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Cellularity of cyclotomic Birman–Wenzl–Murakami algebras

Frederick M. Goodman

Department of Mathematics, University of Iowa, Iowa City, IA, USA

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Dedicated to Gus Lehrer on the occasion of his 60th birthday

Abstract

We show that cyclotomic BMW algebras are cellular algebras.

Keywords: Quantum algebra; Algebras related to braid groups and link invariants; Cellular algebras

1. Introduction

In this paper, we prove that the cyclotomic Birman–Wenzl–Murakami algebras are cellular, in the sense of Graham and Lehrer [8].

The origin of the BMW algebras was in knot theory. Shortly after the invention of the Jones link invariant [10], Kauffman introduced a new invariant of regular isotopy for links in S^3, determined by certain skein relations [11]. Birman and Wenzl [2] and independently Murakami [16] then defined a family braid group algebra quotients from which Kauffman's invariant could be recovered. These (BMW) algebras were defined by generators and relations, but were implicitly modeled on certain algebras of tangles, whose definition was subsequently made explicit by Morton and Traczyk [14], as follows: Let S be a commutative unital ring with invertible elements ρ, q, and δ_0 satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$. The *Kauffman tangle algebra* $KT_{n,S}$ is the S-algebra of framed (n, n)-tangles in the disc cross the interval, modulo Kauffman skein relations:

E-mail address: goodman@math.uiowa.edu.

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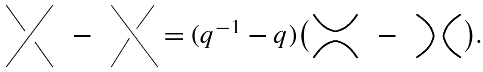
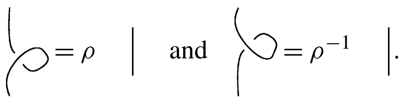
(1) Crossing relation: 

(2) Untwisting relation: 

(3) Free loop relation: $T \cup \bigcirc = \delta_0 T$.

Morton and Traczyk [14] showed that the n-strand algebra $KT_{n,S}$ is free of rank $(2n-1)!!$ as a module over S, and Morton and Wassermann [15] proved that the BMW algebras and the Kauffman tangle algebras are isomorphic.

It is natural to "affinize" the BMW algebras to obtain BMW analogues of the affine Hecke algebras of type A, see [1]. The affine Hecke algebra can be realized geometrically as the algebra of braids in the annulus cross the interval, modulo Hecke skein relations; this suggests defining the affine Kauffman tangle algebra as the algebra of framed (n,n)-tangles in the annulus cross the interval, modulo Kauffman skein relations. However, Turaev [17] showed that the resulting algebra of $(0,0)$-tangles is a (commutative) polynomial algebra in infinitely many variables, so it makes sense to absorb this polynomial algebra into the ground ring. (The ground ring gains infinitely many parameters corresponding to the generators of the polynomial algebra.) One can also define a purely algebraic version of these algebras, by generators and relations [9], the *affine BMW algebras*. In [5], we showed that the two versions are isomorphic.

The affine BMW algebras have a distinguished generator y_1, which, in the geometric (Kauffman tangle) picture is represented by a braid with one strand wrapping around the hole in the annulus cross interval. *Cyclotomic BMW algebras* are quotients of the affine BMW algebras in which the generator y_1 satisfies a monic polynomial equation. The affine and cyclotomic BMW algebras arise naturally in connection with knot theory in the solid torus, braid representations generated by R-matrices of symplectic and orthogonal quantum groups, and the representation theory of the ordinary BMW algebras (where the affine generators become Jucys–Murphy elements). We refer the reader to [6] for further discussion and references.

In order to get a good theory for cyclotomic BMW algebras, it is necessary to impose conditions on the ground ring. An appropriate condition, known as admissibility, was introduced by Wilcox and Yu in [18]. Their condition has a simple formulation in terms of the representation theory of the 2-strand cyclotomic BMW algebra, and also translates into explicit relations on the parameters.

Let $W_{n,S,r}$ denote the cyclotomic quotient of the n-strand affine BMW algebra, in which the affine generator y_1 satisfies a polynomial relation of degree r, defined over a ring S with appropriate parameters. It has been shown in [6,7,19,21] that if S is an admissible integral domain, then $W_{n,S,r}$ is a free S-module of rank $r^n(2n-1)!!$, and is isomorphic to a cyclotomic version of the Kauffman tangle algebra. In this paper, we show that the techniques of [6] can be modified to yield a cellular basis of the cyclotomic BMW algebras.

The cellularity of the ordinary BMW algebras has been shown by Xi [20] and Enyang [3,4]. It is worth pointing out that if we specialize our proof for the cyclotomic case to the ordinary BMW algebras, we end up showing that the tangle basis of [14,15] is cellular; in fact, the proof would require only minor modifications of arguments already present in Morton–Wassermann [15].

Yu [21] has also shown that cyclotomic BMW algebras over admissible ground rings are cellular; her result is slightly more general, since she used a broader definition of admissibility. See also Remark 2.10.

2. Preliminaries

2.1. Definitions

In the following, let S be a commutative unital ring containing elements ρ, q, and δ_j, $j \geqslant 0$, with ρ, q, and δ_0 invertible, satisfying the relation $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$.

Definition 2.1. The affine Kauffman tangle algebra $\widehat{KT}_{n,S,r}$ is the S-algebra of framed (n,n)-tangles in the annulus cross the interval, modulo Kauffman skein relations, namely the *crossing relation* and *untwisting relation*, as given in the introduction, and the *free loop relations*: for $j \geqslant 0$, $T \cup \Theta_j = \rho^{-j}\delta_j T$, where $T \cup \Theta_j$ is the union of an affine tangle T and a disjoint copy of the closed curve Θ_j that wraps j times around the hole in the annulus cross the interval.

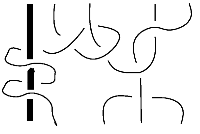


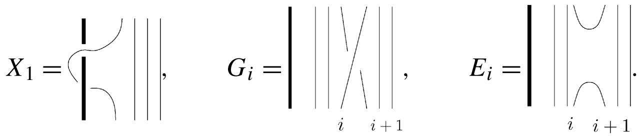
Fig. 2.1. Affine (4, 4)-tangle diagram.

Affine tangles can be represented by *affine tangle diagrams*. These are pieces of link diagrams in the rectangle $\mathcal{R}$, with some number of endpoints of curves on the top and bottom boundaries of $\mathcal{R}$, and a distinguished vertical segment representing the hole in the annulus cross interval. (We call this curve the flagpole.) Affine tangle diagrams are regarded as equivalent if they are regularly isotopic; see [6] for details. An affine (n,n)-tangle diagram is one with n vertices (endpoints of curves) on the top, and n vertices on the bottom edge of $\mathcal{R}$. See Fig. 2.1. We label the vertices on the top edge from left to right as $\mathbf{1}, \dots, \boldsymbol{n}$ and those on the bottom edge from left to right as $\bar{\mathbf{1}}, \dots, \bar{\boldsymbol{n}}$. We order the vertices by $\mathbf{1} < \mathbf{2} < \cdots < \boldsymbol{n} < \bar{\boldsymbol{n}} < \cdots < \bar{\mathbf{2}} < \bar{\mathbf{1}}$.

