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Boundary blow-up solutions in the unit ball: Asymptotics, uniqueness and symmetry

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ABSTRACT

We calculate the full asymptotic expansion of boundary blow-up solutions (see Eq. (1) below), for any nonlinearity f. Our approach enables us to state sharp qualitative results regarding uniqueness and radial symmetry of solutions, as well as a characterization of nonlinearities for which the blow-up rate is universal. Lastly, we study in more detail the standard nonlinearities $f(u) = u^p$, $p > 1$.

1. Introduction

Let B denote the unit ball of $\mathbb{R}^N$, $N \geqslant 1$, and let $f \in C^1(\mathbb{R})$. We study the equation

$$\begin{cases} \Delta u = f(u) & \text{in } B, \\ u = +\infty & \text{on } \partial B, \end{cases} \tag{1}$$

where the boundary condition is understood in the sense that

$$\lim_{x \to x_0,\, x \in B} u(x) = +\infty, \quad \text{for all } x_0 \in \partial B$$

and where f is assumed to be positive at infinity, in the sense that

$$\exists a \in \mathbb{R} \quad \text{s.t.} \quad f(a) > 0 \quad \text{and} \quad f(t) \geqslant 0, \quad \text{for } t > a. \tag{2}$$

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A function u satisfying (1) is called a boundary blow-up solution or simply a large solution. Existence of a solution of (1) is equivalent to the so-called Keller–Osserman condition:

$$\int^{+\infty} \frac{dt}{\sqrt{F(t)}} < +\infty, \quad \text{where } F(t) = \int_a^t f(s)\,ds. \tag{3}$$

For a proof of this fact, see the seminal works of J.B. Keller [8] and R. Osserman [11] for the case of monotone f, as well as [6] for the general case. From here on, we always assume that (3) holds.

Our goal here is to study asymptotics, uniqueness and symmetry properties of solutions. Our approach improves known results in at least two directions: firstly, aside from the necessary condition (3), we need not make *any* additional assumption on f to obtain the sharp asymptotics of solutions. Secondly, we obtain the complete asymptotic expansion of solutions to *all* orders. Here is a summary of our findings.

Theorem 1.1. *Let $f \in C^1(\mathbb{R})$ and assume* (2), (3) *hold. Consider two solutions u_1, u_2 of* (1)*. Then,*

$$\lim_{x \to x_0,\, x \in B} u_1(x) - u_2(x) = 0, \quad \textit{for all } x_0 \in \partial B.$$

More precisely, there exists a constant $C = C(u_1, u_2, N, F) > 0$, such that for all $x \in B$,

$$\left|u_1(x) - u_2(x)\right| \leqslant C \int_{u_2(x)}^{+\infty} \frac{dt}{F(t)}\,dt. \tag{4}$$

In addition,

$$\left|F(u_1) - F(u_2)\right| \in L^\infty(B). \tag{5}$$

Estimates on the gradient of solutions can be obtained for a restricted class of nonlinearities, namely

Theorem 1.2. *Let $f \in C^1(\mathbb{R})$ and assume* (2) *and* (3) *hold. Assume in addition that f is increasing up to a linear perturbation, i.e. there exist an increasing function $\tilde{f}$ and a constant K such that*

$$f(t) = \tilde{f}(t) - Kt, \quad \textit{for all } t \in \mathbb{R}. \tag{6}$$

Consider two solutions u_1 and u_2 of (1)*. Then,*

$$\left|\nabla(u_1 - u_2)\right| \in L^\infty(B). \tag{7}$$

The previous theorems can be used to study qualitative properties of solutions, such as uniqueness and symmetry. We begin with the question of uniqueness of solutions of (1). The following conjecture is due to P.J. McKenna [5].

Conjecture 1.3. *(See [5].) Let $N \geqslant 1$, Ω a smoothly bounded domain of $\mathbb{R}^N$ and $f \in C^1(\mathbb{R})$ a function such that* (2) *and* (3) *hold. Assume in addition that the function $\tilde{f}$ defined by*

$$f(t) = \tilde{f}(t) - \lambda_1 t, \quad \textit{for all } t \in \mathbb{R} \tag{8}$$

is increasing, where $\lambda_1 = \lambda_1(-\Delta;\Omega) > 0$ denotes the principal eigenvalue of the Laplace operator with homogeneous Dirichlet boundary condition. Then, there exists a unique large solution of

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases}$$

As a direct consequence of Theorem 1.1, we prove Conjecture 1.3 in the case $\Omega = B$.

Corollary 1.4. *Let $f \in C^1(\mathbb{R})$ and assume* (2) *and* (3) *hold. Assume in addition that the function $\tilde{f}$ defined by* (8) *is nondecreasing. Then, there exists a unique large solution of* (1).

Remark 1.5. Many uniqueness theorems have been established in the literature (see, e.g., the survey [1]), and they hold for a general class of bounded domains Ω. However, in all of these results, additional assumptions on f are needed, such as convexity.

Proof of Corollary 1.4. Assume first that f is nondecreasing. Let u_1, u_2 denote two large solutions. It suffices to prove that $u_1 \leqslant u_2$. Assume this is not the case and let $\omega = \{x \in B\colon\ w(x) > 0\} \neq \emptyset$, where $w = u_1 - u_2$. Working if necessary on a connected component of ω, we may always assume that ω is connected. Using Theorem 1.1, we see that w solves the equation

$$\begin{cases} \Delta w = f(u_1) - f(u_2) \geqslant 0 & \text{in } \omega, \\ w = 0 & \text{on } \partial\omega. \end{cases}$$

By the Maximum Principle, $w \leqslant 0$ in ω, a contradiction.

Assume now that we only have $f' \geqslant -\lambda_1$. Let $\varphi_1 > 0$ denote an eigenfunction associated to λ_1 and let $\sigma = w/\varphi_1$, where $w = u_1 - u_2$ denotes the difference of two solutions. Assume again that $\omega = \{x \in B\colon\ w(x) > 0\} \neq \emptyset$. By a standard calculation,

$$\nabla \cdot \left(\varphi_1^2 \nabla\sigma\right) = \left(f(u_1) - f(u_2) + \lambda_1(u_1 - u_2)\right)\varphi_1 \geqslant 0 \quad \text{in } \omega.$$

We claim that $\sigma = 0$ on $\partial\omega$, from which the desired contradiction will follow. By (4) and the well-known estimate $\varphi_1 \geqslant c(1 - |x|)$, it suffices to show that

$$\lim_{x \to \partial B} \frac{\int_{u_2(x)}^{+\infty} \frac{dt}{F(t)}\, dt}{1 - |x|} = 0. \tag{9}$$

We shall prove later (see Lemma 2.4) that there exists a radial boundary blow-up solution U of (1) such that $u_2 \geqslant U$. Since U is radial, it follows from (2) that $U'(r) > 0$ for $r = |x|$ close to 1. In particular,

$$U'' \leqslant \Delta U = f(U).$$

Multiplying by U' and integrating the above inequality between r_0 and r close to 1, it follows that $(U')^2/2 \leqslant F(U) + C$. Integrating again between r and 1, we obtain

$$\int_{U(r)}^{+\infty} \frac{dt}{\sqrt{2(F + C)}} \leqslant 1 - r,$$

for r close to 1. So,

$$\frac{\int_{u_2(x)}^{+\infty} \frac{dt}{F(t)}\, dt}{1 - |x|} \leqslant \frac{\int_{U(x)}^{+\infty} \frac{dt}{F(t)}\, dt}{1 - |x|} \leqslant \frac{C}{\sqrt{2F(U(x))}}$$

and (9) follows. □

When $f' \ngeq -\lambda_1$, uniqueness fails in general. One may ask however whether all solutions of (1) are radial. H. Brezis made the following conjecture.

Conjecture 1.6. *(See [4].) Let $f \in C^1(\mathbb{R})$ denote a function such that (2) and (3) hold. Then, every solution of (1) is radially symmetric.*

To our knowledge, the first contribution to the proof of Conjecture 1.6 is due to P.J. McKenna, W. Reichel, and W. Walter (see [10]), using the additional assumption that $\lim_{t\to+\infty} f'(t)/\sqrt{F(t)} = +\infty$. A. Porretta and L. Véron then proved the conjecture, assuming that f is asymptotically convex (see [12]). We improve these results as follows.

Corollary 1.7. *Let $f \in C^1(\mathbb{R})$ and assume that (2) and (3) hold. Let u denote a solution of (1). Assume in addition that, up to a linear perturbation, f is increasing (i.e. (6) holds for some nondecreasing function $\tilde{f}$ and some constant K). Then, u is radially symmetric. Furthermore, $\frac{\partial u}{\partial r} > 0$ in $B \setminus \{0\}$.*

Remark 1.8. In the setting of the classical symmetry result of B. Gidas, W.M. Ni and L. Nirenberg (see [7]), (6) is also assumed in order to prove symmetry. In the same article, the authors give an example of a nonlinearity f failing (6) for which there do exist nonradial solutions of the equation. In the context of large solutions, we do not have such a counter-example. In fact, we expect that none exists, i.e. we believe that Conjecture 1.6 holds. But at this stage, we do not even know whether radial symmetry continues to hold for simple nonlinearities such as $f(u) = u^2(1 + \sin u)$.

Corollary 1.7 is a direct consequence of the moving plane method and Theorem 1.2:

Proof of Corollary 1.7. Let U denote a radial solution of (1). It follows from (2) that U is a nondecreasing function of $r = |x|$ for r close to 1^- and $\frac{dU}{dr}(r) \to +\infty$ as $r \to 1^-$. By (7), we conclude that any solution u of (1) satisfies $\frac{\partial u}{\partial r}(x) \to +\infty$ as $x \to \partial B$, while the tangential part of the gradient of u remains bounded. We then apply Theorem 2.1 in [12]. □

In addition to the *relative* asymptotic information given by (4), (5) and (7), the *exact* asymptotic expansion of a solution can be calculated to all orders. This is what we explain next. Consider (for simplicity only) a radial solution of (1), i.e. a solution of

$$\frac{d^2u}{dr^2} + \frac{N-1}{r}\frac{du}{dr} = f(u), \tag{10}$$

with $\lim_{r\to1^-} u(r) = +\infty$. We want to think of the second term on the left-hand side of (10) as a lower order perturbation as $r \to 1$. Multiplying the equation by $v = du/dr$ and putting the error term on the right-hand side, we get

$$\frac{1}{2}\frac{d}{dr}v^2 = \frac{d}{dr}F(u) - \frac{N-1}{r}v^2.$$

Make the change of independent variable $u = u(r)$. Thinking of v as a function of the new variable u, we have $\frac{d}{dr} = \frac{du}{dr}\frac{d}{du} = v\frac{d}{du}$ and so

$$\frac{1}{2}\frac{d}{du}v^2 = \frac{dF}{du} - \frac{N-1}{r}v.$$

In other words, v solves the nonlinear integral equation

$$v(u)=\sqrt{2\left(F(u)-(N-1)\int_{U_0}^{u}\frac{v}{r}\,dt\right)+C}=:\mathcal{N}(v),$$

where U_0, C are given constants. The above equation turns out to be contractive in a suitable Banach space. In particular, it can be solved using a standard iterative scheme $v_{k+1}=\mathcal{N}(v_k)$. As we shall demonstrate, each v_k contains (in implicit form) the first k terms in the asymptotic expansion of the solution at blow-up. To summarize, we have:

Theorem 1.9. *Let $f \in C^1(\mathbb{R})$ and assume* (2), (3) *hold. Let $U_0 \in \mathbb{R}$, $I=[U_0,+\infty)$ and let v_0 be the function defined for $u \in I$ by*

$$v_0(u)=\sqrt{2F(u)}. \tag{11}$$

Consider the Banach space

$$\mathcal{X}=\left\{v \in C(I;\mathbb{R})\colon\ \exists M>0 \text{ such that } |v| \leqslant M v_0\right\},$$

endowed with the norm $\|v\|=\sup_I |v/v_0|$. If the constant U_0 is chosen sufficiently large, then for some $\rho \in (0,1)$, there exists a unique solution $v \in \mathcal{B}(v_0,\rho) \subset \mathcal{X}$ of the integral equation

$$v(u)=\sqrt{2\left(F(u)-(N-1)\int_{U_0}^{u}\frac{v}{r}\,dt\right)},\quad u \in I, \tag{12}$$

where $r=r(u,v)$ is given for $u \in I$, $v \in \mathcal{B}(v_0,\rho)$ by

$$r(u,v)=1-\int_{u}^{+\infty}\frac{1}{v}\,dt. \tag{13}$$

In addition, v is the limit in X of (v_k) defined for $k=0$ by (11) *and for $k \geqslant 1$, by*

$$v_k(u)=\sqrt{2\left(F(u)-(N-1)\int_{U_0}^{u}\frac{v_{k-1}}{1-\int_t^{+\infty}\frac{1}{v_{k-1}}\,ds}\,dt\right)} \tag{14}$$

and the sequence v_k is asymptotic to v, i.e. as $u \to +\infty$,

$$v_{k+1}(u)=v_k(u)+o\big(v_k(u)\big)$$

and given any $k \in \mathbb{N}$ we have

$$v(u)=v_k(u)+O\big(v_{k+1}(u)-v_k(u)\big).$$

Let now u denote any solution of (1) *and fix $r_0 \in (0,1)$ such that $u(x) \geqslant U_0$ for $|x| \geqslant r_0$. For $k \geqslant 0$, define u_k for $r \geqslant r_0$ as the unique solution*[1] *of*

$$\begin{cases} \dfrac{du_k}{dr} = v_k(u_k), \\ \lim\limits_{r\to 1^-} u_k(r) = +\infty, \end{cases} \tag{15}$$

where v_k is given by (14)*. Then,*

$$\int_{u_k(r)}^{u_{k+1}(r)} \frac{du}{v_0} = o\left(\int_{u_k(r)}^{+\infty} \frac{du}{v_0} \right) \quad \text{as } r \to 1^-$$

and given any $k \in \mathbb{N}$, we have

$$\int_{u_k(|x|)}^{u(x)} \frac{du}{v_0} = o\left(\int_{u_k(|x|)}^{+\infty} \frac{du}{v_0} \right) \quad \text{as } x \to \partial B. \tag{16}$$

Theorem 1.9 enables one to calculate (implicitly) the asymptotic expansion of a solution term by term. But how many terms in this expansion are singular? This is what we discuss in our last set of results.

We begin with the simplest class of nonlinearities f, those for which only one term in the expansion is singular, namely the function u_0 defined by (11) and (15). It turns out, as A.C. Lazer and P.J. McKenna first demonstrated (see [9]), that in this case $u_0(1-d(x))$ is the only singular term in the asymptotics of any blow-up solution on any smoothly bounded domain $\Omega \subset \mathbb{R}^N$ and for any dimension $N \geqslant 1$, where $d(x)$ denotes the distance of a point $x \in \Omega$ to the boundary of Ω. In other words, the blow-up rate is universal. The question is now to determine for which nonlinearities f, this universal blow-up occurs. We characterize these nonlinearities as follows:

Theorem 1.10. *Let $\Omega \subset \mathbb{R}^N$ denote a bounded domain satisfying an inner and an outer sphere condition at each point of its boundary. Let $f \in C^1(\mathbb{R})$, assume* (2), (3) *hold and consider the equation*

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Assume

$$\lim_{u\to+\infty} \sqrt{2F(u)} \int_u^{+\infty} \frac{\int_0^t \sqrt{2F}\,ds}{(2F)^{3/2}}\,dt = 0. \tag{18}$$

Then, any solution of (17) *satisfies*

$$\lim_{x\to\partial\Omega} u(x) - u_0\big(1-d(x)\big) = 0, \tag{19}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$ and u_0 is defined by (11), (15)*.*

[1] Note that (15) can be solved by quadratures and its solution is unique. Indeed, $v_k(u) \sim v_0(u) = \sqrt{2F(u)}$ as $u \to +\infty$ and this implies by (3) that $\int^{+\infty} dt/v_k(t) < +\infty$.

We also have the following partial converse statement: *if*

$$\liminf_{u\to+\infty}\sqrt{2F(u)}\int_u^{+\infty}\frac{\int_0^t\sqrt{2F}\,ds}{(2F)^{3/2}}\,dt>0,\tag{20}$$

then (19) *always fails.*

Remark 1.11. To our knowledge, (18) improves upon all known conditions for (19) to hold (see in particular [9] and [2]). Despite its unappealing technical appearance, (18) only uses information on the asymptotics of F (in particular, no direct information on f is required). Nonlinearities such that $F(u)\sim e^u$ or $F(u)\sim u^p$, $p>4$ as $u\to+\infty$ satisfy (18). For $F(u)\sim u^4$, (20) holds and so the conclusion (19) fails.

Remark 1.12. Condition (18) can be weakened to

$$\lim_{r\to1^-}\sqrt{2F(u_0)}\int_{u_0}^{+\infty}\frac{\int_0^t\sqrt{2F}\,ds}{(2F)^{3/2}}\,dt=0,$$

where $u_0=u_0(r)$ is defined by (15). Similarly, (20) can be weakened to

$$\liminf_{r\to1^-}\sqrt{2F(u_1)}\int_{u_1}^{+\infty}\frac{\int_0^t\sqrt{2F}\,ds}{(2F)^{3/2}}\,dt>0.$$

As an immediate corollary, we obtain uniqueness on general domains, whenever only one singular term appears:

Corollary 1.13. *Assume* (18). *If in addition, f is nondecreasing, then the solution of* (17) *is unique.*

Proof. Simply repeat the proof of Corollary 1.4. □

More than one term can be present in the asymptotic expansion of u. Finding all the (singular) terms in this expansion is of staggering algebraic complexity. To illustrate this, we provide the first three terms (in implicit form).

Proposition 1.14. *Let u_2 be defined by* (15) *for $k=2$. Let also R_1, R_2, R_3 denote three real-valued functions defined for $U\in\mathbb{R}$ sufficiently large by*

$$R_0(U)=\int_U^{+\infty}\frac{du}{\sqrt{2F}},\qquad R_1(U)=(N-1)\int_U^{+\infty}\frac{\int^u\sqrt{2F}\,dt}{(2F)^{3/2}}\,du,$$

$$R_2(U)=(N-1)\int_U^{+\infty}\left(-\int^u\left((N-1)\frac{\int^t\sqrt{2F}\,ds}{\sqrt{2F}}+\sqrt{2F}\int_u^{+\infty}\frac{ds}{\sqrt{2F}}\right)dt\right.$$
$$\left.+\frac{5(N-1)}{4}\frac{(\int^u\sqrt{2F}\,dt)^2}{2F}\right)\frac{du}{(2F)^{3/2}}.$$

Then, for all $r \in (0,1)$, r close to 1, we have

$$1 - r = R_0\big(u_2(r)\big) + R_1\big(u_2(r)\big) + R_2\big(u_2(r)\big)\big(1 + o(1)\big).$$

For specific nonlinearities, it is possible to invert the above identity. This is what we do for $f(u) = u^p$, $p > 1$:

Proposition 1.15. *Let $p > 1$ (with $2/(p-1) \notin \mathbb{N}$) and let $f(u) = u^p$, for $u > 0$. Then, the unique positive solution of (1) satisfies*

$$u = d^{-\frac{2}{p-1}} \sum_{k=0}^{[2/(p-1)]} a_k d^k + o(1) \quad \text{as } r \to 1^-,$$

where $d(r) = 1 - r$ for $r \in (0,1)$, and where each $a_k \in \mathbb{R}$ depends on N and p only.

Remark 1.16. Proposition 1.15 was first proved by S. Berhanu and G. Porru (see [3]). As can be seen from the proof, Proposition 1.15 remains valid for any nonlinearity f such that, for some positive constant c, $F(u) = cu^{p+1} + O(u)$ for large values of u (and any solution of the equation).

Outline of the paper

1. In the next section, we show that any solution u of (1) can be squeezed between two radial solutions U and V, i.e. the inequality $U \leqslant u \leqslant V$ holds throughout B.
2. Thanks to this result, we need only find the asymptotics of *radial* solutions to prove Theorem 1.1. This is what we do in Section 3.
3. To obtain gradient estimates, the squeezing technique is insufficient and more work is needed. In Section 4, we estimate tangential derivatives via a standard comparison argument, while we gain control over the radial component through a more delicate potential theoretic argument.
4. Section 5 is dedicated to the proof of Theorem 1.9, that is we establish an algorithm for computing the asymptotics of solutions to all orders.
5. In Section 6, we characterize nonlinearities for which the blow-up rate is universal.
6. At last, Sections 7 and 8 contain the tedious calculations of the first three terms of the asymptotic expansion of u in implicit form for general f, and of all terms explicitly for $f(u) = u^p$.

Notation

Throughout this paper, the letter C denotes a generic constant, the value of which is immaterial. In the last section of the paper, we use the symbol c_k to denote a quantity indexed by an integer k, thought of being "constant for fixed k," the value of which is again immaterial.

2. Ordering solutions

In this section, we prove that any solution of the equation is bounded above and below by radial blow-up solutions. To do so, we impose the following additional condition: $g(t) := f(-t)$ satisfies (2) and

$$\int^{+\infty} \frac{dt}{\sqrt{G(t)}} = +\infty, \quad \text{where } G'(t) = f(-t). \tag{21}$$

Remark 2.1. Note that (21) is not restrictive. Indeed, if u denotes a solution of (1) and $m = \min_B u$, then u also solves (1) with nonlinearity $\tilde{f}$ defined for $u \in \mathbb{R}$ by

$$\tilde{f}(u) = \begin{cases} f(m) + (m-u) & \text{if } u < m, \\ f(u) & \text{if } u \geqslant m. \end{cases}$$

Then, $\tilde{f}$ clearly satisfies (21).

We now proceed through a series of three lemmas.

Lemma 2.2. *Assume* (21) *holds. For $M \in \mathbb{R}$ sufficiently large, there exists a radial function $\underline{v} \in C^2(B) \cap C(\overline{B})$ satisfying*

$$\Delta \underline{v} \geqslant f(\underline{v}) \quad \textit{in } B$$

and such that

$$\underline{v} \leqslant -M \quad \textit{in } B.$$

Proof. Let $g(t) = f(-t)$ for $t \in \mathbb{R}$ and let $a > M$ be a parameter to be fixed later on. Since g satisfies (2), we may always assume that $g(t) \geqslant 0$ for $t \geqslant M$. Let now w denote a solution of

$$\begin{cases} -w'' = g(w), \\ w(0) = a, \\ w'(0) = 0. \end{cases} \tag{22}$$

Claim. *There exists an $a > M$ sufficiently large such that $w(1) \geqslant M$.*

Note that w is nonincreasing in the set $\{t\colon\ w(t) \geqslant M\}$. We distinguish two cases.

Case 1. $w > M$.

In this case, w is defined on all of $\mathbb{R}^+$. In particular, $w(1) > M$, as desired.

Case 2. There exists $R > 0$ such that $w(R) = M$.

In this case, since w is nonincreasing in $(0, R)$, we just need to prove that $R \geqslant 1$. To do so, multiply (22) by $-w'$ and integrate between 0 and $r \in (0, R)$:

$$-w' = \sqrt{2\bigl(G(a) - G(w)\bigr)},$$

where G is an antiderivative of g. Integrate again between 0 and R:

$$\int_M^a \frac{dt}{\sqrt{2(G(a) - G(t))}} = \int_0^R \frac{-w'}{\sqrt{2(G(a) - G(w))}}\, dr = R.$$

Now,

$$R = \int_M^a \frac{dt}{\sqrt{2(G(a) - G(t))}} \geqslant \int_M^{G^{-1}(G(a)/2)} \frac{dt}{\sqrt{2(G(a) - G(t))}} \geqslant \int_M^{G^{-1}(G(a)/2)} \frac{dt}{\sqrt{2G(t)}}.$$

By (21), we deduce that $R \geqslant 1$ for sufficiently large a. We have just proved that $w|_{(0,1)} \geqslant w(1) \geqslant M$ and the claim follows.

