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A Note on the Total Chromatic Number of Halin Graphs with Maximum Degree 4

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Abstract—In this paper, we prove that $\chi_T(G) = 5$ for any Halin graph G with $\Delta(G) = 4$, where $\Delta(G)$ and $\chi_T(G)$ denote the maximal degree and the total chromatic number of G, respectively.

Keywords—Halin-graph, Total colouring, Total chromatic number.

1. INTRODUCTION

We quote two definitions.

DEFINITION 1.1. *(See [1].) For a 3-connected plane graph $G(V, E, F)$, if all edges on the boundary of one face f_0 of the face set F are removed, it becomes a tree, and the degree of each vertex of $V(f_0)$ is three, then graph G is called a* **Halin graph**, *the vertices on $V(f_0)$ are called* **exterior vertices** *of G, and the others* **interior vertices** *of G.*

DEFINITION 1.2. *(See [2].) If all of the elements in $V \cup E$ of the graph $G(V, E)$ can be coloured by k colours such that no two adjacent or incident elements have the same colour, then this colouring is called a k-***total colouring** *of G; and*

$$\chi_T(G) = \min\{k \mid k\text{-total colouring of } G\}$$

is called the (vertex-edge) **total chromatic number** *of G.*

From this, the total colouring conjecture can be written in the form

$$\chi_T(G) \le \Delta(G) + 2 \text{ for any graph } G.$$

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In [3], the (vertex) chromatic number $\chi(G)$, the edge chromatic number $\chi'(G)$, and the total chromatic number $\chi_T(G)$ of the Halin graphs G were studied in great detail, and it was shown that if G is a Halin graph with $\Delta(G) \geq 5$, then

$$\chi_T(G) = \Delta(G) + 1.$$

In the end of [3], it was pointed out that determining $\chi_T(G)$ for any Halin graph G with $\Delta(G) = 3, 4$ is an open problem. In this paper, the case when $\Delta(G) = 4$ is completely solved.

The other terms and notations can be found in [4–6].

2. THE HALIN GRAPHS WITH $\Delta(G) = 4$

We denote by W_p the wheel graph of order p.

LEMMA 2.1. *Let $G(G \neq W_p)$ be a Halin graph, then there exists an interior vertex w which adjacent vertices only one is interior vertex and the others exterior vertices.*

PROOF. Consider the longest path in graph $T' = G - E(f_0)$, where $E(f_0)$ denotes the edges set in the boundary of outer face f_0, w denote the second vertex or the reverse second vertex, then w has the property in Lemma 2.1.

Denote by W the set of all w that satisfy conditions in Lemma 2.1.

THEOREM 2.1. *For wheel graph W_5 of order five, we have*

$$\chi_T(W_5) = 5.$$

The proof is obvious. A 5-total colouring of W_5 is shown in Figure 1.

THEOREM 2.2. *For any Halin graph G with $\Delta(G) = 4$, have*

$$\chi_T(G) = 5.$$

PROOF. We use induction on the number $p = |V(G)|$.

When $p = 5$, $G = W_5$, by Theorem 2.1, the conclusion is true; when $p = 6$, does not exist Halin graph G with $\Delta(G) = 4$; and when $p = 7$, there is a unique Halin graph G with $\Delta(G) = 4$, and the conclusion is also true. Now we assume that the conclusion is true for $p = k - 1$ $(k \geq 8)$, and consider the case $p = k$.

Obviously, $\chi_T(G) \geq 5$. Hence, it is enough to prove that $\chi_T(G) \leq 5$, that is, only to prove that G has a 5-total colouring.

Among all vertices of W, let w be one of minimum degree, i.e.,

$$d(w) = \min\{d(v) \mid v \in W\},$$

where W is the set of all w that satisfy the condition in Lemma 2.1.

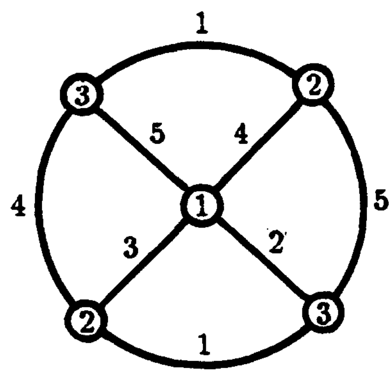

Figure 1. W_5.

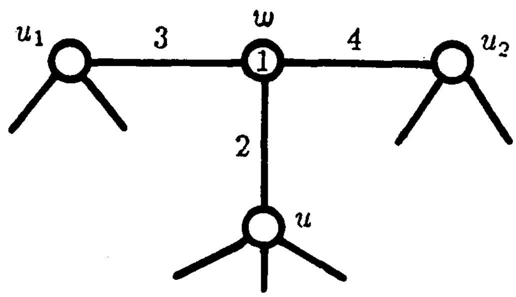

Figure 2. G_{01}.