Definition 2.2. The *affine Birman–Wenzl–Murakami* algebra $\widehat{W}_{n,S}$ is the S algebra with generators $y_1^{\pm 1}$, $g_i^{\pm 1}$ and e_i ($1 \leqslant i \leqslant n-1$) and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$ and $y_1 y_1^{-1} = y_1^{-1} y_1 = 1$.
(2) (Idempotent relation) $e_i^2 = \delta_0 e_i$.
(3) (Type B braid relations)
  (a) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i-j| \geqslant 2$.
  (b) $y_1 g_1 y_1 g_1 = g_1 y_1 g_1 y_1$ and $y_1 g_j = g_j y_1$ if $j \geqslant 2$.
(4) (Commutation relations)
  (a) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i-j| \geqslant 2$.
  (b) $y_1 e_j = e_j y_1$ if $j \geqslant 2$.
(5) (Affine tangle relations)
  (a) $e_i e_{i\pm 1} e_i = e_i$.
  (b) $g_i g_{i\pm 1} e_i = e_{i\pm 1} e_i$ and $e_i g_{i\pm 1} g_i = e_i e_{i\pm 1}$.
  (c) For $j \geqslant 1$, $e_1 y_1^j e_1 = \delta_j e_1$.
(6) (Kauffman skein relation) $g_i - g_i^{-1} = (q^{-1} - q)(e_i - 1)$.
(7) (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$ and $e_i g_{i\pm 1} e_i = \rho e_i$.
(8) (Unwrapping relation) $e_1 y_1 g_1 y_1 = \rho e_1 = y_1 g_1 y_1 e_1$.

Let X_1, G_i, E_i denote the following affine tangle diagrams:



Theorem 2.3. *(See [5].) The affine BMW algebra $\widehat{W}_{n,S}$ is isomorphic to the affine Kauffman tangle algebra $\widehat{KT}_{n,S}$ by a map φ determined by $\varphi(g_i) = G_i$, $\varphi(e_i) = E_i$, and $\varphi(y_1) = \rho X_1$.*

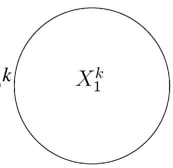
We now suppose S (as above) has additional distinguished invertible elements $u_1, \dots, u_r$.

Definition 2.4. The *cyclotomic BMW algebra* $W_{n,S,r}(u_1, \dots, u_r)$ is the quotient of $\widehat{W}_{n,S}$ by the relation

$$(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0. \tag{2.1}$$

To define the cyclotomic Kauffman tangle algebra, we begin by rewriting the relation Eq. (2.1) in the form $\sum_{k=0}^{r}(-1)^{r-k}\varepsilon_{r-k}(u_1, \dots, u_r)y_1^k = 0$, where ε_j is the jth elementary symmetric function. The corresponding relation in the affine Kauffman tangle algebra is $\sum_{k=0}^{r}(-1)^{r-k}\varepsilon_{r-k}(u_1, \dots, u_r)\rho^k X_1^k = 0$. Now we want to impose this as a local skein relation.

Definition 2.5. The *cyclotomic Kauffman tangle algebra* $KT_{n,S,r}(u_1, \dots, u_r)$ is the quotient of the affine Kauffman tangle algebra $\widehat{KT}_{n,S}$ by the cyclotomic skein relation:

$$\sum_{k=0}^{r}(-1)^{r-k}\varepsilon_{r-k}(u_1, \dots, u_r)\rho^k$$


$$= 0. \tag{2.2}$$

The sum is over affine tangle diagrams which differ only in the interior of the indicated disc and are identical outside of the disc; the interior of the disc contains an interval on the flagpole and a piece of an affine tangle diagram isotopic to X_1^k.

Definition 2.6. Say that S is *weakly admissible* if e_1 is not a torsion element in $W_{2,S,r}$. Say that S is *admissible* if $\{e_1, y_1e_1, \dots, y_1^{r-1}e_1\}$ is linearly independent over S in $W_{2,S,r}$.

These conditions can be translated into explicit conditions on the parameters of S; see [6,7,18].

Theorem 2.7. *(See [6,7,19,21].) If S is an admissible integral domain, then the assignment $e_i \mapsto E_i$, $g_i \mapsto G_i$, $y_1 \mapsto \rho X_i$ determines an isomorphism of $W_{n,S,r}$ and $KT_{n,S,r}$. Moreover these algebras are free S-modules of rank $r^n(2n-1)!!$.*

Because of Theorems 2.3 and 2.7, we will no longer take care to distinguish between affine or cyclotomic BMW algebras and their realizations as algebras of tangles. We identify e_i and g_i

with the corresponding affine tangle diagrams and $x_1 = \rho^{-1} y_1$ with the affine tangle diagram X_1. The ordinary BMW algebra $W_{n,S}$ embeds in the affine BMW algebra $W_{n,S}$ as the subalgebra generated by the e_i's and g_i's.

2.2. *The rank of tangle diagrams*

An ordinary or affine tangle diagram T with n strands is said to have *rank* $\leqslant r$ if it can be written as a product $T = T_1 T_2$, where T_1 is an (ordinary or affine) (r, n) tangle and T_2 is an (ordinary or affine) (n, r) tangle.

2.3. *The algebra involution * on BMW algebras*

Each of the ordinary, affine, and cyclotomic BMW algebras admits a unique involutive algebra anti-automorphism, denoted $a \mapsto a^*$, fixing each of the generators g_i, e_i (and x_1 in the affine or cyclotomic case). For an (ordinary or affine) tangle diagram T representing an element of one of these algebras, T^* is the diagram obtained by flipping T around a horizontal axis.

2.4. *The Hecke algebra and the BMW algebra*

The Hecke algebra $H_{n,S}(q^2)$ of type A is the quotient of the group algebra $S\mathcal{B}_n$ of the braid group, by the relations $\sigma_i - \sigma_i^{-1} = (q - q^{-1})$ $(1 \leqslant i \leqslant n-1)$, where σ_i are the Artin braid generators. Let τ_i denote the image of the braid generator σ_i in the Hecke algebra.

Given a permutation $\pi \in \mathfrak{S}_n$, let β_π be the *positive permutation braid* in the braid group $\mathcal{B}_n$ whose image in $\mathfrak{S}_n$ is π. A positive permutation braid is a braid in which two strands cross at most once, and all crossings are positive, that is the braid is in the monoid generated by the Artin generators σ_i of the braid group. Let g_π be the image of β_π in $W_{n,S}$, and τ_π the image of β_π in $H_{n,S}(q^2)$. If π has a reduced expression $\pi = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, then $g_\pi = g_{i_1} g_{i_2} \cdots g_{i_\ell}$, and $\tau_\pi = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_\ell}$. It is well known that $\{\tau_\pi\colon \pi \in \mathfrak{S}_n\}$ is a basis of the Hecke algebra $H_{n,S}(q^2)$. The Hecke algebra has an involutive algebra anti-automorphism $x \mapsto x^*$ determined by $(\tau_\pi)^* = \tau_{\pi^{-1}}$.