It follows that the function $\underline{v}$ defined for $x \in B$ by $\underline{v}(x) = -w(|x|)$, is the desired subsolution. □

Lemma 2.3. *Let $f \in C^1(\mathbb{R})$ and assume (2) and (3) hold. Assume $\underline{v} \in C(\overline{B})$ satisfies*

$$\Delta \underline{v} \geqslant f(\underline{v}) \quad \text{in } B.$$

Then, there exists a radial large solution V of (1) such that $V \geqslant \underline{v}$.

Proof. Let $\overline{v} := N$. Then, $\underline{v}$ and $\overline{v}$ are respectively a sub and supersolution of

$$\begin{cases} \Delta v = f(v) & \text{in } B, \\ v = N & \text{on } \partial B, \end{cases} \tag{23}$$

provided N is chosen so large that $N > \|\underline{v}\|_{L^\infty(B)}$ and $f(N) \geqslant 0$. Furthermore, $\underline{v} < \overline{v}$ in B for such values of N. By the method of sub and supersolutions (see, e.g., Proposition 2.1 in [6]), there exists a minimal solution V_N of (23) such that $N \geqslant V_N \geqslant \underline{v}$. Note that V_N is radial, as follows from the classical symmetry result of Gidas, Ni and Nirenberg (see [7]). Also, since V_N is minimal, we have that the sequence (V_N) is nondecreasing with respect to N (apply, e.g., the Minimality Principle, Corollary 2.2 in [6]).

It turns out that the sequence (V_N) is uniformly bounded on compact sets of B. Indeed, fix $R_1 < 1$. There exists a solution $\tilde{U}$ blowing up on the boundary of the ball of radius 1 and satisfying $\tilde{U} \geqslant \underline{v}$ in B_{R_1}, see Remark 2.9 in [6]. By minimality, $\underline{v} \leqslant V_N \leqslant \tilde{U}$ in B_{R_1}, whence (V_N) is uniformly bounded on B_{R_2} for any given $R_2 < R_1$.

We have just proved that each V_N is radial and that the sequence (V_N) is nondecreasing and bounded on compact subsets of B. By standard elliptic regularity, it follows that (V_N) converges to a radial solution V of (1), such that $V \geqslant \underline{v}$ in B. □

Lemma 2.4. *Assume (3) and (21) hold. Let u be a solution of (1). Then, there exist two radial functions U, V solving (1) such that*

$$U \leqslant u \leqslant V \quad \text{in } B.$$

Proof. Let $-M$ denote the minimum value of u and let $\underline{v}$ denote the subsolution given by Lemma 2.2. In particular, $\underline{v} \leqslant u$. By Lemma 2.3, there exists a solution $U \geqslant \underline{v}$ of (1) and we may assume that U is the minimal solution relative to $\underline{v}$, i.e. given any other solution $\tilde{u} \geqslant \underline{v}$ of (1), $U \leqslant \tilde{u}$. In particular, $U \leqslant u$. It remains to construct a radial solution V of (1) such that $u \leqslant V$. To do so, we fix $R < 1$. By Lemma 2.3, letting $\underline{v} = u|_{B_R}$, there exists a radial solution $v = V_R$ of

$$\begin{cases} \Delta v = f(v) & \text{in } B_R, \\ v = +\infty & \text{on } \partial B_R, \end{cases} \tag{24}$$

such that $V_R \geqslant u$ in B_R. Since V_R is constructed as the monotone limit of minimal solutions V_N (see the proof of the previous lemma), one can easily check that the mapping $R \mapsto V_R$ is nonincreasing (hence automatically bounded on compact sets of B). Hence, as $R \to 1$, V_R converges to a solution V of (1), which is radial and satisfies $V \geqslant u$ in B, as desired. □

3. Asymptotics of radial solutions

Our next result establishes that the asymptotic expansion of a radial blow-up solution is unique. More precisely, consider the one-dimensional problem

$$\frac{d^2\phi}{dr^2} = f(\phi), \quad r < 1, \qquad \phi(r) \to +\infty \quad \text{as } r \to 1^-. \tag{25}$$

All solutions are given implicitly by

$$\int_{\phi}^{+\infty} \frac{ds}{\sqrt{2F(s)}} = 1 - r, \quad \text{where } F' = f.$$

We recall the following fact, first observed by C. Bandle and M. Marcus in [2]:

Remark 3.1. Let ϕ and ϕ_c denote two solutions of (25) corresponding to the antiderivatives F and $F + c$, respectively. Then $\phi(r) - \phi_c(r) \to 0$ as $r \to 1^-$.

We improve this result in the following way.

Theorem 3.2. *Let $N \geqslant 1$ and let u_1, u_2 denote two strictly increasing functions solving*

$$\begin{cases} \dfrac{d^2u}{dr^2} + \dfrac{N-1}{r}\dfrac{du}{dr} = f(u), \quad r < 1, \\ \lim\limits_{r \to 1^-} u(r) = +\infty. \end{cases} \tag{26}$$

Then,

$$\left|u_1(r) - u_2(r)\right| \leqslant C \int_{u_2(r)}^{+\infty} \frac{dt}{F(t)}\, dt.$$

In addition, the quantity $|F(u_1) - F(u_2)|$ is bounded.

Remark 3.3. Clearly, Theorem 1.1 follows as a direct consequence of Remark 2.1, Lemma 2.4 and Theorem 3.2.

Proof of Theorem 3.2. We want to think of the second term on the left-hand side of Eq. (26) as a lower order perturbation as $r \to 1$. So, we integrate (26) in the same way we would solve (25), namely we let $v = du/dr$ and multiply the equation by v. We get

$$\frac{d}{dr}\left(\frac{v^2}{2}\right) + \frac{N-1}{r} v^2 = \frac{d}{dr}\bigl(F(u)\bigr).$$

We define the resulting error term by

$$g := -\frac{v^2}{2} + F(u), \tag{27}$$

which, seen as function of r, satisfies the differential equation

$$\frac{dg}{dr} = \frac{N-1}{r} v^2. \tag{28}$$

Since u is a strictly increasing function, the change of independent variable $u = u(r)$ is valid. Thinking of g as a function of the variable u, we have $\frac{dg}{du} = \frac{dg}{dr}\frac{dr}{du} = \frac{1}{v}\frac{dg}{dr}$ and so

$$\frac{dg}{du} = \frac{N-1}{r} v. \tag{29}$$

Since (26) holds for r close to 1, the above equation holds for u in a neighborhood of $+\infty$. Solving (27) for v, we finally obtain

$$\begin{cases} \dfrac{dg}{du} = \dfrac{N-1}{r} v = \dfrac{N-1}{r}\sqrt{2\big(F(u) - g\big)}, \\ \dfrac{dr}{du} = 1/v = \big(2\big(F(u) - g\big)\big)^{-1/2}. \end{cases} \tag{30}$$

We start by calculating the leading asymptotic behavior of g at $+\infty$:

Lemma 3.4.

$$\lim_{u\to+\infty} \frac{g(u)}{F(u)} = 0.$$

In addition,

$$\lim_{u\to+\infty} \frac{g(u)}{(N-1)G(u)} = 1, \quad \textit{where } G \textit{ is any antiderivative of } \sqrt{2F}. \tag{31}$$

Proof. First, we claim that

$$\lim_{u\to+\infty} \frac{G(u)}{F(u)} = 0. \tag{32}$$

Indeed, fix $\varepsilon > 0$ and recalling that (3) holds, choose $M > 0$ so large that $\int_M^{+\infty} \frac{dt}{\sqrt{2F(t)}} < \varepsilon$. By the definition of G, there exists a constant C_M such that

$$G(u) = C_M + \int_M^u \sqrt{2F(t)}\, dt.$$

Since F is nondecreasing it follows that

$$G(u) \leqslant C_M + 2F(u) \int_M^u \frac{dt}{\sqrt{2F(t)}} \leqslant C_M + 2\varepsilon F(u).$$

Dividing by $F(u)$ and letting $u \to +\infty$, (32) follows. Next, we claim that

$$\lim_{u\to+\infty} \frac{g(u)}{F(u)} = 0. \tag{33}$$

Note that by (29), $g(u)$ is increasing, thus it is bounded below by a constant c as $u \to +\infty$. Hence, by (30),

$$\frac{dg}{du} \leqslant \frac{N-1}{r}\sqrt{2\big(F(u)-c\big)} \leqslant 2(N-1)\sqrt{2\big(F(u)-c\big)},$$

where the last inequality holds if $r > 1/2$, i.e. if u is sufficiently large. Integrating on a given interval (u_0, u), we obtain

$$c \leqslant g(u) \leqslant g(u_0) + 2(N-1)\int_{u_0}^{u} \sqrt{2\big(F(t)-c\big)}\,dt.$$

Using (32) and the fact that $\lim_{t\to+\infty} \frac{\sqrt{2F(t)}}{\sqrt{2(F(t)-c)}} = 1$, we deduce (33). Now that (33) has been established, we return to (30) and infer that given $\varepsilon > 0$, we have for sufficiently large u,

$$\frac{dg}{du} \geqslant \frac{N-1}{r}\sqrt{2(1-\varepsilon)F(u)} \geqslant (N-1)\sqrt{2(1-\varepsilon)F(u)}$$

and

$$\frac{dg}{du} \leqslant \frac{N-1}{r}\sqrt{2(1+\varepsilon)F(u)} \leqslant \frac{N-1}{1-\varepsilon}\sqrt{2(1+\varepsilon)F(u)}.$$

Integrating the above, we finally obtain for large u,

$$(1-\varepsilon)(N-1)\int_{u_0}^{u} \sqrt{2F(t)}\,dt \leqslant g(u)-g(u_0) \leqslant (1+2\varepsilon)^{3/2}(N-1)\int_{u_0}^{u} \sqrt{2F(t)}\,dt$$

and (31) follows. The fact that $g(u) \to +\infty$ as $u \to +\infty$ follows automatically. □

Next, we prove that given two solutions u_1, u_2, the corresponding error terms g_1, g_2 given by (27) differ by a bounded quantity.

Lemma 3.5. *Let u_1 and u_2 be two solutions of* (26)*. Introduce $v_i = \frac{du_i}{dr}$ and*

$$g_i = -\frac{v_i^2}{2} + F(u_i), \quad \textit{for } i = 1, 2.$$

Then, $g_1 - g_2$ is bounded.

Proof. We begin by rewriting the system (30) as a nonlinear integral equation with unknown g. To do so, solve the first line of (30) for r:

$$r = (N-1)\frac{\sqrt{2(F-g)}}{\frac{dg}{du}}.$$

Differentiate with respect to u:

$$\frac{dr}{du} = (N-1)\left\{\frac{f-\frac{dg}{du}}{\sqrt{2(F-g)}\,\frac{dg}{du}} - \frac{\sqrt{2(F-g)}\,\frac{d^2g}{du^2}}{(\frac{dg}{du})^2}\right\}.$$

Equate the above equation with the second line of (30), to obtain the following second order differential equation:

$$\frac{d^2g}{du^2}+\frac{1}{2(N-1)}\frac{1}{F-g}\left(\frac{dg}{du}\right)^2-\frac{f-\frac{dg}{du}}{2(F-g)}\frac{dg}{du}=0. \tag{34}$$

Define

$$q=\frac{1}{F-g}-\frac{1}{F}=\frac{g}{F(F-g)}. \tag{35}$$

So, $1/(F-g)=1/F+q$ and (34) can be rewritten as

$$\frac{d^2g}{du^2}+\frac{1}{2(N-1)}\left(\frac{1}{F}+q\right)\left(\frac{dg}{du}\right)^2-\frac{f-\frac{dg}{du}}{2}\left(\frac{1}{F}+q\right)\frac{dg}{du}=0.$$

In other words, $k=dg/du$ solves the logistic equation

$$\frac{dk}{du}+\frac{1}{2}\left(\frac{1}{F}+q\right)k\left(\frac{N}{N-1}k-f\right)=0.$$

