Linear Operators Preserving \( L \)-Matrices

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ABSTRACT

We investigate the linear operators that preserve the set of \( L \)-matrices. We show that a linear operator \( T \) preserves the set of \( L \)-matrices and \( T \) is also one-to-one on the set of cells if and only if \( T \) strongly preserves the set of \( L \)-matrices. In each case, \( T \) preserves the set of matrices of term rank 1. Then by a result of Beasley and Pullman on preservers of matrices of term rank 1, we can obtain the structure of \( T \).

INTRODUCTION

Let \( \mathbb{K} \) be a field, and \( \mathbb{M}_{m,n} = \mathbb{M}_{m,n}(\mathbb{K}) \) be the set of all \( m \times n \) matrices over \( \mathbb{K} \). Let \( T : \mathbb{M}_{m,n} \rightarrow \mathbb{M}_{m,n} \) be a linear operator. We say \( T \) preserves the subset \( \mathcal{H} \) of \( \mathbb{M}_{m,n} \) if \( T \) maps each matrix in the set \( \mathcal{H} \) to a matrix in \( \mathcal{H} \). We say \( T \) strongly preserves the subset \( \mathcal{H} \) of \( \mathbb{M}_{m,n} \) if \( T \) preserves both \( \mathcal{H} \) and \( \mathbb{M}_{m,n} \setminus \mathcal{H} \), the complement of \( \mathcal{H} \) in \( \mathbb{M}_{m,n} \). We call such \( T \) an \( \mathcal{H} \) preserver or an \( \mathcal{H} \) strong preserver, respectively.

The study of preservers began in the late nineteenth century when Frobenius [4] investigated the linear operators that preserve the characteristic polynomial and those that preserve the determinant. After more than a half century of relative inactivity in this area, Marcus and Moyls [5, 6] rekindled interest in preserver problems by characterizing the linear operators that preserve the rank function and those that preserve the set of matrices of rank 1. Since 1959, there has been a great deal of activity in this area.


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Recently, the study of what various authors call combinatorial matrix theory, qualitative matrix theory, or structured matrix theory has become popular. Some information may be obtained about a matrix by just knowing the location of its zero entries or by knowing the relative location of its positive, negative, and zero entries. One class of matrices that has been heavily studied is the set of all \( n \times n \) matrices with the property that every matrix with the same \((0, +, -)\) sign pattern is nonsingular. These are called sign-nonsingular matrices. In [2], the linear operators that preserve the set of sign-nonsingular matrices were investigated. In this paper we shall investigate the preservers of a set of related matrices.

An \( L \)-matrix is an \( m \times n \) real matrix \( A \) such that every \( m \times n \) real matrix with the same \((0, +, -)\) sign pattern as \( A \) has linearly independent rows. If \( m = n \), an \( m \times m \) \( L \)-matrix is a sign-nonsingular matrix. Since linear operators which preserve sign-nonsingular matrices were investigated in [2] and [7], throughout this paper we assume \( m < n \).

In this paper, we investigate the linear operators that preserve \( L \)-matrices. We show that a linear operator \( T \) strongly preserves \( L \)-matrices if and only if \( T \) preserves the set of \( L \)-matrices and is one-to-one on the set of cells. We show that if \( T \) strongly preserves the set of \( L \)-matrices, then \( T \) preserves the set of matrices of term rank 1. Then by [1], we can obtain the structure of \( T \).

2. PRELIMINARIES

We restrict our matrices to \( m \times n \) \((m < n)\) real matrices. We let \( \mathbb{M}_{m,n} = \mathbb{M}_{m,n}(\mathbb{R}) \) and \( \mathbb{M}_n = \mathbb{M}_{n,n}(\mathbb{R}) \).

The number of nonzero entries in a matrix \( A \) is denoted \(|A|\). The number of elements in a set \( S \) is also denoted \(|S|\).