2.5. *Affine and cyclotomic Hecke algebras*

Definition 2.8. (See [1].) Let S be a commutative unital ring with an invertible element q. The *affine Hecke algebra* $\widehat{H}_{n,S}(q^2)$ over S is the S-algebra with generators $t_1, \tau_1, \dots, \tau_{n-1}$, with relations:

(1) The generators τ_i are invertible, satisfy the braid relations, and $\tau_i - \tau_i^{-1} = (q - q^{-1})$.
(2) The generator t_1 is invertible, $t_1 \tau_1 t_1 \tau_1 = \tau_1 t_1 \tau_1 t_1$ and t_1 commutes with τ_j for $j \geqslant 2$.

Let $u_1, \dots, u_r$ be additional invertible elements in S. The *cyclotomic Hecke algebra* $H_{n,S,r}(q^2; u_1, \dots, u_r)$ is the quotient of the affine Hecke algebra $\widehat{H}_{n,S}(q^2)$ by the polynomial relation $(t_1 - u_1) \cdots (t_1 - u_r) = 0$.

Define elements t_j $(1 \leqslant j \leqslant n)$ in the affine or cyclotomic Hecke algebra by

$$t_j = \tau_{j-1} \cdots \tau_1 t_1 \tau_1 \cdots \tau_{j-1}.$$

It is well known that the ordinary Hecke algebra $H_{n,S}(q^2)$ embeds in the affine Hecke algebra and that the affine Hecke algebra $\widehat{H}_{n,S}(q^2)$ is a free S-module with basis the set of elements $\tau_\pi \boldsymbol{t}^b$, where $\pi \in S_n$ and $\boldsymbol{t}^b$ denotes a Laurent monomial in $t_1, \ldots, t_n$. Similarly, a cyclotomic Hecke algebra $H_{n,S,r}(q; u_1, \ldots, u_r)$ is a free S-module with basis the set of elements $\tau_\pi \boldsymbol{t}^b$, where now $\boldsymbol{t}^b$ is a monomial with restricted exponents $0 \leqslant b_i \leqslant r-1$.

Let S be a commutative ring with appropriate parameters ρ, q, δ_j. There is an algebra homomorphism $p : \widehat{W}_{n,S} \to \widehat{H}_{n,S}(q^2)$ determined by $g_i \mapsto \tau_i$, $e_i \mapsto 0$, and $x_1 \mapsto t_1$. The kernel of p is the ideal I_n spanned by affine tangle diagrams with rank strictly less than n. Suppose that S has additional parameters $u_1, \ldots, u_r$. Then p induces a homomorphism of the cyclotomic quotients $p : W_{n,S,r}(u_1, \ldots, u_r) \to H_{n,S,r}(q^2; u_1, \ldots, u_r)$.

The affine and cyclotomic Hecke algebras have unique involutive algebra anti-automorphisms * fixing the generators τ_i and t_1. (The image of a word in the generators is the reversed word.) The quotient map p respects the involutions, $p(x^*) = p(x)^*$.

We have a linear section $t : \widehat{H}_{n,S}(q^2) \to \widehat{W}_{n,S}$ of the map p determined by $t(\tau_\pi \boldsymbol{t}^b) = g_\pi \boldsymbol{x}^b$. Moreover, $t(x^*) \equiv t(x)^* \mod I_n$ and $t(x)t(y) \equiv t(xy) \mod I_n$ for any $x, y \in \widehat{H}_{n,S}(q^2)$. Analogous statements hold for the cyclotomic algebras.

2.6. Cellular bases

We recall the definition of *cellularity* from [8]; see also [13]. The version of the definition given here is slightly weaker than the original definition in [8]; we justify this below.

Definition 2.9. Let R be an integral domain and A a unital R-algebra. A *cell datum* for A consists of an R-linear algebra involution $*$ of A; a partially ordered set $(\Lambda, \geqslant)$ and for each $\lambda \in \Lambda$ a set $\mathcal{T}(\lambda)$; and a subset $\mathcal{C} = \{c^\lambda_{s,t} \colon \lambda \in \Lambda \text{ and } s, t \in \mathcal{T}(\lambda)\} \subseteq A$; with the following properties:

(1) $\mathcal{C}$ is an R-basis of A.
(2) For each $\lambda \in \Lambda$, let $\breve{A}^\lambda$ be the span of the $c^\mu_{s,t}$ with $\mu > \lambda$. Given $\lambda \in \Lambda$, $s \in \mathcal{T}(\lambda)$, and $a \in A$, there exist coefficients $r^s_v(a) \in R$ such that for all $t \in \mathcal{T}(\lambda)$:

$$a c^\lambda_{s,t} \equiv \sum_v r^s_v(a) c^\lambda_{v,t} \mod \breve{A}^\lambda.$$

(3) $(c^\lambda_{s,t})^* \equiv c^\lambda_{t,s} \mod \breve{A}^\lambda$ for all $\lambda \in \Lambda$ and, $s, t \in \mathcal{T}(\lambda)$.

A is said to be a *cellular algebra* if it has a cell datum.

For brevity, we say write that $\mathcal{C}$ is a cellular basis of A.

Remark 2.10.

(1) The original definition in [8] requires that $(c^\lambda_{s,t})^* = c^\lambda_{t,s}$ for all λ, s, t. However, one can check that the basic consequences of the definition ([8], pp. 7–13) remain valid with our weaker axiom.
(2) In case $2 \in R$ is invertible, one can check that our definition is equivalent to the original.
(3) One can formulate a version of the "basis–free" definition of cellularity of König and Xi (see for example [12]) equivalent to our modified definition.

(4) Suppose A is an R-algebra with involution $*$, and J is a $*$-closed ideal; then we have an induced algebra involution $*$ on A/J. Let us say that J is a cellular ideal in A if it satisfies the axioms of a cellular algebra (except for being unital) with cellular basis $\{c^{\lambda}_{s,t}\colon \lambda \in \Lambda_J \text{ and } s, t \in \mathcal{T}(\lambda)\} \subseteq J$ and we have, as in point (2) of the definition of cellularity, $ac^{\lambda}_{s,t} \equiv \sum_v r^s_v(a) c^{\lambda}_{v,t} \mod \breve{J}^{\lambda}$ not only for $a \in J$ but also for $a \in A$. If J is a cellular ideal in A, and A/J is cellular (with respect to the given involutions), then A is cellular. With the original definition of [8], this statement would be true only if J has a $*$-invariant R-module complement in A.

(5) Yu [21] has also proved cellularity of the cyclotomic BMW algebras, using the original definition of cellularity of [8]; at one point, her proof requires a more delicate analysis, in order to obtain a $*$-invariant complement in $W_{n,S,r}$ of the kernel of $p: W_{n,S,r} \to H_{n,S,r}$.

3. Some new bases of the affine and cyclotomic BMW algebras

The basis of cyclotomic BMW algebras that we produced in [6] involved ordered monomials in the non-commuting but mutually conjugate elements

$$x'_j = g_{j-1} \cdots g_1 x_1 g_1^{-1} \cdots g_{j-1}^{-1}.$$

To obtain this basis, we first produced a basis of the *affine* BMW algebra consisting of affine tangle diagrams satisfying certain topological conditions.