The general solution of such an equation is well known and is given by

$$k=\frac{2(N-1)}{N}\frac{e^{\frac{1}{2}\int_{u_0}^{u}(\frac{1}{F}+q)f\,dt}}{\int_{u_0}^{u}((\frac{1}{F}+q)e^{\frac{1}{2}\int_{u_0}^{t}(\frac{1}{F}+q)f\,ds})\,dt+C},$$

where u_0, C are arbitrary constants. Since all integrands are positive for large u, we may take $u_0=+\infty$ in the above formula and obtain

$$\begin{aligned}k&=-\frac{2(N-1)}{N}\frac{e^{-\frac{1}{2}\int_{u}^{+\infty}(\frac{1}{F}+q)f\,dt}}{\int_{u}^{+\infty}(\frac{1}{F}+q)e^{-\frac{1}{2}\int_{t}^{+\infty}qf\,ds}\,dt+C}\\&=-\frac{2(N-1)}{N}\sqrt{F}\frac{e^{-\frac{1}{2}\int_{u}^{+\infty}qf\,dt}}{\int_{u}^{+\infty}(\frac{1}{F}+q)\sqrt{F}e^{-\frac{1}{2}\int_{t}^{+\infty}qf\,ds}\,dt+C}\\&=-\frac{(N-1)}{N}\sqrt{2F}\frac{e^{-\frac{1}{2}\int_{u}^{+\infty}qf\,dt}}{\int_{u}^{+\infty}(\frac{1}{\sqrt{2F}}+\frac{q}{2}\sqrt{2F})e^{-\frac{1}{2}\int_{t}^{+\infty}qf\,ds}\,dt+C}.\end{aligned} \tag{36}$$

Next, we identify the leading asymptotics of the quantity $\int_u^{+\infty}qf\,dt$. To do so, simply recall the definition of q given by (35), as well as the leading asymptotics of g given by Lemma 3.4:

$$\int_u^{+\infty}qf\,dt=\int_u^{+\infty}\frac{g}{F(F-g)}f\,dt\sim(N-1)\int_u^{+\infty}G\frac{f}{F^2}\,dt, \tag{37}$$

where $G' = \sqrt{2F}$. Integrating by parts, we discover that

$$\int_u^{+\infty} G\frac{f}{F^2}\,dt = \frac{G}{F} + \int_u^{+\infty} \frac{\sqrt{2F}}{F}\,dt = \frac{G}{F} + 2\int_u^{+\infty} \frac{1}{\sqrt{2F}}\,dt = o(1). \tag{38}$$

Using this in (36), we deduce that

$$k \sim -\left(\frac{N-1}{N}\right)\frac{1}{C}\sqrt{2F}.$$

In addition, $k = dg/du = (N-1)/r\sqrt{2(F-g)} \sim (N-1)\sqrt{2F}$. So, we must have $C = -1/N$ and so

$$\frac{dg}{du} = k = -\frac{(N-1)}{N}\sqrt{2F}\frac{e^{-\frac{1}{2}\int_u^{+\infty} qf\,dt}}{\int_u^{+\infty}(\frac{1}{\sqrt{2F}} + \frac{q}{2}\sqrt{2F})e^{-\frac{1}{2}\int_t^{+\infty} qf\,ds}\,dt - \frac{1}{N}}. \tag{39}$$

Take now two solutions u_1, u_2 of (26) and let g_1, g_2 denote the associated error terms. By (39), we have

$$\begin{aligned}\frac{dg_1}{du} - \frac{dg_2}{du} = -\frac{N-1}{N}\sqrt{2F}\Bigg(&\frac{e^{-\frac{1}{2}\int_u^{+\infty} q_1 f\,dt}}{\int_u^{+\infty}(\frac{1}{\sqrt{2F}} + \frac{q_1}{2}\sqrt{2F})e^{-\frac{1}{2}\int_t^{+\infty} q_1 f\,ds}\,dt - \frac{1}{N}}\\ &- \frac{e^{-\frac{1}{2}\int_u^{+\infty} q_2 f\,dt}}{\int_u^{+\infty}(\frac{1}{\sqrt{2F}} + \frac{q_2}{2}\sqrt{2F})e^{-\frac{1}{2}\int_t^{+\infty} q_2 f\,ds}\,dt - \frac{1}{N}}\Bigg),\end{aligned}$$

where $q = q_i$ satisfies (35) for $g = g_i$. Reducing to the same denominator and using (37), (38), it follows that

$$\begin{aligned}\frac{dg_1}{du} - \frac{dg_2}{du} \sim -\frac{N-1}{N^3}\sqrt{2F}\Bigg(&e^{-\frac{1}{2}\int_u^{+\infty} q_1 f\,dt}\left[\int_u^{+\infty}\left(\frac{1}{\sqrt{2F}} + \frac{q_2}{2}\sqrt{2F}\right)e^{-\frac{1}{2}\int_t^{+\infty} q_2 f\,ds}\,dt - \frac{1}{N}\right]\\ &- e^{-\frac{1}{2}\int_u^{+\infty} q_2 f\,dt}\left[\int_u^{+\infty}\left(\frac{1}{\sqrt{2F}} + \frac{q_1}{2}\sqrt{2F}\right)e^{-\frac{1}{2}\int_t^{+\infty} q_1 f\,ds}\,dt - \frac{1}{N}\right]\Bigg).\end{aligned}$$

To simplify the above expression, we write $e_i = e^{-\frac{1}{2}\int_u^{+\infty} q_i f\,dt}$. We obtain

$$\begin{aligned}\frac{dg_1}{du} - \frac{dg_2}{du} &\sim -\frac{N-1}{N^3}\sqrt{2F}\Bigg(e_1\left[\int_u^{+\infty}\left(\frac{1}{\sqrt{2F}} + \frac{q_2}{2}\sqrt{2F}\right)e_2\,dt - \frac{1}{N}\right]\\ &\quad - e_2\left[\int_u^{+\infty}\left(\frac{1}{\sqrt{2F}} + \frac{q_1}{2}\sqrt{2F}\right)e_1\,dt - \frac{1}{N}\right]\Bigg)\\ &= \frac{N-1}{N^4}\sqrt{2F}(e_1 - e_2) - \frac{N-1}{N^3}\sqrt{2F}\Bigg(e_1\left[\int_u^{+\infty}\left(\frac{1}{\sqrt{2F}} + \frac{q_2}{2}\sqrt{2F}\right)e_2\,dt\right]\end{aligned}$$

$$
\begin{aligned}
&- e_2\left[\int_u^{+\infty}\left(\frac{1}{\sqrt{2F}}+\frac{q_1}{2}\sqrt{2F}\right)e_1\,dt\right]\Bigg)\\
&= \frac{N-1}{N^4}\sqrt{2F}(e_1-e_2) - \frac{N-1}{N^3}\sqrt{2F}\Bigg((e_1-e_2)\left[\int_u^{+\infty}\left(\frac{1}{\sqrt{2F}}+\frac{q_2}{2}\sqrt{2F}\right)e_2\,dt\right]\\
&\quad + e_2\left[\int_u^{+\infty}\left(\frac{1}{\sqrt{2F}}+\frac{q_1}{2}\sqrt{2F}\right)e_1\,dt - \int_u^{+\infty}\left(\frac{1}{\sqrt{2F}}+\frac{q_2}{2}\sqrt{2F}\right)e_2\,dt\right]\Bigg).
\end{aligned}
$$

Canceling lower order terms in the above expression and noting that $e_2 \sim 1$, we obtain

$$
\begin{aligned}
\frac{dg_1}{du} - \frac{dg_2}{du} &\sim \frac{N-1}{N^4}\sqrt{2F}(e_1-e_2) - \frac{N-1}{N^3}\sqrt{2F}\\
&\quad \times \left[\int_u^{+\infty}\left(\frac{1}{\sqrt{2F}}+\frac{q_1}{2}\sqrt{2F}\right)e_1\,dt - \int_u^{+\infty}\left(\frac{1}{\sqrt{2F}}+\frac{q_2}{2}\sqrt{2F}\right)e_2\,dt\right].
\end{aligned}
$$

The right-hand side in the above expression can be rewritten as

$$
\begin{aligned}
&\frac{N-1}{N^4}\sqrt{2F}(e_1-e_2) - \frac{N-1}{N^3}\sqrt{2F}\left[\int_u^{+\infty}\frac{1}{\sqrt{2F}}(e_1-e_2)\,dt + \frac{1}{2}\int_u^{+\infty}\sqrt{2F}(q_1e_1-q_2e_2)\,dt\right]\\
&\sim \frac{N-1}{N^4}\sqrt{2F}(e_1-e_2) - \frac{N-1}{2N^3}\sqrt{2F}\int_u^{+\infty}\sqrt{2F}(q_1e_1-q_2e_2)\,dt\\
&= \frac{N-1}{N^4}\sqrt{2F}(e_1-e_2) - \frac{N-1}{2N^3}\sqrt{2F}\int_u^{+\infty}\sqrt{2F}\big(q_1(e_1-e_2)+e_2(q_1-q_2)\big)\,dt.
\end{aligned}
$$

Since $e_i \sim 1$, we have, using the mean value formula,

$$
e_1 - e_2 \sim -\frac{1}{2}\int_u^{+\infty}(q_1-q_2)f\,dt. \tag{40}
$$

In addition, by (35),

$$
q_i \sim \frac{g_i}{F^2} \quad \text{and} \quad q_1 - q_2 \sim \frac{g_1-g_2}{F^2}.
$$

So,

$$
e_1 - e_2 \sim -\frac{1}{2}\int_u^{+\infty}(g_1-g_2)\frac{f}{F^2}\,dt
$$

and it follows that

$$\frac{dg_1}{du}-\frac{dg_2}{du}\sim-\frac{1}{2}\frac{N-1}{N^4}\sqrt{2F}\int_u^{+\infty}(g_1-g_2)\frac{f}{F^2}\,dt$$
$$-\frac{N-1}{2N^3}\sqrt{2F}\int_u^{+\infty}\sqrt{2F}\left(-\frac{1}{2}\frac{g_1}{F^2}\int_t^{+\infty}(g_1-g_2)\frac{f}{F^2}\,ds+\frac{g_1-g_2}{F^2}\right)dt.$$

Hence,

$$\left|\frac{dg_1}{du}-\frac{dg_2}{du}\right|$$
$$\leqslant C\sqrt{2F}\left(\int_u^{+\infty}|g_1-g_2|\frac{f}{F^2}\,dt+\int_u^{+\infty}\frac{G}{F^{3/2}}\int_t^{+\infty}|g_1-g_2|\frac{f}{F^2}\,ds\,dt+\int_u^{+\infty}\frac{|g_1-g_2|}{(2F)^{3/2}}\,dt\right)$$
$$\leqslant C\sqrt{2F}\left(\int_u^{+\infty}|g_1-g_2|\frac{f}{F^2}\,dt+\int_u^{+\infty}\frac{|g_1-g_2|}{(2F)^{3/2}}\,dt\right),$$

where $G'=\sqrt{2F}$. We want to estimate further each of the two terms on the right-hand side of the above inequality. Since $g_i=O(G)$, one can easily check that all integrals are convergent. In particular, we may always find $U>u$ so large that

$$\int_U^{+\infty}|g_1-g_2|\frac{f}{F^2}\,dt\leqslant\int_u^U|g_1-g_2|\frac{f}{F^2}\,dt,$$
$$\int_U^{+\infty}\frac{|g_1-g_2|}{(2F)^{3/2}}\,dt\leqslant\int_u^U\frac{|g_1-g_2|}{(2F)^{3/2}}\,dt.$$

It follows that

$$\left|\frac{dg_1}{du}-\frac{dg_2}{du}\right|\leqslant C\sqrt{2F}\left(\int_u^U|g_1-g_2|\frac{f}{F^2}\,dt+\int_u^U\frac{|g_1-g_2|}{(2F)^{3/2}}\,dt\right)$$
$$\leqslant C\Big(\sup_{t\in[u,U]}|g_1-g_2|\Big)\left(\frac{1}{\sqrt{2F}}+\sqrt{2F}\int_u^{+\infty}\frac{dt}{(2F)^{3/2}}\right)$$
$$\leqslant\frac{C}{\sqrt{2F}}\sup_{t\in[u,U]}|g_1-g_2|.$$

Integrating the above expression between a given constant u_0 and u, we obtain

$$|g_1-g_2|(u)\leqslant|g_1-g_2|(u_0)+C\Big(\sup_{t\in[u_0,U]}|g_1-g_2|\Big)\int_{u_0}^u\frac{dt}{\sqrt{2F}}.$$