An \( m \times n \) matrix whose only nonzero entry is a 1, say the \((i,j)\)th entry, is called a cell and is denoted \( E_{i,j} \). We also denote the set of all cells whose nonzero entries all lie in the \( i \)th row by \( R_i = \{E_{i,1}, E_{i,2}, \ldots, E_{i,n}\} \), and the set of cells whose nonzero entries lie in the \( j \)th column by \( C_j = \{E_{1,j}, E_{2,j}, \ldots, E_{m,j}\} \). A multiple of a cell is called a weighted cell, and so the set of weighted cells is \( \{x_{ij}E_{ij} | x_{ij} \in \mathbb{R}, 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \).

If \( S \) is a set and \( \mathcal{P} \) is the space spanned by \( S \), then we denote \( \mathcal{P} = \langle S \rangle \).

We denote the Hadamard product of \( A = (a_{i,j}) \) and \( B = (b_{i,j}) \) in \( \mathbb{M}_{m,n} \) by \( A \odot B \). That is, \( A \odot B = (a_{i,j}b_{i,j}) \).

If \( A = (a_{i,j}) \) and \( B = (b_{i,j}) \) are in \( \mathbb{M}_{m,n} \), we say that \( B \) dominates \( A \) (written \( B \geq A \) or \( A \leq B \)) if for each \( i, j \), \( b_{i,j} = 0 \) implies \( a_{i,j} = 0 \).

Given a matrix \( A \), the column (the row) term rank is the number of nonzero columns (rows) of \( A \). The term rank is the minimum number, \( t(A) \), of lines (columns or rows) which contain all the nonzero entries of \( A \).
A characterization of preservers of term-rank-1 matrices due to Beasley and Pullman [1, Corollary 3.1.2] is used throughout this paper: Suppose that $T$ is a nonsingular linear operator on $\mathbb{M}_{m,n}$. The operator preserves the set of matrices of term rank 1 if and only if $T$ is one of or a composition of some of the following operators.

1. $X \to X^t$ if $m = n$;
2. $X \to PXQ$ for some fixed but arbitrary permutation matrices $P$ in $\mathbb{S}_m$ and $Q$ in $\mathbb{S}_n$;
3. $X \to A \circ X$ for some fixed but arbitrary matrix $A$ in $\mathbb{M}_{m,n}$ with no zero entries.

From the definition of $L$-matrices we observe that if $A \in \mathbb{M}_{m,n}$ and $A$ contains a sign-nonsingular submatrix of order $m$, then $A$ is an $L$-matrix. We note that the converse of this is not true for $m > 3$, that is, there are $m \times n$ $L$-matrices which do not contain any sign-nonsingular submatrix of order $m$. Let

$$S_3 = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$  

Then $S_3$ is an $L$-matrix, but $S_3$ does not contain any sign-nonsingular submatrix of order 3, since any sign-nonsingular matrix of order 3 has a zero entry (Brualdi, Chavey, and Shader [3]).

Let $M = (m_{i,j}) \in \mathbb{M}_{m,n}$ with $m_{i,j} > 0$ for all $(i, j)$, $P \in \mathbb{S}_m$ and $Q \in \mathbb{S}_n$ be permutation matrices, and $S_1 \in \mathbb{S}_m$ and $S_2 \in \mathbb{S}_n$ be diagonal matrices with all nonzero entries on the main diagonal. If $A \in \mathbb{M}_{m,n}$ is an $L$-matrix, then:

1. every row of $A$ has nonzero entries, and
2. $PA$, $AQ$, $S_1 A$, $AS_2$, and $A \circ M$ are $L$-matrices.

These facts will be used without reference throughout the sequel.

3. $L$-MATRIX PRESERVERS

In this section we will investigate the linear operators preserving $m \times n$ $L$-matrices. Let $T$ be a linear operator that is one-to-one on the set of weighted cells. We show that $T$ preserves $L$-matrices if and only if $T$ strongly preserves $L$-matrices. We then characterize linear operators that are one-to-one on the set of weighted cells and preserve $L$-matrices as those of the form $T(X) = P_1 S_1 (X \circ M) S_2 P_2$, where $P_1$ and $P_2$ are permutation matrices, $S_1$ and $S_2$ are signature matrices, and $M$ is a positive matrix.
Let $T: \mathfrak{m}_{m,n} \to \mathfrak{m}_{m,n}$ be a linear operator that preserves $L$-matrices. First we note that $T$ is not necessarily a nonsingular operator. For example, if $m = 2$, then we define a linear operator $T : M_{2,2n} \to M_{2,2n}$ by