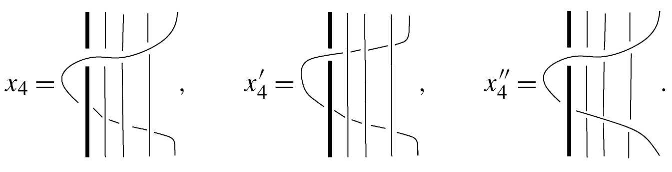
Here we want to produce a new finite basis of the cyclotomic BMW algebras involving monomials in the commuting, but non-conjugate, elements

$$x_j = g_{j-1} \cdots g_1 x_1 g_1 \cdots g_{j-1}.$$

At an intermediate stage of the exposition, we will also use the elements

$$x''_j = g_{j-1}^{-1} \cdots g_1^{-1} x_1 g_1 \cdots g_{j-1},$$

see the following figure:



3.1. Flagpole descending affine tangle diagrams

Definition 3.1. An *orientation* of an affine (n,n)-tangle diagram is a linear ordering of the strands, a choice of an orientation of each strand, and a choice of an initial point on each closed loop.

An orientation determines a way of traversing the tangle diagram; namely, the strands are traversed successively, in the given order and orientation (the closed loops being traversed starting at the assigned initial point).

Definition 3.2. An oriented affine (n, n)-tangle diagram is *stratified* if

(1) there is a linear ordering of the strands such that if strand s precedes strand t in the order, then each crossing of s with t is an over-crossing.
(2) each strand is totally descending, that is, each self-crossing of the strand is encountered first as an over-crossing as the strand is traversed according to the orientation.

We call the corresponding ordering of the strands the *stratification order*.

Note that a stratification order need not coincide with the ordering of strands determined by the orientation. *In the rest of the paper, we are going to use the following orientation and stratification order on affine tangle diagrams*; *when we say an affine tangle diagram is oriented or stratified, we mean with respect to this orientation and stratification order.*

Definition 3.3. A *verticals-second orientation* of affine tangle diagrams is one in which:

(1) Non-closed strands are oriented from lower to higher numbered vertex.
(2) Horizontal strands with vertices at the top of the diagram precede vertical strands, and vertical strands precede horizontal strands with vertices at the bottom of the diagram. Non-closed strands precede closed loops.
(3) Horizontal strands with vertices at the top of the diagram are ordered according to the order of their *final* vertices. Vertical strands and horizontal strands with vertices at the bottom of the diagram are each ordered according to the order of their *initial* vertices.

A *verticals-second stratification order* is one in which the order of strands agrees with that of a verticals-second orientation, except that vertical strands are ordered according to the *reverse* order of their *initial* vertices.

An affine tangle diagram without closed loops has a unique verticals-second orientation and a unique verticals-second stratification order.

A *simple winding* is a piece of an affine tangle diagram with one ordinary strand, without self-crossings, regularly isotopic to the intersection of one of the affine tangle diagrams x_1 or x_1^{-1} with a neighborhood of the flagpole.

Definition 3.4. An affine tangle diagram is in *standard position* (see Fig. 3.1) if:

(1) It has no crossings to the left of the flagpole.

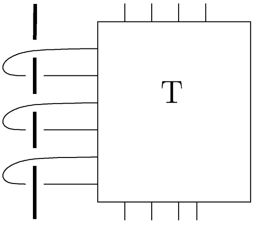


Fig. 3.1. Affine tangle diagram in standard position.

(2) There is a neighborhood of the flagpole whose intersection with the tangle diagram is a union of simple windings.
(3) The simple windings have no crossings and are not nested. That is, between the two crossings of a simple winding with the flagpole, there is no other crossing of a strand with the flagpole.

Definition 3.5. An oriented, stratified affine tangle diagram T in standard position is said to be *flagpole descending* if it satisfies the following conditions:

(1) T is not regularly isotopic to an affine tangle diagram in standard position with fewer simple windings.
(2) The strands of T have no self-crossings.
(3) As T is traversed according to the orientation, successive crossings of ordinary strands with the flagpole descend the flagpole.

Proposition 3.6. *The affine BMW algebra $\widehat{W}_{n,S}$ is spanned by affine tangle diagrams without closed loops that are flagpole descending and stratified.*

Proof. This follows from [6], Proposition 2.19. □

3.2. $\mathbb{Z}$-Brauer diagrams and liftings in the affine BMW algebras

We recall that a Brauer diagram is a tangle diagram in the plane, in which information about over- and under-crossings is ignored. Let G be a group. A *G-Brauer diagram* (or *G-connector*) is a Brauer diagram in which each strand is endowed with an orientation and labeled by an element of the group G. Two labelings are regarded as the same if the orientation of a strand is reversed and the group element associated to the strand is inverted.

Define a map c (the connector map) from oriented affine (n, n)-tangle diagrams *without closed loops* to $\mathbb{Z}$-Brauer diagrams as follows. Let a be an oriented affine (n, n)-tangle diagram without closed loops. If s connects two vertices $\boldsymbol{v}_1$ to $\boldsymbol{v}_2$, include a curve $c(s)$ in $c(a)$ connecting the same vertices with the same orientation, and label the oriented strand $c(s)$ with the *winding number* of s with respect to the flagpole.[1]

Lemma 3.7. *(See [6, Lemma 2.21].) Two affine tangle diagrams without closed loops, with the same $\mathbb{Z}$-Brauer diagram, both stratified and flagpole descending, are regularly isotopic.*

The symmetric group $\mathfrak{S}_n$ can be regarded as the subset of (n, n)-Brauer diagrams consisting of diagrams with only vertical strands. $\mathfrak{S}_n$ acts on ordinary or $\mathbb{Z}$-labeled (n, n)-Brauer diagrams on the left and on the right by the usual multiplication of diagrams, that is, by stacking diagrams.

We consider a particular family of permutations in $\mathfrak{S}_n$. Let s be an integer, $0 \leqslant s \leqslant n$, with s congruent to $n \mod 2$. Write $f = (n - s)/2$. Following Enyang [4], let $\mathcal{D}_{f,n}$ be the set of permutations $\pi \in \mathfrak{S}_n$ satisfying:

[1] The winding number $n(s)$ is determined combinatorially as follows: traversing the strand in its orientation, list the over-crossings $(+)$ and under-crossings $(-)$ of the strand with the flagpole. Cancel any two successive $+$'s or $-$'s in the list, so the list now consists of alternating $+$'s and $-$'s. Then $n(s)$ is $\pm(1/2)$ the length of the list, $+$ if the list begins with a $+$, and $-$ if the list begins with a $-$.

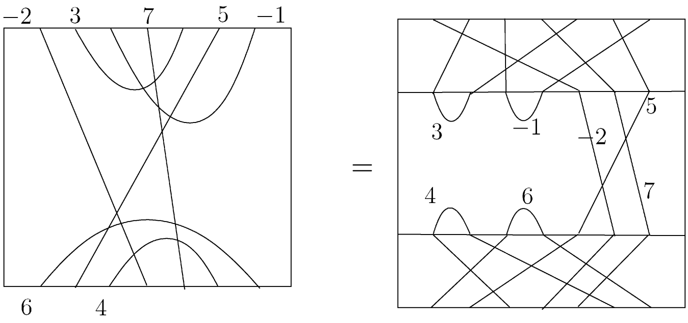


Fig. 3.2. Factorization of $\mathbb{Z}$-Brauer diagrams.