Choose now u_0 so large that $C\int_{u_0}^{+\infty}\frac{dt}{\sqrt{2F}} < 1/2$. It follows that

$$\sup_{t\in[u_0,U]} |g_1 - g_2| \leqslant 2|g_1 - g_2|(u_0) = C_0.$$

This being true for U arbitrarily large, we finally deduce that $g_1 - g_2$ is bounded, as desired. □

Completion of the proof of Theorem 3.2. Let u_1, u_2 denote two solutions of (26). By (30), each u_i, $i = 1, 2$, solves

$$\frac{du_i/dr}{\sqrt{2(F(u_i) - g_i)}} = 1.$$

Integrating, we obtain

$$\int_{u_1}^{+\infty} \frac{1}{\sqrt{2(F(t) - g_1)}}\, dt = 1 - r = \int_{u_2}^{+\infty} \frac{1}{\sqrt{2(F(t) - g_2)}}\, dt.$$

Without loss of generality, for a given r we may assume $u_2(r) \geqslant u_1(r)$. Split the left-hand side integral: $\int_{u_1}^{+\infty} = \int_{u_1}^{u_2} + \int_{u_2}^{+\infty}$. It follows that

$$\begin{aligned}
\int_{u_1}^{u_2} \frac{1}{\sqrt{2(F(t) - g_1)}}\, dt &= \int_{u_2}^{+\infty} \left(\frac{1}{\sqrt{2(F(t) - g_2)}} - \frac{1}{\sqrt{2(F(t) - g_1)}} \right) dt \\
&= \int_{u_2}^{+\infty} \frac{\sqrt{2(F(t) - g_1)} - \sqrt{2(F(t) - g_2)}}{\sqrt{2(F(t) - g_1)}\sqrt{2(F(t) - g_2)}}\, dt \\
&= \int_{u_2}^{+\infty} \frac{g_2 - g_1}{\sqrt{2(F(t) - g_1)}\sqrt{2(F(t) - g_2)}(\sqrt{2(F(t) - g_1)} + \sqrt{2(F(t) - g_2)})}\, dt.
\end{aligned}$$

Recall that by Lemma 3.4, $g_i = o(F)$ as $t \to +\infty$. Recall also that $g_2 - g_1$ is bounded. So, for sufficiently large values of u_2, we deduce

$$\int_{u_1}^{u_2} \frac{1}{\sqrt{2F(t)}}\, dt \leqslant C \int_{u_2}^{+\infty} \frac{dt}{F(t)^{3/2}}. \tag{41}$$

Since F is increasing, it follows that

$$0 \leqslant \frac{u_2 - u_1}{\sqrt{F(u_2)}} \leqslant C \int_{u_1}^{u_2} \frac{1}{\sqrt{2F(t)}}\, dt \leqslant C \int_{u_2}^{+\infty} \frac{dt}{F(t)^{3/2}} \leqslant \frac{C}{\sqrt{F(u_2)}} \int_{u_2}^{+\infty} \frac{dt}{F(t)}.$$

Hence,

$$0 \leqslant u_2 - u_1 \leqslant C \int_{u_2}^{+\infty} \frac{dt}{F(t)},$$

as stated in Theorem 3.2. It remains to prove (5). Without loss of generality, we assume $u_1(r) \leqslant u_2(r)$, so

$$\begin{aligned}
\int_{u_1}^{u_2} \frac{dt}{\sqrt{F(t)}} &= \int_{u_1}^{+\infty} \frac{dt}{\sqrt{F(t)}} - \int_{u_2}^{+\infty} \frac{dt}{\sqrt{F(t)}} \\
&= \int_{u_2}^{+\infty} \frac{dt}{\sqrt{F(t-(u_2-u_1))}} - \int_{u_2}^{+\infty} \frac{dt}{\sqrt{F(t)}} \\
&= \int_{u_2}^{+\infty} \frac{\sqrt{F(t)} - \sqrt{F(t-(u_2-u_1))}}{\sqrt{F(t)F(t-(u_2-u_1))}}\, dt \\
&= \int_{u_2}^{+\infty} \frac{F(t) - F(t-(u_2-u_1))}{\sqrt{F(t)F(t-(u_2-u_1))}(\sqrt{F(t)} + \sqrt{F(t-(u_2-u_1))})}\, dt \\
&\geqslant \big(F(u_2) - F(u_1)\big) \int_{u_2}^{+\infty} \frac{dt}{F(t)^{3/2}}.
\end{aligned}$$

Recalling (41), (5) follows. □

4. Gradient estimates

Proof of Theorem 1.2. Let $w = u_1 - u_2$ denote the difference of two solutions. Without loss of generality, we may assume that u_2 is the minimal solution of (1), so that $u_1 \geqslant u_2$ and u_2 is radial.

Step 1: estimate of tangential derivatives. We begin by proving that any tangential derivative of w is bounded. Since the problem is invariant under rotation and since u_2 is radial, we need only to show that $\frac{\partial u_1}{\partial x_2}(r, 0, \ldots, 0)$ remains bounded as $r \to 1^-$. Given $x = (x_1, x_2, x') \in B$ and $\theta > 0$ small, we denote by $x_\theta = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta, x')$ the image of x under the rotation of angle θ above the x_1-axis in the (x_1, x_2) plane. By the rotation invariance of the Laplace operator, the function u_θ defined for $x \in B$ by $u_\theta(x) = u_1(x_\theta)$, solves (1). Using (4) and assumption (6), we deduce that $w_\theta = u_1 - u_\theta$ solves

$$\begin{cases} \Delta w_\theta + K w_\theta = \tilde{f}(u_1) - \tilde{f}(u_\theta) & \text{in } B, \\ w_\theta = 0 & \text{on } \partial B. \end{cases} \tag{42}$$

By the Maximum Principle on small domains, there exists $R_0 \in (0, 1)$ such that the operator $L = \Delta + K$ is coercive on $B \setminus B_{R_0}$. As a consequence, we claim that there exists a constant $C > 0$ such that for all $x \in B \setminus B_{R_0}$,

$$\big|w_\theta(x)\big| \leqslant C \sup_{\partial B_{R_0}} |w_\theta|. \tag{43}$$

Let indeed $\zeta > 0$ denote the solution of

$$\begin{cases} \Delta\zeta + K\zeta = 0 & \text{in } B \setminus B_{R_0}, \\ \zeta = 1 & \text{on } \partial B_{R_0}, \\ \zeta = 0 & \text{on } \partial B. \end{cases}$$

We shall prove that $z^{\pm} := w_\theta - \pm \sup_{\partial B_{R_0}} |w_\theta|\zeta$ are respectively nonpositive and nonnegative, which implies that (43) holds for the constant $C = \|\zeta\|_\infty$. We work with z^+ and assume by contradiction that the open set $\omega = \{x \in B \setminus B_{R_0}\colon\ z^+(x) > 0\}$ is non-empty. Restricting the analysis to a connected component, we have

$$\begin{cases} \Delta z^+ + Kz^+ = \tilde f(u_1) - \tilde f(u_\theta) \geqslant 0 & \text{in } \omega,\\ z^+ \leqslant 0 & \text{on } \partial\omega.\end{cases}$$

By the Maximum Principle, we conclude that $z^+ \leqslant 0$ in ω, a contradiction. We have thus proved (43). Since $u_1 \in C^1(\overline{B_{R_0}})$, we deduce that for some constant $C > 0$ and all $x \in B \setminus B_{R_0}$,

$$\left|w_\theta(x)\right| \leqslant C\theta.$$

Applying the above inequality at the point $x = (r, 0, \dots, 0)$, $r \in (R_0, 1)$, and letting $\theta \to 0$, we finally deduce that

$$\left|\frac{\partial u_1}{\partial x_2}(r, 0, \dots, 0)\right| \leqslant C, \quad \text{for all } r \in (R_0, 1),$$

as desired.

Step 2: estimate of the radial derivative. It remains to control $\partial w/\partial r$. Fix $R \in (0, 1)$. Let $G_R(x, y)$ denote Green's function in the ball of radius R. Then, for $x \in B_R$,

$$\begin{aligned} w(x) &= \int_{\partial B_R} \frac{\partial G_R}{\partial \nu_y}(x, \cdot)\, w\, d\sigma + \int_{B_R} G_R(x, \cdot)\big(f(u_1) - f(u_2)\big)\, dy \\ &=: w_1(x) + w_2(x). \end{aligned} \tag{44}$$

We want to let $R \to 1$ in the above identity. To do so, we first observe that w_1 is harmonic. By the Maximum Principle, $|w_1| \leqslant \|w\|_{L^\infty(\partial B_R)}$. By estimate (4), we conclude that $w_1 \to 0$ as $R \to 1$. To estimate w_2, we need the following crucial estimate:

Lemma 4.1. *Assume* (6). *Then,*

$$\sup_{\theta \in S^{N-1}} \int_0^1 \left|f(u_1) - f(u_2)\right|(r, \theta)\, dr < +\infty.$$

We shall also need the following elementary estimates.

Lemma 4.2. *There exists a constant $C > 0$ such that for all $1/2 < r, R < 1$ and all $x, y \in B_R$,*

$$G_R(x, y) = R^{2-N} G_1\left(\frac{x}{R}, \frac{y}{R}\right), \tag{45}$$

$$\begin{cases} \displaystyle\int_{\partial B_r} G_1(x, \cdot)\, d\sigma \leqslant 1, \\ \displaystyle\int_{\partial B_r} \left|\frac{\partial G_1}{\partial |x|}(x, \cdot)\right| d\sigma \leqslant C. \end{cases} \tag{46}$$

We postpone the proofs of the above two lemmas and return to (44). Using polar coordinates,

$$w_2(x) = \int_{B_R} G_R(x,\cdot)\big(f(u_1) - f(u_2)\big)\,dy$$
$$= \int_0^R \left(\int_{\partial B_r} G_R(x,\cdot)\big(f(u_1) - f(u_2)\big)\,d\sigma \right) dr.$$

By Lemmas 4.1 and 4.2, we may easily pass to the limit in the above expression as $R \to 1$, so

$$w(x) = \int_B G_1(x,\cdot)\big(f(u_1) - f(u_2)\big)\,dy.$$

Using again Lemmas 4.1 and 4.2, we also have that w is differentiable in the $r = |x|$ variable and

$$\frac{\partial w}{\partial r}(x) = \int_B \frac{\partial G_1}{\partial |x|}(x,\cdot)\big(f(u_1) - f(u_2)\big)\,dy.$$

Using polar coordinates again and Lemmas 4.1 and 4.2, we finally obtain

$$\left|\frac{\partial w}{\partial r}\right| \leqslant C + \sup_{r\in(1/2,1)} \left(\int_{\partial B_r} \left| \frac{\partial G_1}{\partial |x|}(x,\cdot) \right| d\sigma \right) \sup_{\theta\in S^{N-1}} \left(\int_0^1 \big|f(u_1) - f(u_2)\big|(r,\theta)\,dr \right) \leqslant C.$$

It only remains to prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. We first deal with the case where u_1, u_2 are radial and $u_1 \geqslant u_2$. By assumption (6), we have

$$\int_0^1 \big|f(u_1) - f(u_2)\big|\,dr \leqslant \int_0^1 \big(\tilde f(u_1) - \tilde f(u_2)\big)\,dr + K\|u_1 - u_2\|_{L^\infty(B)}.$$