$$T \begin{pmatrix} a_1 & b_1 & a_2 & b_2 & \cdots & a_n & b_n \\ \star & \star & \star & \star & \cdots & \star & \star \end{pmatrix} = \begin{pmatrix} a_1 & -b_1 & a_2 & -b_2 & \cdots & a_n & -b_n \\ b_1 & a_1 & b_2 & a_2 & \cdots & b_n & a_n \end{pmatrix}$$

for any

$$A = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 & \cdots & a_n & b_n \\ \star & \star & \star & \star & \cdots & \star & \star \end{pmatrix} \in \mathfrak{m}_{2,2n}.$$

If $A$ is an $L$-matrix, then the first row of $A$ must be nonzero. Therefore, some $a_i$ or $b_i$ is nonzero, and $T(A)$ is an $L$-matrix if and only if some $a_i$ or $b_i$ is nonzero. Hence $T$ preserves $L$-matrices. But $T$ is singular, since $T(\sum_{i=1}^{m} E_{2,i}) = 0$.

Now we consider the linear operators that preserve $L$-matrices and that are also one-to-one on the set of weighted cells. Observe that if $T$ is one-to-one on the set of weighted cells, then $T$ preserves a set $\mathcal{A}$ if and only if $T$ strongly preserves $\mathcal{A}$. This is because some power of $T$, say the $p$th, has the property that for every $(i,j), T^p(E_{ij}) = x_{ij}E_{ij}$ for some nonzero $x_{ij}$, i.e., $T^p$ is a weighted permutation.

**Lemma 1.** Let $E_1$ and $E_2$ be two cells whose nonzero entries lie in the same row. If $T$ is a linear operator which is one-to-one on the set of weighted cells and preserves $L$-matrices, then the nonzero entries of the weighted cells $T(E_1)$ and $T(E_2)$ lie in the same row or column.

**Proof.** Suppose not. Without loss of generality we may assume that $T(E_1) = x_{11}E_{11}$ and that $T(E_2) = y_{22}E_{22}$. Then $T^{-1}(I_m \otimes I_n) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$ has term rank less than $m$ and hence is not an $L$-matrix. This contradicts that $T$, and hence $T^{-1}$, preserves $L$-matrices.

A similar argument to the proof above shows that $T$ maps two cells whose nonzero entries lie in the same column to weighted cells whose nonzero entries lie in the same row or column. Since $m < n$, Lemma 1 implies that if a linear operator $T$ is one-to-one on the set of weighted cells and also preserves $L$-matrices, then $T$ maps the cells of a given row to weighted cells
whose nonzero entries are all in a single row, and maps the cells of a given column to weighted cells whose nonzero entries are all in a single column.

An immediate consequence of the above is:

**Theorem 1.** If \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) is a linear operator that preserves \( L \)-matrices and \( T \) is also one-to-one on the set of weighted cells, then \( T \) preserves the set of matrices of term rank 1.

**Theorem 2.** Let \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) be a linear operator. Then \( T \) preserves \( L \)-matrices and \( T \) is also one-to-one on the set of cells, if and only if for any \( X \in \mathcal{M}_{m,n} \), \( T(X) = P_1 S_1 (X \circ M) S_2 P_2 \), where \( P_1 \in \mathcal{M}_m \) and \( P_2 \in \mathcal{M}_n \) are permutation matrices, \( S_1 \in \mathcal{M}_m \) and \( S_2 \in \mathcal{M}_n \) are diagonal matrices of \( \pm 1 \)'s and \( M = (m_{i,j}) \in \mathcal{M}_{m,n} \) with \( m_{i,j} > 0 \).