(1) If i, j are even numbers with $2 \leqslant i < j \leqslant 2f$, then $\pi(i) < \pi(j)$.
(2) If i is odd with $1 \leqslant i \leqslant 2f - 1$, then $\pi(i) < \pi(i+1)$.
(3) If $2f + 1 \leqslant i < j \leqslant n$, then $\pi(i) < \pi(j)$.

Then $\mathcal{D}_{f,n}$ is a complete set of left coset representatives of

$$\big((\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \rtimes \mathfrak{S}_f\big) \times \mathfrak{S}_s \subseteq \mathfrak{S}_n,$$

where the f copies of $\mathbb{Z}_2$ are generated by the transpositions $(2i-1, 2i)$ for $1 \leqslant i \leqslant f$; $\mathfrak{S}_f$ permutes the f blocks $[2i-1, 2i]$ among themselves; and $\mathfrak{S}_s$ acts on the last s digits $\{2f+1, \ldots, n\}$.

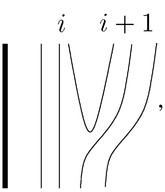
An element π of $\mathcal{D}_{f,n}$ factors as $\pi = \pi_1\pi_2$, where $\pi_2 \in \mathcal{D}_{f,f}$, and π_1 is a $(2f, s)$ shuffle; i.e., π_1 preserves the order of $\{1, 2, \ldots, 2f\}$ and of $\{2f+1, \ldots, 2f+s = n\}$. Moreover, $\ell(\pi) = \ell(\pi_1) + \ell(\pi_2)$.

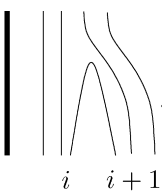
For any $\mathbb{Z}$-Brauer diagram D, let D_0 denote the underlying ordinary Brauer diagram; that is, D_0 is obtained from D by forgetting the integer valued labels of the strands. If D is a $\mathbb{Z}$-Brauer diagram with exactly s vertical strands, then D has a unique factorization

$$D = \alpha d \beta^{-1}, \tag{3.1}$$

where α and β are elements of $\mathcal{D}_{f,n}$, and d has underlying Brauer diagram of the form $d_0 = e_1 e_2 \cdots e_{2f-1}\pi$, where π is a permutation of $\{2f+1, \ldots, n\}$. This factorization is illustrated in Fig. 3.2.

It will be convenient to work in the affine BMW category, that is, the category whose objects are the natural numbers $0, 1, 2, \ldots$, with $\mathrm{Hom}(k, \ell)$ being the S-span of affine (k, ℓ)-tangle diagrams, modulo Kauffman skein relations. Let us introduce the elements $\cup_i$ and $\cap_i$ which are the lower and upper half of e_i,

$\cup_i =$ 

, $\cap_i =$ 

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We collect several elementary observations. Fix integers n and s with $0 \leqslant s \leqslant n$ and $s \equiv n \bmod 2$. Set $f = (n-s)/2$. Each of the following statements is justified by picture proofs.

Lemma 3.8.

(1) $e_1 e_3 \cdots e_{2f-1} = (\cap_{2f-1} \cdots \cap_3 \cap_1)(\cup_1 \cup_3 \cdots \cup_{2f-1})$.
(2) *For k odd,* $1 \leqslant k \leqslant 2f-1$, $(\cup_1 \cup_3 \cdots \cup_{2f-1}) x'_k = (\cup_1 \cup_3 \cdots \cup_{2f-1}) x_k$.
(3) *For k odd,* $1 \leqslant k \leqslant 2f-1$, $x''_k (\cap_{2f-1} \cdots \cap_3 \cap_1) = x_k (\cap_{2f-1} \cdots \cap_3 \cap_1)$.
(4) *If* π *is a permutation of* $\{2f+1, 2f+2, \ldots, n\}$, *then*

$$(\cup_1 \cup_3 \cdots \cup_{2f-1}) g_\pi = g_{\tilde{\pi}} (\cup_1 \cup_3 \cdots \cup_{2f-1}),$$

where $\tilde{\pi}$ *is the permutation of* $\{1, 2, \ldots, s\}$ *defined by* $\tilde{\pi}(j) = \pi(j+2f) - 2f$. *More generally, if* T *is an ordinary tangle on the strands* $\{2f+1, 2f+2, \ldots, n\}$, *then*

$$(\cup_1 \cup_3 \cdots \cup_{2f-1}) T = \tilde{T} (\cup_1 \cup_3 \cdots \cup_{2f-1}),$$

where $\tilde{T}$ *is the shift of* T *to the strands* $\{1, 2, \ldots, s\}$.
(5) *For* $1 \leqslant k \leqslant s$, $(\cup_1 \cup_3 \cdots \cup_{2f-1}) x_{k+2f} = x_k (\cup_1 \cup_3 \cdots \cup_{2f-1})$.

Now we can obtain a lifting of $\mathbb{Z}$-Brauer diagrams to affine tangle diagrams that are flagpole descending and stratified, using the factorization of Eq. (3.1). Let D be a $\mathbb{Z}$-Brauer diagram with exactly s vertical strands. Set $f = (n-s)/2$. Consider the factorization $D = \alpha\, d \beta^{-1}$, where $\alpha, \beta \in \mathcal{D}_{f,n}$, and $d_0 = e_1 \cdots e_{2f-1}\pi$, with π a permutation of $\{2f+1, \ldots, n\}$.

First, there is a unique (up to regular isotopy) stratified ordinary (n,n)-tangle diagram T_{d_0} without closed loops or self-crossings of strands with Brauer diagram

$$c(T_{d_0}) = d_0 = e_1 e_3 \cdots e_{2f-1} \pi,$$

namely

$$\begin{aligned} T_{d_0} &= e_1 e_3 \cdots e_{2f-1} g_\pi = (\cap_{2f-1} \cdots \cap_3 \cap_1)(\cup_1 \cup_3 \cdots \cup_{2f-1}) g_\pi \\ &= (\cap_{2f-1} \cdots \cap_3 \cap_1) g_{\tilde{\pi}} (\cup_1 \cup_3 \cdots \cup_{2f-1}), \end{aligned} \tag{3.2}$$

where $\tilde{\pi}$ is the permutation of $\{1, 2, \ldots, s\}$ defined by $\tilde{\pi}(j) = \pi(j+2f) - 2f$.

Next, we set

$$\begin{aligned} T'_d = {} & \left(x''_1\right)^{a_1} \cdots \left(x''_{2f-1}\right)^{a_{2f-1}} (\cap_{2f-1} \cdots \cap_3 \cap_1) g_{\tilde{\pi}} \left(x''_s\right)^{b_s} \cdots \left(x''_1\right)^{b_1} \\ & (\cup_1 \cup_3 \cdots \cup_{2f-1}) \left(x'_{2f-1}\right)^{c_{2f-1}} \cdots \left(x'_1\right)^{c_1}, \end{aligned} \tag{3.3}$$

where the exponents are determined as follows:

(1) For i odd, $i \leqslant 2f-1$, if d has a strand beginning at $\boldsymbol{i}$ with label k, then $c_i = k$; otherwise $c_i = 0$.
(2) For $i \geqslant 2f+1$, if d has a strand beginning at $\boldsymbol{i}$ with label k, then $b_{i-2f} = k$; otherwise $b_{i-2f} = 0$.