CASE 1. $d(w) = 3$.

Denote by y_1, y_2 two exterior vertices adjacent to w, and denote by u an interior vertex adjacent to w. Suppose that u_1, u_2 which are different from y_1, y_2 are adjacent exterior vertices of y_1, y_2, respectively, that is,

$$u_1 \in N(y_1) - \{y_2, w\}, \qquad u_2 \in N(y_2) - \{y_1, w\}.$$

We consider the graph

$$G_{01} = G - \{y_1, y_2\} + \{u_1 w, u_2 w\}.$$

Obviously, G_{01} is a Halin graph with $\Delta(G_{01}) = 4$ and $|V(G_{01})| = k - 2$.

By the induction hypothesis, G_{01} has a 5-total colouring σ_0. On the basis of σ_0, we make a 5-total colouring σ of G.

Denote by $C = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ the set of five colours in σ_0, and let

$$Y = \Big(V(G) \bigcup E(G)\Big) - \{y_1, y_2, u_1y_1, y_1y_2, u_2y_2, wy_1, wy_2\}.$$

We may suppose without loss of generality that

$$\sigma_0(w) = \alpha_1, \quad \sigma_0(uw) = \alpha_2, \quad \sigma_0(u_1w) = \alpha_3, \quad \sigma_0(u_2w) = \alpha_4.$$

A sketch of the 5-total colouring σ_0 of G_{01} is shown in Figure 2, where the number i denotes the colour α_i $(i = 1, 2, \ldots, 5)$. Thus, $\sigma_0(u_1)$ can only take α_1, or α_2, or α_4, or α_5, and $\sigma_0(u_2)$ can only take α_1, or α_2, or α_3, or α_5, that is, by notation of vector, $(\sigma_0(u_1), \sigma_0(u_2))$ can only take the following 16 pairs of colours:

$$\begin{array}{llll}
(\alpha_1,\alpha_1), & (\alpha_1,\alpha_2), & (\alpha_1,\alpha_3), & (\alpha_1,\alpha_5), \\
(\alpha_2,\alpha_1), & (\alpha_2,\alpha_2), & (\alpha_2,\alpha_3), & (\alpha_2,\alpha_5), \\
(\alpha_4,\alpha_1), & (\alpha_4,\alpha_2), & (\alpha_4,\alpha_3), & (\alpha_4,\alpha_5), \\
(\alpha_5,\alpha_1), & (\alpha_5,\alpha_2), & (\alpha_5,\alpha_3), & (\alpha_5,\alpha_5).
\end{array}$$

Since $\alpha_1 \notin \{\sigma_0(u_1), \sigma_0(u_2)\}$, and by symmetry of u_1 and u_2, it is enough to prove that G has a 5-total colouring σ for each case of the following 6 pairs of colours:

$$\begin{array}{lll}
(\alpha_2,\alpha_2), & (\alpha_2,\alpha_3), & (\alpha_2,\alpha_5), \\
 & (\alpha_4,\alpha_3), & (\alpha_4,\alpha_5), \\
 & & (\alpha_5,\alpha_5).
\end{array}$$

SUBCASE 1.1. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_2, \alpha_2)$ or (α_5, α_5).

Let

$$\begin{aligned}
&\sigma(y_1y_2) = \sigma_0(w), \quad \sigma(y_2) = \sigma(u_1y_1) = \alpha_3, \\
&\sigma(u_2y_2) = \sigma(wy_1) = \alpha_4, \quad \sigma(wy_2) = \alpha_5, \\
&\sigma(y_1) = \alpha_5, \qquad \text{if } \sigma_0(u_1) = \alpha_2, \text{ or} \\
&\sigma(y_1) = \alpha_2, \qquad \text{if } \sigma_0(u_1) = \alpha_5, \\
&\sigma(y) = \sigma_0(y), \qquad y \in Y.
\end{aligned}$$

Obviously, this σ is a 5-total colouring of G.

SUBCASE 1.2. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_2, \alpha_3)$.

Let

$$\begin{aligned} \sigma(y_1y_2) &= \alpha_1, \quad \sigma(y_2) = \alpha_2, \\ \sigma(u_1y_1) &= \sigma(wy_2) = \alpha_3, \\ \sigma(u_2y_2) &= \sigma(wy_1) = \alpha_4, \\ \sigma(y_1) &= \alpha_5, \quad \sigma(y) = \sigma_0(y), \qquad y \in Y. \end{aligned}$$

Such a colouring σ is obviously a 5-total colouring of G.