**Proof.** The sufficiency is easily established.

By Theorem 1, if \( T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) is a linear operator that preserves \( L \)-matrices and \( T \) is also one-to-one on the set of weighted cells, then \( T \) preserves the set of matrices of term rank 1. By the linearity of \( T \), since \( T \) is one-to-one on the set of weighted cells, \( T \) is nonsingular, and hence by Beasley and Pullman [1, Corollary 3.1.2], for any \( X \in \mathcal{M}_{m,n} \) we have \( T(X) = P_1 (X \circ M) P_2 \), where \( P_1 \in \mathcal{M}_m \) and \( P_2 \in \mathcal{M}_n \) are permutation matrices and \( M = (m_{i,j}) \) with \( m_{i,j} \neq 0 \). Let \( T_1 : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) be a linear operator defined by

\[
T_1(X) = P_1^t (X \circ M) P_2^t = P_1^t P_1 (X \circ M) P_2 P_2^t = X \circ M.
\]

Clearly \( T_1 \) preserves \( L \)-matrices, since \( T \) preserves \( L \)-matrices and \( P_1^t, P_2^t \) are permutation matrices. Let \( T_2 : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n} \) be a linear operator defined by

\[
T_2(X) = S_1 T_1(X) S_2 = S_1 (X \circ M) S_2,
\]

where

\[
S_1 = \text{diag} \left( \frac{m_{1,1}}{|m_{1,1}|}, \frac{m_{2,1}}{|m_{2,1}|}, \ldots, \frac{m_{m,1}}{|m_{m,1}|} \right)
\]

and

\[
S_2 = \text{diag} \left( 1, \frac{m_{1,2}}{|m_{1,2}|}, \frac{m_{1,1}}{|m_{1,1}|}, \ldots, \frac{m_{1,n}}{|m_{1,n}|}, \frac{m_{1,1}}{|m_{1,1}|} \right).
\]
Clearly \( T_2 \) preserves \( L \)-matrices, since \( T_1 \) preserves \( L \)-matrices and \( S_1, S_2 \) are diagonal matrices with all entries on the main diagonal equal 1 or \(-1\). Also, since \( S_1 \) and \( S_2 \) are diagonal matrices, we have that

\[
T_2(X) = S_1T_1(X)S_2 = X \circ (S_1MS_2).
\]

Let \( N = S_1MS_2 \). Then \( N = (n_{i,j}) \in \mathbb{M}_{m,n} \) with \( n_{i,j} \neq 0 \), \( n_{i,1} > 0 (1 \leq i \leq m) \), and \( n_{1,j} > 0 (1 \leq j \leq n) \). Since \( T_2 \) preserves \( L \)-matrices, we must have that \( T_2(A) = A \circ (S_1MS_2) = A \circ N \) is an \( L \)-matrix for any \( L \)-matrix \( A \). If \( n_{2,2} < 0 \), we let

\[
A = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 \\
1 & -1 & 0 & \cdots & 0
\end{pmatrix} \oplus I_{m-2}.
\]

Then \( A \) is an \( L \)-matrix, but

\[
A \circ N = \begin{pmatrix}
n_{1,1} & n_{1,2} & 0 & \cdots & 0 \\
n_{2,1} & -n_{2,2} & 0 & \cdots & 0
\end{pmatrix} \oplus \text{diag}(n_{3,n-m+3}, \ldots, n_{m,n})
\]

is not an \( L \)-matrix, a contradiction. Hence \( n_{2,2} > 0 \). By permuting rows and columns we have that all entries in the matrix \( N \) are positive. So \( N = (n_{i,j}) \in \mathbb{M}_{m,n} \) with \( n_{i,j} > 0 \). Thus

\[
T(X) = P_1(X \circ M)P_2 = P_1\left[ X \circ (S_1^{-1})NS_2^{-1} \right]P_2 = P_1S_1^{-1}(X \circ N)S_2^{-1}P_2,
\]

which completes the proof.

Now we consider linear operators that strongly preserve \( L \)-matrices. We have the following theorem.

**Theorem 3.** If \( T : \mathbb{M}_{m,n} \rightarrow \mathbb{M}_{m,n} \) is a linear operator that strongly preserves \( L \)-matrices, then \( T \) is nonsingular.