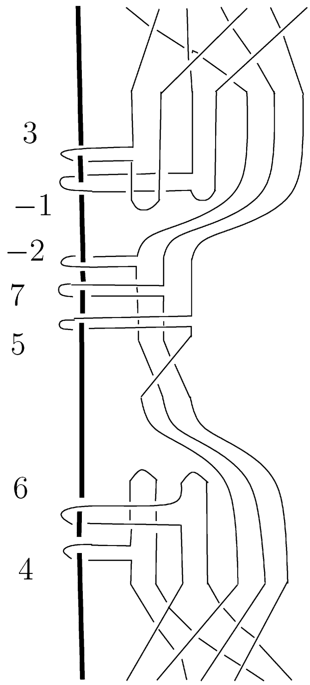

Fig. 3.3. Lifting of an affine Brauer diagram.

(3) For i odd, $i \leqslant 2f - 1$, if d has a horizontal strand ending at $\bar{i}$ with label k, then $a_i = k$; otherwise, $a_i = 0$.

Finally, we set

$$T'_D = g_\alpha T'_d (g_\beta)^*, \tag{3.4}$$

endowed with the verticals-second orientation.

Example 3.9. For the $\mathbb{Z}$-Brauer diagram D illustrated in Fig. 3.2, T'_D is illustrated in Fig. 3.3 (where the winding number of the strands are indicated by the integers written at the left of the figure). We have

$$T'_d = \left(x''_1\right)^4 \left(x''_3\right)^6 (\cap_1 \cap_3) g_1 g_2 \left(x''_3\right)^5 \left(x''_2\right)^7 \left(x''_1\right)^{-2} (\cup_1 \cup_3) \left(x'_3\right)^{-1} \left(x'_1\right)^3.$$

Lemma 3.10. *T'_D is flagpole descending and stratified, and has $\mathbb{Z}$-Brauer diagram equal to D.*

Proof. Straightforward. □

Proposition 3.11. $\mathbb{U}' = \{T'_D$: *D is a $\mathbb{Z}$-Brauer diagram*$\}$ *spans* $\widehat{W}_{n,S}$.

Proof. This follows from Proposition 3.6, Lemma 3.7, and Lemma 3.10. □

Proposition 3.12. $\mathbb{U}' = \{T'_D\colon$ *D is a* $\mathbb{Z}$*-Brauer diagram*$\}$ *is a basis of* $\widehat{W}_{n,S}$.

Proof. Essentially the same as the proof of Theorem 2.25 in [6]. □

Now fix an integer $r \geqslant 1$. Let $\mathbb{U}'_r$ be the set of $U'_D \in \mathbb{U}'$ such that the integer valued labels on the strands of D are restricted to lie in the interval $0 \leqslant k \leqslant r-1$. Equivalently, the exponents of the x'_j, x''_j appearing in U'_D are restricted to be in the same interval of integers.

Proposition 3.13. *The cyclotomic BMW algebra* $W_{n,S,r}(u_1,\dots,u_r)$ *is spanned over* S *by* $\mathbb{U}'_r$.

Proof. Same as the proof of Proposition 3.6 in [6]. □

Proposition 3.14. *For any integral domain* S *with admissible parameters,* $\mathbb{U}'_r$ *is an* S*-basis of the cyclotomic BMW algebra* $W_{n,S,r}(u_1,\dots,u_r)$.

Proof. Same as the proof of Theorem 5.5 in [6]. □

Remark 3.15. It is straightforward to generalize the content of this section to affine (k,ℓ)-tangle diagrams. The notions of standard position, orientation, and stratification, and in particular the verticals-second orientation and stratification extend to affine (k,ℓ)-tangle diagrams. Likewise, the notion of flagpole descending extends.

Define (k,ℓ)-connectors to be "Brauer diagrams" with k upper vertices and ℓ lower vertices, and likewise define $\mathbb{Z}$-weighted (k,ℓ)-connectors. We can extend the definition of the connector map c to a map from oriented affine (k,ℓ)-tangle diagrams without closed loops to $\mathbb{Z}$-weighted (k,ℓ)-connectors. Then the analogue of Lemma 3.7 holds.

Other results in this section can also be generalized, but we will need only a weak version of Proposition 3.11, and only for affine $(0,2f)$-tangle diagrams. Consider the set of affine $(0,2f)$-tangle diagrams of the form

$$g_\alpha\bigl(x''_1\bigr)^{a_1}\cdots\bigl(x''_{2f-1}\bigr)^{a_{2f-1}}(\cap_{2f-1}\cdots\cap_3\cap_1), \tag{3.5}$$

where $\alpha \in \mathcal{D}_{f,f}$. These affine tangle diagrams are stratified and flagpole descending, and have no closed loops. Moreover, every $\mathbb{Z}$-weighted $(0,2f)$-connector has a lifting in this set. Therefore, by the analogue of Lemma 3.7, every totally descending, flagpole descending affine $(0,2f)$-tangle diagram without closed loops, is regularly isotopic to one of the diagrams represented in Eq. (3.5).

3.3. New bases

So far, we have produced bases $\mathbb{U}'$ of the affine BMW algebras and $\mathbb{U}'_r$ of the cyclotomic BMW algebras involving (as did our previous bases in [6]) ordered monomials in the non-commuting but conjugate elements x'_j and x''_j.

We will now use these bases to obtain new bases involving instead monomials in the commuting elements x_j.

Consider the definition of T'_D in Eqs. (3.3) and (3.4). Note that

$$\begin{aligned}&(\cup_1 \cup_3 \cdots \cup_{2f-1})\big(x'_{2f-1}\big)^{c_{2f-1}} \cdots \big(x'_3\big)^{c_3}\big(x'_1\big)^{c_1}\\ &\quad = (\cup_1 \cup_3 \cdots \cup_{2f-1})(x_{2f-1})^{c_{2f-1}} \cdots \big(x'_3\big)^{c_3}\big(x'_1\big)^{c_1}\\ &\quad = (\cup_1 \cup_3 \cdots \cup_{2f-1})\big(\big(x'_{2f-3}\big)^{c_{2f-3}} \cdots \big(x'_3\big)^{c_3}\big(x'_1\big)^{c_1}\big)(x_{2f-1})^{c_{2f-1}},\end{aligned}$$

using Lemma 3.8(2), and the commutivity of x_{2f-1} with $\widehat{W}_{2f-2,S}$. Applying this step repeatedly, we end with

$$\begin{aligned}&(\cup_1 \cup_3 \cdots \cup_{2f-1})\big(x'_{2f-1}\big)^{c_{2f-1}} \cdots \big(x'_3\big)^{c_3}\big(x'_1\big)^{c_1}\\ &\quad = (\cup_1 \cup_3 \cdots \cup_{2f-1})(x_1)^{c_1}(x_3)^{c_3} \cdots (x_{2f-1})^{c_{2f-1}}.\end{aligned}$$

Likewise, using Lemma 3.8(3),

$$\big(x''_1\big)^{a_1} \cdots \big(x''_{2f-1}\big)^{a_{2f-1}}(\cap_{2f-1} \cdots \cap_3 \cap_1) = (x_1)^{a_1} \cdots (x_{2f-1})^{a_{2f-1}}(\cap_{2f-1} \cdots \cap_3 \cap_1).$$

Thus the element T'_D has the form

$$\begin{aligned}T'_D = {}& g_\alpha (x_1)^{a_1} \cdots (x_{2f-1})^{a_{2f-1}}(\cap_{2f-1} \cdots \cap_3 \cap_1)T\\ &\times (\cup_1 \cup_3 \cdots \cup_{2f-1})(x_1)^{c_1}(x_3)^{c_3} \cdots (x_{2f-1})^{c_{2f-1}}(g_\beta)^*, \qquad (3.6)\end{aligned}$$

where T is an affine tangle diagram on s strands with no horizontal strands.