SUBCASE 1.3. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_2, \alpha_5)$.

This case is the same as Subcase 1.2 except the colour on the vertex u_2.

SUBCASE 1.4. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_4, \alpha_3)$.

Let

$$\begin{aligned} \sigma(y_1y_2) &= \alpha_1, \quad \sigma(y_1) = \alpha_2, \\ \sigma(u_1y_1) &= \sigma(wy_2) = \alpha_3, \\ \sigma(u_2y_2) &= \sigma(wy_1) = \alpha_4, \\ \sigma(y_2) &= \alpha_5, \quad \sigma(y) = \sigma_0(y), \qquad y \in Y. \end{aligned}$$

This σ is obviously a 5-total colouring of G.

SUBCASE 1.5. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_4, \alpha_5)$.

This case is the same as Subcase 1.2 except the colours on u_1 and u_2.

CASE 2. $d(w) = 4$.

Suppose that y_1, y_2, y_3 are exterior vertices adjacent to w, and that u is an interior vertex adjacent to w, and $u_1y_1, u_2y_3 \in E(G)$. We consider the graph

$$G_{02} = G - \{y_1, y_2, y_3\} + \{u_1w, u_2w\}.$$

Obviously, G_{02} is a Halin graph with $\Delta(G_{02}) = 3$ or 4, and $|V(G_{02})| = k - 3$.

By the induction hypothesis, there exists a 5-total colouring σ_0 of G_{02}. Now, on the basis of σ_0, we make a 5-total colouring σ of G. Let

$$Y = \left(V \bigcup E\right) - \{y_1, y_2, y_3, u_1y_1, u_2y_3, wy_1, wy_2, wy_3, y_1y_2, y_2y_3\}.$$

We may suppose without loss of generality that a 5-total colouring σ_0 of G_{02} is the same as that in Case 1, and it is enough to prove that there exists a 5-total colouring σ of G for each of the six subcases in Case 1. The proofs of six subcases are similar to those in Case 1.

SUBCASE 2.1. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_2, \alpha_2)$.

Let

$$\begin{aligned} \sigma(y_1y_2) &= \alpha 1, \quad \sigma(y_2y_3) = \alpha_2, \\ \sigma(u_1y_2) &= \sigma(wy_3) = \sigma(y_2) = \alpha_3, \\ \sigma(wy_1) &= \sigma(u_2y_3) = \alpha_4, \\ \sigma(y_1) &= \sigma(y_3) = \sigma(wy_2) = \alpha_5, \\ \sigma(y) &= \sigma_0(y), \qquad y \in Y. \end{aligned}$$

Obviously, this σ is a 5-total colouring of G.

SUBCASE 2.2. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_5, \alpha_5)$.

Let

$$\begin{aligned} \sigma(y_1y_2) &= \alpha_1, \quad \sigma(y_1) = \sigma(y_2y_3) = \alpha_2, \\ \sigma(y_3) &= \sigma(u_1y_1) = \sigma(wy_2) = \alpha_3, \\ \sigma(wy_1) &= \sigma(u_2y_3) = \alpha_4, \\ \sigma(y_2) &= \sigma(wy_3) = \alpha_5, \quad \sigma(y) = \sigma_0(y), \qquad y \in Y. \end{aligned}$$

Such a colouring σ is obviously a 5-total colouring of G.

SUBCASE 2.3. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_2, \alpha_3)$.

This case is the same as Subcase 2.1 except the colour on the vertex u_2.

SUBCASE 2.4. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_2, \alpha_5)$.

Let

$$\begin{aligned} \sigma(y_2y_3) &= \alpha_1, \quad \sigma(y_1y_2) = \sigma(y_3) = \alpha_2, \\ \sigma(y_2) &= \sigma(u_1y_1) = \sigma(wy_3) = \alpha_3, \\ \sigma(u_2y_3) &= \sigma(wy_1) = \alpha_4, \\ \sigma(y_1) &= \sigma(wy_2) = \alpha_5, \quad \sigma(y) = \sigma_0(y), \qquad y \in Y. \end{aligned}$$

This σ is obviously a 5-total colouring of G.

SUBCASE 2.5. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_4, \alpha_3)$.

This case is the same as Subcase 2.4 except the colours on u_1 and u_2.

SUBCASE 2.6. $(\sigma_0(u_1), \sigma_0(u_2)) = (\alpha_4, \alpha_5)$.

This case is same as Subcase 2.4 except the color on u_1.

Combining Case 1 and Case 2 for $p = k$, there exists the 5-total colouring σ of G.

By the induction principle, Theorem 2.2 is proved.

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