**Proof.** If not, then there exists an \( m \times n \) matrix \( A = (a_{i,j}) \neq O \) such that \( T(A) = O \). Without loss of generality, we assume \( a_{1,1} \neq 0 \). Now we define an \( m \times n \) matrix \( B = (b_{i,j}) \) by \( b_{1,j} = 0 (1 \leq j \leq n) \), \( b_{1,j} = 0 (2 \leq i < j \leq n) \), \( b_{i,j} = -a_{i,j} (1 \leq j < i \leq m) \), and \( b_{i,i} + a_{i,i} \neq 0 (2 \leq i \leq n) \). Thus \( A + B \) is an \( L \)-matrix. So \( T(A + B) = T(A) + T(B) = T(B) \) must be an \( L \)-matrix. Since \( T \) strongly preserves \( L \)-matrices, we must have that \( B \)
is an $L$-matrix. This is a contradiction because $B$ has first row zero. Therefore, $T(A) \neq O$, so $T$ is nonsingular.

Now let $T : \mathbb{M}_{m,n} \to \mathbb{M}_{m,n}$ be a linear operator that strongly preserves $L$-matrices. Let $R_i = \sum_{j=1}^{n} E_{i,j}$. That is, $R_i$ is the matrix whose $i$th row is all ones and whose other entries are 0. We define a linear operator $T_i$ on $\mathbb{M}_{m,n}$ by $T_i(X) = T(X) \circ R_i$. Let

$$V_i = \{ T_i(E_{k,i}) : 1 \leq k \leq m, 1 \leq l \leq n \}, \quad \mathcal{V}_i = \langle V_i \rangle,$$

and

$$W_i = \{ E_{k,i} : T_i(E_{k,i}) \neq O \}.$$

**Lemma 2.** $\dim \mathcal{V}_i = n$ and $|W_i| \geq n$.

**Proof.** It is evident that $\dim \mathcal{V}_i \leq n$. Since by Theorem 3, $T$ is nonsingular, each $T_i$ must be onto. That is, the dimension of the image of $T_i$ is $n$. Since $\{ E_{k,l} : 1 \leq k \leq m, 1 \leq l \leq n \}$ is a basis for $\mathbb{M}_{m,n}$, we have $\dim \mathcal{V}_i = n$. Since $\dim \mathcal{V}_i \leq |W_i|$, the proof is complete.

**Lemma 3.** If $S = \{ E_1, E_2, \ldots, E_n \}$ is a subset of $W_i$ such that $\mathcal{V}_i = \langle \{ T_i(E_1), T_i(E_2), \ldots, T_i(E_n) \} \rangle$ and $E_i$ has a zero column. Without loss of generality, we assume that $\Sigma_{i=1}^{n} E_i$ has a zero row. Therefore $T((I_m O) + \Sigma_{i=1}^{n} \alpha_i E_i)$ must be an $L$-matrix for any choice of $\alpha_i$'s. But for some choice of $\alpha_i$'s, $T((I_m O) + \Sigma_{i=1}^{n} \alpha_i E_i)$ has a zero $i$th row and hence is not an $L$-matrix, a contradiction.

**Theorem 4.** If $T : \mathbb{M}_{m,n} \to \mathbb{M}_{m,n}$ is a linear operator that strongly preserves $L$-matrices, then $T$ preserves the set of matrices of row term rank 1.

Given any matrix $A$, we denote by $A_i$ the matrix whose $i$th row is the same as the $i$th row of $A$ and all other rows are zero. That is, $A_i = \Sigma_{j=1}^{n} a_{ij} E_{i,j} = A \circ R_i$.
Proof. We show that for each $k$ there is a unique $i$ such that $R_k \subseteq W_i$. The theorem then follows from the nonsingularity of $T$ assured by Theorem 3.

Let $E_1, E_2, \ldots, E_n$ be cells such that $\langle T(E_1), \ldots, T(E_n) \rangle = \mathcal{Y}_i$. Suppose that $F$ is a cell in the same column as $E_1$ and that $F \neq E_1$. Then $T(F) = c_1T_i(E_1) + \cdots + c_nT_i(E_n)$ for some constants $c_1, c_2, \ldots, c_n$. If $c_j \neq 0$ for some $j \neq 1$, then $\langle T_i(F), T_i(E_1), \ldots, T_i(E_{j-1}), T_i(E_{j+1}), \ldots, T_i(E_n) \rangle = \mathcal{Y}_i$, contradicting Lemma 3. Thus, $T_i(F)$ is a multiple of $T_i(E_1)$. It follows that the space spanned by the images under $T_i$ of the cells in any column has dimension 1.