We consider the cyclotomic BMW algebra $W_n = W_{n,S,r}(u_1, \ldots, u_r)$; the arguments for the affine BMW algebras are similar. If $T'_D \in \mathbb{U}'_r$, then the exponents a_i and c_i of x_i in Eq. (3.6) satisfy $0 \leqslant a_i, c_i \leqslant r-1$.

Recall from Section 2.5 that the quotient of W_s by the ideal I_s spanned by affine tangle diagrams with rank strictly less than s is isomorphic to the cyclotomic Hecke algebra $H_{s,S,r} = H_{s,S,r}(q^2; u_1, \ldots, u_r)$. The cyclotomic Hecke algebra $H_{s,S,r}$ is a free S-module with basis the set of $\tau_\omega t_1^{b_1} \cdots t_s^{b_s}$, with $0 \leqslant b_i \leqslant r-1$ and $\omega \in \mathfrak{S}_s$. Therefore, the element $T \in W_s$ is congruent modulo I_s to a linear combination of elements $t(\tau_\omega t_1^{b_1} \cdots t_s^{b_s}) = g_\omega x_1^{b_1} \cdots x_s^{b_s}$.

If we replace T with $g_\omega x_1^{b_1} \cdots x_s^{b_s}$ in Eq. (3.6), and then apply Lemma 3.8(1), (4) and (5), we obtain (in place of T'_D) an expression

$$g_\alpha\, x_1^{a_1} \cdots x_{2f-1}^{a_{2f-1}}(e_1 e_3 \cdots e_{2f-1} g_\omega) x_1^{c_1} x_3^{c_3} \cdots x_{2f-1}^{c_{2f-1}} x_{2f+1}^{b_1} \cdots x_n^{b_s}\ (g_\beta)^*. \qquad (3.7)$$

On the other hand, if we replace T by an element of I_s, then we obtain (in place of T'_D) a linear combination of affine tangle diagrams with rank strictly less than s.

Given a $\mathbb{Z}$-Brauer diagram D, we define an element T_D of the form displayed in Eq. (3.7) whose associated $\mathbb{Z}$-Brauer diagram $c(U_D)$ is equal to D, as follows: Suppose D has $2n$ vertices and s vertical strands, and let $f = (n-s)/2$. Let D have the factorization $D = \alpha d \beta^{-1}$, where $\alpha, \beta \in \mathcal{D}_{f,n}$, and $d_0 = e_1 \cdots e_{2f-1}\pi$, with π a permutation of $\{2f+1, \ldots, n\}$.

Define

$$T_d = x_1^{a_1} \cdots x_{2f-1}^{a_{2f-1}}(e_1 e_3 \cdots e_{2f-1} g_\pi) x_1^{c_1} x_3^{c_3} \cdots x_{2f-1}^{c_{2f-1}} x_{2f+1}^{b_{2f+1}} \cdots x_n^{b_n}, \qquad (3.8)$$

where the exponents are determined as follows: If d has a horizontal strand beginning at $\boldsymbol{i}$ with integer valued label k, then $c_i = k$; and $c_i = 0$ otherwise. If d has a vertical strand beginning at $\boldsymbol{i}$ with integer valued label k, then $b_i = k$; and $b_i = 0$ otherwise. If d has a horizontal strand ending at $\bar{\boldsymbol{i}}$ with integer valued label k, then $a_i = k$; and $a_i = 0$ otherwise.

Finally, set $T_D = g_\alpha T_d (g_\beta)^*$. Then $c(U_D) = D$.

Theorem 3.16. *Let S be an admissible integral domain. Let $\mathbb{U}_r$ be the set of T_D corresponding to $\mathbb{Z}$-Brauer diagrams D with integer valued labels in the interval $0 \leqslant k \leqslant r-1$. Then $\mathbb{U}_r$ is an S-basis of $\widehat{W}_{n,S,r}(u_1,\dots,u_r)$.*

Proof. We will show that $\mathbb{U}_r$ is spanning. Linear independence is proved as in the proof of Theorem 5.5 in [6]. It suffices to show that $\mathbb{U}'_r$ is contained in the linear span of $\mathbb{U}_r$.

Let $T'_D \in \mathbb{U}'_r$, where D has exactly s vertical strands. We show by induction on s that T'_D is in the span of $\mathbb{U}_r$. If $s = 0$, then in Eq. (3.6), the tangle T is missing, and T'_D is already an element of $\mathbb{U}_r$. If $s = 1$, then in Eq. (3.6), the tangle T is equal to a power of x_1, and again $T'_D \in \mathbb{U}_r$, by Lemma 3.8(5).

Assume that $s > 1$ and that all elements of $\mathbb{U}'_r$ with fewer than s vertical strands are in the span of $\mathbb{U}_r$. It follows from Eq. (3.6), and the discussion following it, that T'_D is in the span of $\mathbb{U}_r$, modulo the ideal $I_n^{(s-1)}$ in $\widehat{W}_{n,S,r}(u_1,\dots,u_r)$ spanned by of affine tangle diagrams with rank strictly less than s.

Now it only remains to check that $I_n^{(s-1)}$ is spanned by elements of $\mathbb{U}'_r$ with fewer than s vertical strands. Here one only has to observe that smoothing any crossing in a tangle diagram with k vertical strands produces a tangle diagram with at most k vertical strands. Therefore, the algorithm from [6], Propositions 2.18 and 2.19, for writing an affine tangle diagram (with k vertical strands) as a linear combination of flagpole descending affine tangle diagrams produces only affine tangle diagrams with at most k vertical strands. □

For the affine case, we have the following result, with essentially the same proof:

Theorem 3.17. *Let S be any ring with appropriate parameters. $\mathbb{U} = \{T_D\colon$ D is a $\mathbb{Z}$-Brauer diagram$\}$ is a basis of $\widehat{W}_{n,S}$.*

Let D be a $\mathbb{Z}$-Brauer diagram, with $2n$ vertices and s vertical strands, having factorization $D = \alpha d \beta$, and let T_d be defined as in Eq. (3.8) and let $T_D = g_\alpha\, T_d\, (g_\beta)^*$. We can rewrite T_D as follows. Factor α as $\alpha = \alpha_1\alpha_2$, with α_1 a $(2f, s)$-shuffle, and $\alpha_2 \in \mathcal{D}_{f,f}$, and factor β similarly. Then

$$T_D = g_{\alpha_1}\big[g_{\alpha_2}\boldsymbol{x}^a(e_1e_3\cdots e_{2f-1})\boldsymbol{x}^c(g_{\beta_2})^*\big]\big(g_\pi \boldsymbol{x}^b\big)(g_{\beta_1})^*, \tag{3.9}$$

where $\boldsymbol{x}^a$ is short for $x_1^{a_1}\cdots x_{2f-1}^{a_{2f-1}}$, and similarly for $\boldsymbol{x}^c$, while $\boldsymbol{x}^b$ denotes $x_{2f+1}^{b_{2f+1}}\cdots x_n^{b_n}$.