Using (4), we see that $u_1 - u_2$ is bounded and so it remains to estimate $\tilde f(u_1) - \tilde f(u_2)$. By (30), each u_i, $i = 1, 2$, solves

$$\frac{du_i/dr}{\sqrt{2(F(u_i) - g_i)}} = 1.$$

We also know by Lemma 3.4 that $g_i = o(F(u_i))$. So,

$$\lim_{r\to 1} \frac{du_i/dr}{\sqrt{2F(u_i)}} = 1.$$

Using this fact, as well as Lemma 3.5 and (5), we obtain for $R \in (1/2, 1)$,

$$\begin{aligned}
\int_0^R \big(\tilde f(u_1)-\tilde f(u_2)\big)\,dr &\leqslant \int_0^R \big(f(u_1)-f(u_2)\big)\,dr + K\|u_1-u_2\|_{L^\infty(B)}\\
&\leqslant C\int_0^R \big(f(u_1)-f(u_2)\big)\frac{du_1/dr}{\sqrt{2F(u_1)}}\,dr + C\\
&\leqslant C\int_0^R \left(f(u_1)\frac{du_1/dr}{\sqrt{2F(u_1)}} - f(u_2)\frac{du_2/dr}{\sqrt{2F(u_2)}}\right)dr\\
&\quad + C\int_0^R f(u_2)\left(\frac{du_2/dr}{\sqrt{2F(u_2)}} - \frac{du_1/dr}{\sqrt{2F(u_1)}}\right)dr + C\\
&\leqslant C\big(\sqrt{F(u_1)}-\sqrt{F(u_2)}\big)(R) + C\\
&\quad + C\int_0^R f(u_2)\left(\frac{\sqrt{2(F(u_2)-g_2)}}{\sqrt{2F(u_2)}} - \frac{\sqrt{2(F(u_1)-g_1)}}{\sqrt{2F(u_1)}}\right)dr\\
&\leqslant C\left(\frac{F(u_1)-F(u_2)}{\sqrt{F(u_1)}+\sqrt{F(u_2)}}\right)(R) + C\\
&\quad + C\int_0^R f(u_2)\frac{\sqrt{F(u_2)F(u_1)-g_2F(u_1)} - \sqrt{F(u_1)F(u_2)-g_1F(u_2)}}{\sqrt{F(u_2)F(u_1)}}\,dr\\
&\leqslant \frac{C}{\sqrt{F(u_1(R))}}\big\|F(u_1)-F(u_2)\big\|_{L^\infty(B)} + C\\
&\quad + C\int_0^R \frac{f(u_2)}{\sqrt{F(u_2)F(u_1)}}\frac{g_1F(u_2)-g_2F(u_1)}{\sqrt{F(u_2)F(u_1)}}\,dr\\
&\leqslant C + C\int_0^R \frac{f(u_2)}{\sqrt{F(u_2)F(u_1)}}\frac{(g_1-g_2)F(u_2)+g_2(F(u_2)-F(u_1))}{\sqrt{F(u_2)F(u_1)}}\,dr\\
&\leqslant C + C\|g_1-g_2\|_{L^\infty(B)}\int_0^R \frac{f(u_2)}{F(u_2)}\,dr + C\big\|F(u_1)-F(u_2)\big\|_{L^\infty(B)}\int_0^R \frac{g_2}{F(u_2)}\,dr\\
&\leqslant C + C\int_{1/2}^R \frac{f(u_2)}{F(u_2)}\frac{du_2/dr}{\sqrt{2F(u_2)}}\,dr + C\\
&\leqslant C + C\big(F^{-1/2}(1/2) - F^{-1/2}(R)\big) \leqslant C.
\end{aligned}$$

This proves the lemma for radial solutions. To obtain the estimate in the general case, we may always assume that u_2 is the minimal solution of (1), so that $u_2 \leqslant u_1$ and u_2 is radial. By Lemma 2.4, up to replacing f by $\tilde f$ given by Remark 2.1, there exists another radial solution V such that $V \geqslant u_1 \geqslant u_2$. Using assumption (6), we have

$$\begin{aligned}\int_0^1 \left|f(u_1)-f(u_2)\right| dr &\leqslant \int_0^1 \left(\tilde f(u_1)-\tilde f(u_2)\right) dr + K\|u_1-u_2\|_{L^\infty(B)}\\ &\leqslant \int_0^1 \left(\tilde f(V)-\tilde f(u_2)\right) dr + K\|u_1-u_2\|_{L^\infty(B)}\\ &\leqslant \int_0^1 \left(f(V)-f(u_2)\right) dr + 2K\|u_1-u_2\|_{L^\infty(B)}.\end{aligned}$$

By (4), $u_1 - u_2$ is bounded and the result follows from the radial case. □

Proof of Lemma 4.2. (45) is standard: write the representation formula (44) both in B_R and in B_1, change variables in the B_1 integral and identify the kernels. Next, we prove that given any $r \in (0,1)$, $\int_{\partial B_r} G_1(x,\cdot)\,d\sigma \leqslant 1$. It suffices to show that for any $\phi \in C_c(0,1)$,

$$\int_0^1 \phi(r)\left(\int_{\partial B_r} G_1(x,\cdot)\,d\sigma\right) dr \leqslant \|\phi\|_{L^1(0,1)}. \tag{47}$$

By definition of Green's function, the left-hand side of the above inequality is the function v solving

$$\begin{cases} -\Delta v = \phi & \text{in } B,\\ v = 0 & \text{on } \partial B.\end{cases}$$

The above equation can also be integrated directly:

$$v'(r) = r^{1-N}\int_0^r \phi(t)t^{N-1}\,dt,$$

whence $|v'| \leqslant \|\phi\|_{L^1(0,1)}$ and $|v| \leqslant \|\phi\|_{L^1(0,1)}$, i.e. (47) holds. This proves that $\int_{\partial B_r} G_1(x,\cdot)\,d\sigma \leqslant 1$.

We turn to the second estimate in (46). Recall that the Green's function in the unit ball is expressed for $x, y \in B$, $x \neq y$, by

$$G_1(x,y) = \Gamma\left(\left(R^2+r^2-2Rr\cos\varphi\right)^{1/2}\right) - \Gamma\left(\left(1+R^2r^2-2Rr\cos\varphi\right)^{1/2}\right), \tag{48}$$

where $R=|x|$, $r=|y|$, φ is the angle formed by the vectors x and y and Γ is the fundamental solution of the Laplace operator. Differentiating with respect to R, we obtain for some $C_N > 0$,

$$\begin{aligned} C_N\frac{\partial G_1}{\partial |x|}(x,y) &= \frac{R-r\cos\varphi}{(R^2+r^2-2Rr\cos\varphi)^{N/2}} - \frac{Rr^2-r\cos\varphi}{(1+R^2r^2-2Rr\cos\varphi)^{N/2}}\\ &= \frac{R-r+r(1-\cos\varphi)}{((R-r)^2+2Rr(1-\cos\varphi))^{N/2}} - \frac{Rr^2-r+r(1-\cos\varphi)}{((1-Rr)^2+2Rr(1-\cos\varphi))^{N/2}}\\ &= A-B. \end{aligned} \tag{49}$$

We estimate A and leave the reader perform similar calculations for B. Clearly, given $\varepsilon > 0$, the expression (49) remains uniformly bounded in the range $1/2 < R, r < 1$, $\varepsilon < \varphi < 2\pi - \varepsilon$. Hence,

$$\int_{\partial B_r} |A|\, d\sigma \leqslant C_\varepsilon + C \int_{\partial B_r \cap [0<\varphi<\varepsilon]} |A|\, d\sigma.$$

For $y \in \partial B_r \cap [0 < \varphi < \varepsilon]$, let $z = z(y)$ denote the intersection of the line (Oy) and the hyperplane P passing through x and tangent to the hypersphere ∂B_R. Then, there exist constants $c_1, c_2 > 0$ such that for all $y \in \partial B_r \cap [0 < \varphi < \varepsilon]$,

$$c_1(1 - \cos\phi) \leqslant |z - x|^2 \leqslant c_2(1 - \cos\phi).$$

Hence, letting $B^{N-1}(x, \rho) \subset P$ denote the $N - 1$-dimensional ball of radius $\rho > 0$ centered at x, we obtain

$$\begin{aligned}
\int_{\partial B_r} |A|\, d\sigma &\leqslant C\left(1 + \int_{B^{N-1}(x, R\sin\varepsilon)} \frac{|R-r| + Cr|z-x|^2}{(|R-r|^2 + c|z-x|^2)^{N/2}}\, dz\right) \\
&\leqslant C\left(1 + \int_{B^{N-1}(O, R_\varepsilon)} \frac{|R-r| + C|z|^2}{(|R-r|^2 + c|z|^2)^{N/2}}\, dz\right) \\
&\leqslant C\left(1 + \int_{B^{N-1}(O, \frac{R_\varepsilon}{|R-r|})} \frac{|R-r| + C|R-r|^2|z|^2}{|R-r|^N(1 + c|z|^2)^{N/2}} |R-r|^{N-1}\, dz\right) \\
&\leqslant C\left(1 + \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + c|z|^2)^{N/2}}\, dz + |R-r| \int_{B^{N-1}(O, \frac{R_\varepsilon}{|R-r|})} |z|^{2-N}\, dz\right) \\
&\leqslant C.
\end{aligned}$$

Working similarly with the B term in (49), we finally obtain the desired estimate (46). □

5. Asymptotics to all orders

This section is devoted to the proof of Theorem 1.9. Our first task consists in applying the Fixed Point Theorem to the functional $\mathcal{N}$ defined for $v \in \mathcal{B}(v_0, \rho)$, $u \in I$ by

$$\big[\mathcal{N}(v)\big](u) = \sqrt{2\left(F(u) - (N-1)\int_{U_0}^{u} \frac{v}{r}\, dt\right)}, \tag{50}$$

where r is given by (13). Let us check first that $\mathcal{N}(\mathcal{B}(v_0, \rho)) \subset \mathcal{B}(v_0, \rho)$. Take $v \in \mathcal{B}(v_0, \rho)$. Then,

$$1 \geqslant r \geqslant 1 - \frac{1}{1-\rho} \int_{U_0}^{+\infty} \frac{1}{v_0}\, dt = 1 - \frac{1}{1-\rho} \int_{U_0}^{+\infty} \frac{1}{\sqrt{2F}}\, dt. \tag{51}$$

By (3), it follows that for $\rho < 1/4$ and U_0 sufficiently large, $1 \geqslant r \geqslant 1/2$. Hence,

$$\left|\int_{U_0}^{u} \frac{v}{r}\,dt\right| \leqslant C\int_{U_0}^{u} \sqrt{2F}\,dt = o\big(F(u)\big),$$

where we used Lemma 3.4. So, for U_0 large and $u \geqslant U_0$,

$$\left|\frac{N-1}{F(u)}\int_{U_0}^{u} \frac{v}{r}\,dt\right| \leqslant \rho.$$

We deduce that

$$\left|\frac{\mathcal{N}(v)-v_0}{v_0}\right| = 1-\sqrt{1-\frac{N-1}{F(u)}\int_{U_0}^{u}\frac{v}{r}\,dt} \leqslant \frac{1}{2}\left|\frac{N-1}{F(u)}\int_{U_0}^{u}\frac{v}{r}\,dt\right| < \rho. \tag{52}$$

Next, we prove that $\mathcal{N}$ is contractive. Given $v_1, v_2 \in \mathcal{B}(v_0,\rho)$, let $r_1 = r(u,v_1), r_2 = r(u,v_2)$ (where r is given by (13)). Then, by estimate (51), $1/2 \leqslant r_1, r_2 \leqslant 1$ and

$$\begin{aligned}
\left|\frac{\mathcal{N}(v_1)-\mathcal{N}(v_2)}{v_0}\right| &= \left|\sqrt{1-\frac{N-1}{F(u)}\int_{U_0}^{u}\frac{v_1}{r_1}\,dt} - \sqrt{1-\frac{N-1}{F(u)}\int_{U_0}^{u}\frac{v_2}{r_2}\,dt}\right| \\
&\leqslant C\frac{N-1}{F(u)}\int_{U_0}^{u}\left|\frac{v_1}{r_1}-\frac{v_2}{r_2}\right|dt \\
&\leqslant \frac{C}{F(u)}\left(\int_{U_0}^{u}|v_1-v_2|\,dt + \int_{U_0}^{u} v_0\left|\frac{1}{r_1}-\frac{1}{r_2}\right|dt\right) \\
&\leqslant \frac{C}{F(u)}\left(\rho\int_{U_0}^{u}\sqrt{2F}\,dt + \int_{U_0}^{u}\sqrt{2F}|r_1-r_2|\,dt\right) \\
&\leqslant \frac{C}{F(u)}\left(\rho\int_{U_0}^{u}\sqrt{2F}\,dt + \int_{U_0}^{u}\sqrt{2F}\left|\int_{t}^{+\infty}\left(\frac{1}{v_1}-\frac{1}{v_2}\right)ds\right|dt\right) \\
&\leqslant \frac{C\rho}{F(u)}\left(\int_{U_0}^{u}\sqrt{2F}\,dt + \int_{U_0}^{u}\sqrt{2F}\left|\int_{t}^{+\infty}\frac{1}{\sqrt{2F}}\,ds\right|dt\right) \\
&\leqslant \frac{C\rho}{F(u)}\int_{U_0}^{u}\sqrt{2F}\,dt.
\end{aligned}$$

Using Lemma 3.4, we conclude that $\mathcal{N}$ is contractive in $\mathcal{B}(v_0,\rho)$ if U_0 was chosen large enough in the first place. We may thus apply the Fixed Point Theorem.