Suppose that $E_{k,1} \in W_i$ and $E_{k,i} \notin W_i$. By permuting rows and columns we may assume without loss of generality that $E_{1,1} \notin W_i$ and $E_{1,2} \in W_i$, and further, since by Lemma 3 for each $j$ there is a cell in $W_i$ whose nonzero entry is in column $j$, that for each $j \geq 3$, there is $k < j$ such that $E_j = E_{k,j} \in W_i$. Thus, since the images under $T_i$ of the cells in any column generate a space of dimension 1, it is possible to choose scalars $c_j$, $j \geq 2$, such that $T_i((I_m O) + \sum_{j=2}^{n} c_j E_j) = O$. But $[I_m O] + \sum_{j=2}^{n} c_j E_j$ is an $L$-matrix, and hence its image cannot have a zero row, a contradiction. Thus if $E_{k,1} \in W_i$ then $R_k \subseteq W_i$.

We now show that each $R_k$ is contained in only one $W_i$. Suppose to the contrary that $R_k \subseteq W_k$ and $R_k \subseteq W_j$ for some $j \neq i$. Then by permuting rows in both the domain and range of $T$, we may assume without loss of generality that $R_1 \subseteq W_1$, $R_1 \subseteq W_2$. By permuting we may assume that for each $i \geq 3$, there is $j < i$ such that $R_j \subseteq W_i$. We can then permute rows so that for each $i \leq m$, $W_i$ contains $R_j$ for some $j \leq l$, and further, such that if $S$ is a proper subset of $\{R_1, \ldots, R_i\}$, there is some $i$ such that no $R_j$ in $S$ is in $W_i$. Note that $l < m$.

Let $K$ be an $L$-matrix with $|K| \geq |B|$ for all $L$-matrices $B$ and such that the number of nonzero entries in row $i$ of $K$ is at least as many as the number of nonzero entries in row $i+1$. That is, $|K_i| \geq |K_{i+1}|$ for $i = 1, 2, \ldots, m - 1$. Now choose scalars $v_{ij}$ such that $V = (v_{ij})$ has the same sign pattern as $K$ and such that $T(V_i) \geq T(B_i)$ for every $B$ with the same sign pattern as $K$. Since $R_1 \subseteq W_1$ and $R_1 \subseteq W_2$, $V$ is chosen so that $|T(V_i)| \geq |K_1| + |K_2|$. Now, choose scalars $y_2, y_3, \ldots, y_l$ so that $|T(V_1 + y_2 V_2 + \cdots + y_l V_l)| \geq |K_1| + |K_2| + \cdots + |K_m|$. This can be done, since $|V_i| \geq |V_{i+1}|$ and each $W_i$ contains $R_j$ for some $j \leq l$. That is, $|T(V_1 + y_2 V_2 + \cdots + y_l V_l)| \geq |K|$. Now, $V_1 + y_2 V_2 + \cdots + y_l V_l + \epsilon (V_{l+1} + \cdots + V_m)$ is an $L$-matrix for all nonzero $\epsilon$; and for some nonzero $\epsilon_0$ we have $T[V_1 + y_2 V_2 + \cdots + y_l V_l + \epsilon_0 (V_{l+1} + \cdots + V_m)] \geq T(V_1 + y_2 V_2 + \cdots + y_l V_l)$; and since $|T(V_1 + y_2 V_2 + \cdots + y_l V_l)| \geq |K|$ and $|K| \geq |B|$ for all $L$-matrices $B$, we must have that $|T[V_1 + y_2 V_2 + \cdots + y_l V_l]| = |K|$. That is, $T[V_1 + y_2 V_2 + \cdots + y_l V_l + \epsilon_0 (V_{l+1} + \cdots + V_m)]$ and $T(V_1 + y_2 V_2 + \cdots + y_l V_l)$ have the same sign pat-
tern. This is a contradiction to the fact that $T$ strongly preserves L-matrices, since $V_1 + y_2 V_2 + \cdots + y_l V_l$ has a zero row because $l < m$.

