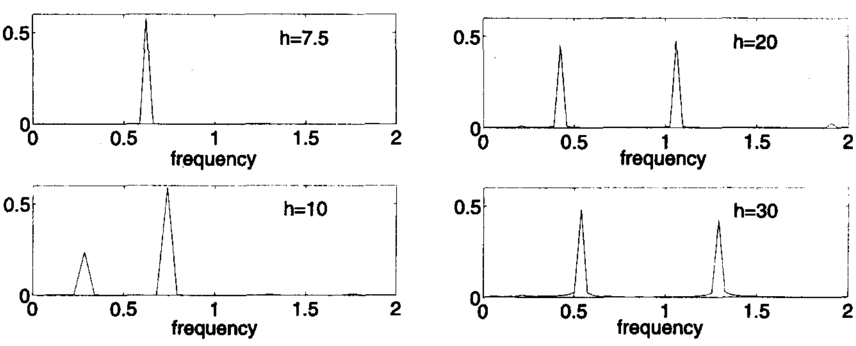

Figure 6. Fourier transform of the tip motion.

Decreasing h leads to an increase in the spiral period (see Figure 4). When h becomes sufficiently small, the spiral disappears as its period becomes infinite.

The results of our computations are summarized in Figure 5. Here, circles correspond to simple periodic spirals and stars to meandering spirals. The solid line $h = h_c \approx 0.385$ corresponds to the analytically predicted stability boundary $h = (Z-1)^{-1}$ of the homogeneous solution for $Z = 3.6$, which is stable for $h < h_c$. The boundary between periodic and meandering spirals depends on h and on Le as seen in Figure 5. The period(s) of the spiral tip motion are shown in Figure 4 for Le = 4. We see that the transition from periodic to meandering spirals corresponds to the introduction of a second (larger) period. Indeed, a Fourier analysis of the meandering spiral tip motion reveals that there are two dominant frequencies, as seen in Figure 6. The level of any additional frequencies is so low that they are barely seen in the figures. Thus, the motion is essentially the sum of the two sinusoidal motions whose periods are shown in Figure 4. Harmonics and combination frequencies were found to be less than 7% of the smaller of the two primary spectral components. Figure 6 illustrates the growth of the meander frequency (the lower frequency) as h increases.

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