So, it only remains to prove (16). We first observe that the sequence (v_k) defined by (14) is asymptotic, i.e. $v_{k+1}(u) = v_k(u)(1 + o(1))$, as $u \to +\infty$. Since $v_{k+1} = \mathcal{N}(v_k)$, it suffices to prove that $\mathcal{N}(v_0) - v_0 = o(v_0)$ and iterate. By (52),

$$\left|\frac{\mathcal{N}(v_0) - v_0}{v_0}\right| \leqslant \frac{C}{F(u)} \int_{U_0}^{u} \sqrt{2F}\, dt$$

and the claim follows by Lemma 3.4. So, the sequence (v_k) is asymptotic and so must be the sequence (u_k) defined by (15). We are now in a position to prove (16). By Theorem 1.1, we may restrict to the case where u is radially symmetric. Let $v = du/dr$. By (26), v solves

$$\frac{dv}{dr} + \frac{N-1}{r} v = f(u).$$

Use the change of variable $u = u(r)$ to get

$$v\frac{dv}{du} + \frac{N-1}{r} v = f(u).$$

Integrating, it follows that for some constant C

$$\frac{v^2}{2} = F(u) + C - \int_{U_0}^{u} \frac{N-1}{r} v\, dt. \tag{53}$$

Up to replacing $F(u)$ by $\tilde{F}(u) = F(u) + C$ (which is harmless from the point of view of asymptotics), we may assume $C = 0$. So it suffices to prove that $v \in \mathcal{B}(v_0, \rho)$ to conclude that v coincides with the unique fixed point of $\mathcal{N}$, whence (16) will follow. By (53) (with $C = 0$), $v \leqslant v_0$ and so

$$0 \leqslant v_0 - v \leqslant \sqrt{2F(u)} - \sqrt{2\left(F(u) - \int_{U_0}^{u} \frac{N-1}{r} v_0\, dt\right)} \leqslant C\frac{\int_{U_0}^{u} \sqrt{2F}\, dt}{\sqrt{2F(u)}}.$$

By Lemma 3.4, it follows that

$$0 \leqslant \frac{v_0 - v}{v_0} \leqslant v_0 - v < \rho$$

and $v \in B(v_0, \rho)$ as desired.

6. Universal blow-up rate

In this section, we prove Theorem 1.10, that is we characterize nonlinearities for which the blow-up rate is universal.

Proof of Theorem 1.10.

Step 1. We begin by establishing the theorem when $\Omega = B$ is the unit ball. In light of Theorem 1.1, it suffices to prove (19) for one given solution u of (1), which we may therefore assume to be radial. By (30), we have after integration that

$$\int_u^{+\infty} \frac{1}{\sqrt{2(F(t)-g)}}\,dt = 1-r. \tag{54}$$

By definition of u_0, we also have

$$\int_{u_0}^{+\infty} \frac{1}{\sqrt{2F(t)}}\,dt = 1-r. \tag{55}$$

Observe that $u \geqslant u_0$, split the integral in (55) as $\int_{u_0}^{+\infty} = \int_{u_0}^{u} + \int_u^{+\infty}$ and equate (54) and (55). It follows that

$$\begin{aligned}
\int_{u_0}^{u} \frac{1}{\sqrt{2F(t)}}\,dt &= \int_u^{+\infty} \left(\frac{1}{\sqrt{2(F(t)-g)}} - \frac{1}{\sqrt{2F(t)}} \right) dt \\
&= \int_u^{+\infty} \frac{\sqrt{2F(t)} - \sqrt{2(F(t)-g)}}{\sqrt{2F(t)}\sqrt{2(F(t)-g)}}\,dt \\
&= \int_u^{+\infty} \frac{g}{\sqrt{2F(t)}\sqrt{2(F(t)-g)}(\sqrt{2F(t)} + \sqrt{2(F(t)-g)})}\,dt.
\end{aligned}$$

Recall that by Lemma 3.4, $g = o(F)$ as $t \to +\infty$ and $g(u) \sim (N-1)G(u) = (N-1)\int_0^u \sqrt{2F}\,dt$. So, for sufficiently large values of u, we deduce

$$\int_{u_0}^{u} \frac{1}{\sqrt{2F(t)}}\,dt \leqslant C \int_u^{+\infty} \frac{\int_0^t \sqrt{2F}\,ds}{(2F(t))^{3/2}}\,dt. \tag{56}$$

Since F is nondecreasing, it follows that

$$0 \leqslant \frac{u-u_0}{\sqrt{2F(u)}} \leqslant \int_{u_0}^{u} \frac{1}{\sqrt{2F(t)}}\,dt \leqslant C \int_u^{+\infty} \frac{\int_0^t \sqrt{2F}\,ds}{(2F(t))^{3/2}}\,dt.$$

Hence,

$$0 \leqslant u - u_0 \leqslant C\sqrt{2F(u)} \int_u^{+\infty} \frac{\int_0^t \sqrt{2F}\,ds}{(2F(t))^{3/2}}\,dt,$$

and (19) follows from (18).

Step 2. Next, we prove that (19) holds for general domains Ω. To this end, we combine a standard approximation argument by inner and outer spheres (see, e.g., [9]) and the comparison technique of [6]. Let u denote a solution of (17) and take a point $x_0 \in \partial\Omega$. Let $B \subset \Omega$ denote a ball which is tangent to $\partial\Omega$ at x_0. Shrink B somewhat by letting $B_\varepsilon = (1-\varepsilon)B$, $\varepsilon > 0$. Observe that $u \in C(\overline{B_\varepsilon})$ is a subsolution of

$$\begin{cases} \Delta U = f(U) & \text{in } B_\varepsilon, \\ U = +\infty & \text{on } \partial B_\varepsilon. \end{cases} \tag{57}$$

By Lemma 2.3, there exists a solution V_ε of (57), such that $V_\varepsilon \geqslant u$ in B_ε. Furthermore, V_ε can be chosen to be the minimal solution of (57) such that $V_\varepsilon \geqslant u$ in B_ε. In particular, V_ε is radial and $\varepsilon \to V_\varepsilon$ is nondecreasing. In addition, $\varepsilon \to V_\varepsilon$ is uniformly bounded on compact subsets of B (working as in the proof of Lemma 2.3), so V_ε converges as $\varepsilon \to 0$, to a solution V of (1) such that $V \geqslant u$ in B. By Step 1,

$$\lim_{\substack{x\to x_0\\ x\in B}} V(x) - u_0\big(1 - d_B(x)\big) = 0,$$

where d_B denotes the distance to ∂B. Since $V \geqslant u$ and since the above discussion is valid for any point $x_0 \in \partial\Omega$, we finally obtain

$$\limsup_{x\to\partial\Omega}\big[u(x) - u_0\big(1 - d(x)\big)\big] \leqslant 0, \tag{58}$$

where $d(x)$ is the distance to $\partial\Omega$. Choose now an exterior ball $B \subset \mathbb{R}^N \setminus \overline{\Omega}$ which is tangent to $\partial\Omega$ at x_0. For $\varepsilon > 0$ small and $R > 0$ large, the annulus $A_\varepsilon = RB \setminus (1-\varepsilon)B$ contains Ω. Let U_ε denote a large solution on A_ε, which we may assume to be minimal, radial and bounded above on Ω by u. Again $U_\varepsilon \to U$ as $\varepsilon \to 0$ where U is a radial large solution in $A = RB \setminus B \supset \Omega$. Repeating the analysis of Step 1 (which was purely local) for the case of a radial solution defined on an annulus rather than a ball, we easily deduce that

$$\lim_{\substack{x\to x_0\\ x\in B}} U(x) - u_0\big(1 - d_B(x)\big) = 0.$$

Since $u \geqslant U$ and since the above discussion is valid for any point $x_0 \in \partial\Omega$, we obtain

$$\liminf_{x\to\partial\Omega}\big[u(x) - u_0\big(1 - d(x)\big)\big] \geqslant 0. \tag{59}$$

So, by (59) and (58), we have that (19) holds in any smoothly bounded domain Ω.

Step 3. It only remains to prove that (19) fails when (20) holds. We use Theorem 1.9 to compute the second term in the asymptotic expansion of a solution. By (14),

$$\begin{aligned} v_1(u) &= \sqrt{2\left(F(u) - (N-1)\int_0^u \sqrt{2F}\,dt\,\big(1+o(1)\big)\right)} \\ &= \sqrt{2F(u)}\left(1 - (N-1)\frac{\int_0^u \sqrt{2F}\,dt}{2F(u)}\big(1+o(1)\big)\right), \end{aligned}$$

whence

$$\frac{1}{v_1} = \frac{1}{\sqrt{2F(u)}}\left(1 + (N-1)\frac{\int_0^u \sqrt{2F}\,dt}{2F(u)}\big(1+o(1)\big)\right)$$
$$= \frac{1}{\sqrt{2F(u)}} + (N-1)\frac{\int_0^u \sqrt{2F}\,dt}{(2F(u))^{3/2}}\big(1+o(1)\big).$$

Integrating (15) for $k = 1$, it follows that for r close enough to 1,

$$\int_{u_1}^{+\infty} \frac{dt}{\sqrt{2F}} + (N-1)\big(1+o(1)\big)\int_{u_1}^{+\infty} \frac{\int_0^t \sqrt{2F}\,ds}{(2F)^{3/2}}\,dt = 1 - r. \tag{60}$$

Recall (55), split the integral in (55) as $\int_{u_0}^{+\infty} = \int_{u_0}^{u} + \int_{u}^{+\infty}$ and equate (60) and (55) to get

$$\int_{u_0}^{u_1} \frac{dt}{\sqrt{2F}} = (N-1)\big(1+o(1)\big)\int_{u_1}^{+\infty} \frac{\int_0^t \sqrt{2F}\,ds}{(2F)^{3/2}}\,dt.$$

Since F is nondecreasing, we deduce that

$$\frac{u_1 - u_0}{\sqrt{2F(u_0)}} \geqslant (N-1)\big(1+o(1)\big)\int_{u_1}^{+\infty} \frac{\int_0^t \sqrt{2F}\,ds}{(2F)^{3/2}}\,dt. \tag{61}$$

Note also that

$$\int_{u_0}^{u_1} \frac{\int_0^t \sqrt{2F}\,ds}{(2F)^{3/2}}\,dt \leqslant \int_{u_0}^{u_1} \frac{t}{2F}\,dt \leqslant \frac{(u_1-u_0)^2}{4F(u_0)}. \tag{62}$$

Assume by contradiction that $\lim_{r\to 1^-}(u_1 - u_0)(r) = 0$. Then, (62) implies that

$$\int_{u_0}^{u_1} \frac{\int_0^t \sqrt{2F}\,ds}{(2F)^{3/2}}\,dt = o\left(\frac{u_1 - u_0}{\sqrt{2F(u_0)}}\right).$$

Using this information in (61), we obtain that

$$\frac{u_1 - u_0}{\sqrt{2F(u_0)}} \geqslant (N-1)\big(1+o(1)\big)\int_{u_0}^{+\infty} \frac{\int_0^t \sqrt{2F}\,ds}{(2F)^{3/2}}\,dt.$$

But (20) would then lead us to a contradiction with the assumption $\lim_{r\to 1^-}(u_1 - u_0)(r) = 0$. So we must have

$$\liminf_{r\to 1^-}(u_1 - u_0) > 0$$

and so (19) fails. □

7. The first three singular terms

In the previous section, we characterized nonlinearities for which only one term in the expansion is singular. In the present section, we calculate implicitly the next two terms in the expansion. We have not tried to characterize those f for which all remaining terms are nonsingular, but this can certainly be achieved. We leave the tenacious reader try her/his hand at this computational problem.