Thus for each $k$ there is a unique $i$ such that $\Xi_k \subseteq W_i$, and by the nonsingularity assured by Theorem 3, the theorem follows. \hfill \blacksquare

**Theorem 5.** If $T: \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{m,n}$ is a linear operator that strongly preserves L-matrices, then $T$ is one-to-one on the set of cells.

**Proof.** First we show $T(E)$ is a cell for any cell $E$. If not, then there exists a cell $E$ such that $|T(E)| \neq 1$. By Theorem 3, $T$ is nonsingular, so $|T(E)| \neq 0$. By Theorem 4, $T$ preserves the set of matrices of row term rank 1, so we must have that $2 \leq |T(E)| \leq n$. By permuting, we may assume that $T(E_{1,1}) \geq E_{1,1} + E_{1,2}$ and for each $j \geq 3$, there is some $i < j$ such that $T(E_{i,j}) \geq E_{1,j}$. Thus

$$T(E_{1,n}) \leq \sum_{j=1}^{n} E_{1,j} \leq \sum_{j=1}^{n-1} T(\beta_{ij}E_{1,j})$$

for some choice of $\beta_{ij} \neq 0$. Hence for some $\alpha_k \neq 0$ ($k = 1, 2, \ldots, n - 1$), $\sum_{j=1}^{n-1} T(\alpha_j E_{1,j})$ has the entries of its first row all nonzero. Thus, we can choose $\alpha_k \neq 0$ ($k = 1, 2, \ldots, n - 1$) and $e \neq 0$ such that $\sum_{j=1}^{n-1} T(\alpha_j E_{1,j}) + T(eE_{1,n})$ has the same sign pattern as $\sum_{j=1}^{n-1} T(\alpha_j E_{1,j})$. Now we let

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n\text{-}m} & \cdots & \alpha_{n-2} & \alpha_{n-1} & e \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n\text{-}m} & \cdots & \alpha_{n-2} & \alpha_{n-1} & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \vdots & \vdots & 0 \\ 1 \end{pmatrix}$$

Clearly $A$ is an L-matrix. So $T(A)$ is also an L-matrix. Since $\sum_{j=1}^{n-1} T(\alpha_j E_{1,j}) + T(eE_{1,n})$ has the same sign pattern as $\sum_{j=1}^{n-1} T(\alpha_j E_{1,j})$ and $T$ preserves the set of matrices of row term rank 1, we have that $T(A - eE_{1,n})$ has the same sign pattern as $T(A)$. So $T(A - eE_{1,n})$ is also an L-matrix. Since $T$ strongly preserves L-matrices, we must have that $A - eE_{1,n}$ is an L-matrix. But

$$A - eE_{1,n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n\text{-}m} & \cdots & \alpha_{n-2} & \alpha_{n-1} & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n\text{-}m} & \cdots & \alpha_{n-2} & \alpha_{n-1} & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \vdots & \vdots & 0 \\ 1 \end{pmatrix}$$
has linearly dependent rows and hence is not an L-matrix, a contradiction. Therefore, if $E$ is a cell, then $T(E)$ is also a cell.

Since $T$ is nonsingular and maps each cell to a weighted cell, it follows that $T$ is one-to-one on the set of weighted cells. □

From the above theorem and Theorem 2, we have the following corollaries:

**Corollary 1.** Let $T: \mathcal{M}_{m,n} \to \mathcal{M}_{m,n}$ be a linear operator. Then $T$ strongly preserves L-matrices if and only if for any $X \in \mathcal{M}_{m,n}$, $T(X) = P_1 S_1 (X \circ M) S_2 P_2$, where $P_1 \in \mathcal{S}_m$ and $P_2 \in \mathcal{S}_n$ are permutation matrices, $S_1 \in \mathcal{S}_m$ and $S_2 \in \mathcal{S}_n$ are diagonal matrices of $\pm 1$'s, and $M = (m_{i,j}) \in \mathcal{M}_{m,n}$ with $m_{i,j} > 0$.

**Corollary 2.** A linear operator $T: \mathcal{M}_{m,n} \to \mathcal{M}_{m,n}$ strongly preserves L-matrices if and only if $T$ preserves L-matrices and $T$ is also one-to-one on the set of cells.

**References**


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