We begin by calculating the leading asymptotics of v_1, v_2. By (14), we have

$$\frac{v_1^2}{2} = F - (N-1)\int^u \sqrt{2F}\big(1+o(1)\big)\,dt.$$

So,

$$\frac{v_1}{\sqrt{2}} = \sqrt{F}\left(1-(N-1)\frac{\int^u \sqrt{2F}\,dt}{F}\big(1+o(1)\big)\right)^{1/2} = \sqrt{F} - \frac{N-1}{2}\frac{\int^u \sqrt{2F}\,dt}{\sqrt{F}}\big(1+o(1)\big).$$

In other words,

$$v_1 = \sqrt{2F} - (N-1)\frac{\int^u \sqrt{2F}\,dt}{\sqrt{2F}}\big(1+o(1)\big).$$

To calculate v_2, we introduce some notation. Given a positive measurable function v, set

$$Pv = \int^u v\,dt, \qquad Qv = \frac{Pv}{v}, \qquad Rv = \int_u^{+\infty} \frac{dt}{v},$$

and

$$Tv = (N-1)PQv + P(vRv).$$

v_1 is then expressed by

$$v_1 = v_0 - (N-1)\big(1+o(1)\big)Qv_0,$$

while v_2 is given by

$$\begin{aligned}
\frac{v_2^2}{2} &= F - (N-1)\int^u \frac{v_1}{1-\int_t^{+\infty}\frac{ds}{v_0}(1+o(1))}\,dt \\
&= F - (N-1)\int^u \big(v_0 - (N-1)Qv_0 + o(Qv_0)\big)\big(1 - Rv_0 + o(Rv_0)\big)\,dt \\
&= F - (N-1)Pv_0 + (N-1)Tv_0\big(1+o(1)\big).
\end{aligned}$$

So,

$$
\begin{aligned}
v_2 &= \big(2F - 2(N-1)Pv_0 + 2(N-1)Tv_0\big(1+o(1)\big)\big)^{1/2} \\
&= v_0\left(1 - (N-1)\frac{Pv_0}{F} + (N-1)\frac{Tv_0}{F}\big(1+o(1)\big)\right)^{1/2} \\
&= v_0\left(1 - \frac{N-1}{2}\frac{Pv_0}{F} + \frac{N-1}{2}\frac{Tv_0}{F} - \frac{3}{8}(N-1)^2\left(\frac{Pv_0}{F}\right)^2 + o\big(Tv_0/F + (Pv_0/F)^2\big)\right).
\end{aligned}
$$

And so,

$$
\begin{aligned}
\frac{1}{v_2} &= \frac{1}{v_0}\left(1 + \frac{N-1}{2}\frac{Pv_0}{F} - \frac{N-1}{2}\frac{Tv_0}{F} + \frac{5}{8}(N-1)^2\left(\frac{Pv_0}{F}\right)^2 + o\big(Tv_0/F + (Pv_0/F)^2\big)\right) \\
&= \frac{1}{v_0} + (N-1)\frac{Pv_0}{v_0^3} + (N-1)\left(-\frac{Tv_0}{v_0^3} + \frac{5}{4}(N-1)\frac{(Pv_0)^2}{v_0^5}\right)\big(1+o(1)\big) \\
&= \frac{1}{\sqrt{2F}} + (N-1)\frac{\int^u \sqrt{2F}\,dt}{(2F)^{3/2}} \\
&\quad + \frac{(N-1)}{(2F)^{3/2}}\left(-\int^u \left((N-1)\frac{\int^t \sqrt{2F}\,ds}{\sqrt{2F}} + \sqrt{2F}\int_u^{+\infty}\frac{ds}{\sqrt{2F}}\right)dt\right. \\
&\quad \left. + \frac{5(N-1)}{4}\frac{(\int^u \sqrt{2F}\,dt)^2}{2F}\right)\big(1+o(1)\big).
\end{aligned}
$$

Integrating once more, we finally obtain

$$
\begin{aligned}
1 - r = \int_{u_2(r)}^{+\infty} \frac{du}{\sqrt{2F}} &+ (N-1)\int_{u_2(r)}^{+\infty} \frac{\int^u \sqrt{2F}\,dt}{(2F)^{3/2}}\,du + \big(1+o(1)\big) \\
&\times (N-1)\int_{u_2(r)}^{+\infty}\left(-\int^u \left((N-1)\frac{\int^t \sqrt{2F}\,ds}{\sqrt{2F}} + \sqrt{2F}\int_u^{+\infty}\frac{ds}{\sqrt{2F}}\right)dt\right. \\
&\left. + \frac{5(N-1)}{4}\frac{(\int^u \sqrt{2F}\,dt)^2}{2F}\right)\frac{du}{(2F)^{3/2}}.
\end{aligned}
$$

This proves Proposition 1.14.

8. An example: $f(u) = u^p$, $p > 1$

Finding the n-th term in the expansion for arbitrary $n \in \mathbb{N}$ is out of reach for general f, simply because of the algorithmic complexity of calculations. However, when additional information on f is available, one can guess the general form of the expansion and then try to establish it. This is precisely what we do in this section, with the nonlinearity $f(u) = u^p$, $p > 1$.

For notational convenience, we shall work with $F(u) = \frac{1}{2}u^{2q}$, where $2q - 1 = p$, which simply amounts to working with a constant multiple of the original solution.

Recall (14) and (15). We want to prove inductively that there exist numbers a_k, b_k depending on k, p, N only such that

$$v_n = u^q \sum_{k=0}^{n} b_k u^{-k(q-1)} + o\bigl(u^{q-n(q-1)}\bigr), \tag{63}$$

$$u_n = d^{-\frac{1}{q-1}} \sum_{k=0}^{n} a_k d^k + E_n\bigl(d^{-\frac{1}{q-1}+n+1}\bigr), \tag{64}$$

where $E_n(d^{-\frac{1}{q-1}+n+1}) \sim e_n d^{-\frac{1}{q-1}+n+1}$ for some $e_n \in \mathbb{R}$, as $d \to 0^+$. We have $v_0 = \sqrt{2F} = u^q$. Solving for u_0 in (15) yields $u_0 = cd^{-\frac{1}{q-1}}$. So, (64) and (63) hold for $n = 0$. Suppose now the result is true for a given $n \in \mathbb{N}$. In the computations below, the letter c_k denotes a number depending on k, p, N only, which value may change from line to line. By (63), we have

$$\begin{aligned}\frac{1}{v_n} &= u^{-q}\left(1 + \sum_{k=1}^{n} b_k u^{-k(q-1)} + o\bigl(u^{-n(q-1)}\bigr)\right)^{-1} \\ &= u^{-q}\left(\sum_{k=0}^{n} c_k u^{-k(q-1)} + o\bigl(u^{-n(q-1)}\bigr)\right).\end{aligned}$$

So,

$$\begin{aligned}\int_t^{+\infty} \frac{ds}{v_n} &= t^{1-q}\left(\sum_{k=0}^{n} c_k t^{-k(q-1)}\right) + o\bigl(t^{-(n+1)(q-1)}\bigr) \\ &= \sum_{k=1}^{n+1} c_k t^{-k(q-1)} + o\bigl(t^{-(n+1)(q-1)}\bigr).\end{aligned}$$

It follows that

$$\frac{1}{1 - \int_t^{+\infty} \frac{ds}{v_n}} = \sum_{k=0}^{n+1} c_k t^{-k(q-1)} + o\bigl(t^{-(n+1)(q-1)}\bigr).$$

Whence,

$$\frac{v_n(t)}{1 - \int_t^{+\infty} \frac{ds}{v_n}} = t^q \sum_{k=0}^{n} c_k t^{-k(q-1)} + o\bigl(t^{q-n(q-1)}\bigr).$$

And so,

$$\begin{aligned}v_{n+1} &= \sqrt{2F - (N-1)\int^{u} \frac{v_n}{1 - \int_t^{+\infty} \frac{ds}{v_n}}\, dt} \\ &= \sqrt{u^{2q} + u^{q+1} \sum_{k=0}^{n} c_k u^{-k(q-1)} + o\bigl(u^{1+q-n(q-1)}\bigr)}\end{aligned}$$

$$= u^q \left(1 + \sum_{k=1}^{n+1} c_k u^{-k(q-1)} + o\left(u^{-(n+1)(q-1)}\right)\right)^{1/2}$$

$$= u^q \sum_{k=0}^{n+1} c_k u^{-k(q-1)} + o\left(u^{q-(n+1)(q-1)}\right).$$

This proves (63). Integrating (15), we obtain

$$\int_{u_n}^{+\infty} \frac{du}{v_n} = d. \tag{65}$$

Now, $v_{n+1} = v_n + c_{n+1} u^{q-(n+1)(q-1)}(1 + o(1))$. So,

$$\frac{1}{v_{n+1}} = \frac{1}{v_n} + c_{n+1} u^{-q-(n+1)(q-1)}\left(1 + o(1)\right).$$

It follows that

$$d = \int_{u_{n+1}}^{+\infty} \frac{du}{v_{n+1}} = \int_{u_{n+1}}^{+\infty} \frac{du}{v_n} + c_{n+1} u_{n+1}^{-(n+2)(q-1)}\left(1 + o(1)\right).$$

In addition, $v_n \sim v_0$, so $u_n \sim u_0$, and so $u_{n+1}^{-(q-1)} \sim d$. Using this in the above equation, we get

$$d + c_{n+1} d^{n+2}\left(1 + o(1)\right) = \int_{u_{n+1}}^{+\infty} \frac{du}{v_n}.$$

Recalling that v_n is defined by (65) and satisfies (64) by induction hypothesis, we conclude that

$$v_{n+1} = \left(d + c_{n+1} d^{n+2}\left(1 + o(1)\right)\right)^{-\frac{1}{q-1}} \sum_{k=0}^{n} a_k \left(d + c_{n+1} d^{n+2}(1 + o(1))\right)^k + E_n\left(d^{-\frac{1}{q-1}+n+1}\right).$$

Expanding again the above expression, we finally obtain

$$v_{n+1} = d^{-\frac{1}{q-1}} \sum_{k=0}^{n+1} a_k d^k + E_{n+1}\left(d^{-\frac{1}{q-1}+n+2}\right),$$

which proves (64). Proposition 1.15 follows.

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References

[1] Catherine Bandle, Asymptotic behavior of large solutions of elliptic equations, An. Univ. Craiova Ser. Mat. Inform. 32 (2005) 1–8, MR2215890 (2006j:35066).
[2] Catherine Bandle, Moshe Marcus, On second-order effects in the boundary behaviour of large solutions of semilinear elliptic problems, Differential Integral Equations 11 (1) (1998) 23–34, MR1607972 (98m:35049).
[3] S. Berhanu, G. Porru, Qualitative and quantitative estimates for large solutions to semilinear equations, Commun. Appl. Anal. 4 (1) (2000) 121–131, MR1855986 (2002j:35099).
[4] Haim Brezis, Personal communication to L. Véron, 2005.
[5] P.J. McKenna, Conference, Bergamo, 2002.
[6] Serge Dumont, Louis Dupaigne, Olivier Goubet, Vicentiu Rădulescu, Back to the Keller–Osserman condition for boundary blow-up solutions, Adv. Nonlinear Stud. 7 (2) (2007) 271–298, MR2308040 (2008e:35062).
[7] B. Gidas, Wei Ming Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (3) (1979) 209–243, MR544879 (80h:35043).
[8] J.B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Math. 10 (1957) 503–510, MR0091407 (19,964c).
[9] A.C. Lazer, P.J. McKenna, Asymptotic behavior of solutions of boundary blowup problems, Differential Integral Equations 7 (3–4) (1994) 1001–1019, MR1270115 (95c:35084).
[10] P.J. McKenna, W. Reichel, W. Walter, Symmetry and multiplicity for nonlinear elliptic differential equations with boundary blow-up, Nonlinear Anal. 28 (7) (1997) 1213–1225, MR1422811 (97h:35049).
[11] Robert Osserman, On the inequality $\Delta u \geqslant f(u)$, Pacific J. Math. 7 (1957) 1641–1647, MR0098239 (20#4701).
[12] Alessio Porretta, Laurent Véron, Symmetry of large solutions of nonlinear elliptic equations in a ball, J. Funct. Anal. 236 (2) (2006) 581–591, MR2240175 (2007b:35131).