4. Cellular bases of cyclotomic BMW algebras

4.1. Tensor products of affine tangle diagrams

The category of affine (k,ℓ)-tangle diagrams is not a tensor category in any evident fashion. Nevertheless, we can define a tensor product of affine tangle diagrams, as follows. Let T_1 and

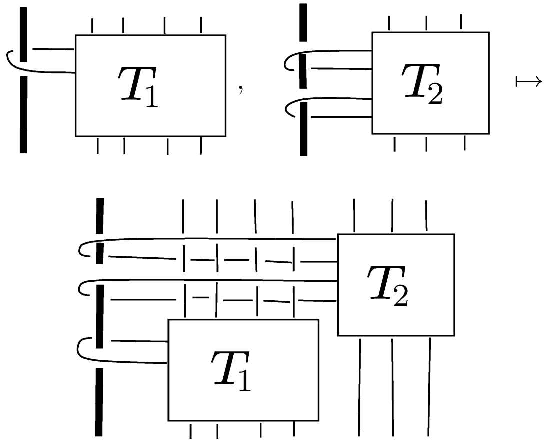

Fig. 4.1. $T_1 \odot T_2$.

T_2 be affine tangle diagrams (say of size (a,a) and (b,b), respectively), and suppose that T_2 has no closed loops. Then $T_1 \odot T_2$ is obtained by replacing the flagpole in the affine tangle diagram T_2 with the entire affine tangle diagram T_1. See Fig. 4.1. If we regard T_1 and T_2 as representing framed tangles in the annulus cross the interval $A \times I$, then $T_1 \odot T_2$ is obtained by inserting the entire copy of $A \times I$ containing T_1 into the hole of the copy of $A \times I$ containing T_2.

Then $T_1 \otimes T_2 \mapsto T_1 \odot T_2$ determines a linear map from $\widehat{W}_{a,S} \otimes \widehat{W}_{b,S}$ into $\widehat{W}_{a+b,S}$, or from $W_{a,S,r} \otimes W_{b,S,r}$ into $W_{a+b,S,r}$. Note that $(T_1 \odot T_2)^* = T_1^* \odot T_2^*$.

These maps of affine and cyclotomic BMW algebras are not algebra homomorphisms. In fact,

$$(1 \odot e_1)(1 \odot x_1)(1 \odot e_1) = z \odot e_1,$$

where z is a (non-scalar) central element in $\widehat{W}_{a,S}$. Nevertheless, we have

$$(A \odot B)(S \odot T) = AS \odot BT,$$

if no closed loops are produced in the product BT, in particular, if at least one of B and T has no horizontal strands.

4.2. *Cellular bases*

Using Eq. (3.9) and our remarks in Section 4.1, we can rewrite the elements T_D of Section 3.3 in the form

$$T_D = g_{\alpha_1}\big(\big[g_{\alpha_2}\boldsymbol{x}^a(e_1e_3\cdots e_{2f-1})\boldsymbol{x}^c(g_{\beta_2})^*\big] \odot g_\pi \boldsymbol{x}^b\big)(g_{\beta_1})^*. \tag{4.1}$$

Here, D is a $\mathbb{Z}$-Brauer diagram with s vertical strands and $f = (n-s)/2$; α_1 and β_1 are $(n-s,s)$-shuffles; $\pi \in \mathfrak{S}_s$ and $\boldsymbol{x}^b = x_1^{b_1}\cdots x_s^{b_s}$. Moreover, α_2 and β_2 are elements of $\mathcal{D}_{f,f}$, $\boldsymbol{x}^a = x_1^{a_1}x_3^{a_3}\cdots x_{2f-1}^{a_{2f-1}}$, and similarly for $\boldsymbol{x}^c$.

The affine $(2f,2f)$-tangle diagram

$$T = g_{\alpha_2}\boldsymbol{x}^a(e_1e_3\cdots e_{2f-1})\boldsymbol{x}^c(g_{\beta_2})^*$$

is stratified and flagpole descending, with no vertical strands and no closed loops. Conversely, any stratified and flagpole descending affine $(2f, 2f)$-tangle diagram with no vertical strands and no closed loops is regularly isotopic to one of this form.

Note that we can factor T as $T = xy^*$, where x and y are stratified and flagpole descending affine $(0, 2f)$-tangle diagram with no closed loops, namely

$$x = g_{\alpha_2}\boldsymbol{x}^a(\cap_{2f-1}\cdots\cap_3\cap_1) \quad \text{and} \quad y = g_{\beta_2}\boldsymbol{x}^c(\cap_{2f-1}\cdots\cap_3\cap_1).$$

By Remark 3.15, any stratified and flagpole descending affine $(0, 2f)$-tangle diagram with no closed loops is regularly isotopic to one of this form.

Lemma 4.1. *The set of $T_D \in \mathbb{U}$ with s vertical strands equals the set of elements*

$$g_\alpha\big(xy^* \odot g_\pi\boldsymbol{x}^b\big)(g_\beta)^*,$$

where x, y are stratified, flagpole descending affine $(0, n-s)$-tangle diagrams without closed loops or self-crossings of strands; α and β are $(n-s, s)$-shuffles; $\pi \in \mathfrak{S}_s$, and $\boldsymbol{x}^b = x_1^{b_1}\cdots x_s^{b_s}$.

Moreover, $T_D \in \mathbb{U}_r$ if, and only if, the exponents b_i are in the range $0 \leqslant b_i \leqslant r-1$, and the winding numbers of x and y with the flagpole are in the same range.

We will show that the cyclotomic BMW algebras defined over integral, admissible rings are cellular. We fix an integral domain S with admissible parameters, and write $W_{n,S,r}$ for $W_{n,S,r}(u_1,\ldots,u_r)$ and $H_{n,S,r}$ for $H_{n,S,r}(q^2; u_1,\ldots,u_r)$.

For each s with $s \leqslant n$ and $n-s$ even, let V_n^s be the span in $W_{n,S,r}$ of the set of elements $T_D \in \mathbb{U}_r$ with s vertical strands.

Lemma 4.2. *For each s, let $\mathbb{B}_s$ be a basis of $H_{s,S,r}$. Let Σ_s be the set of elements*

$$g_\alpha\big(xy^* \odot t(b)\big)(g_\beta)^* \in W_{n,S,r},$$

such that x, y are stratified, flagpole descending affine $(0, n-s)$-tangle diagrams without closed loops; α and β are $(n-s, s)$-shuffles; and $b \in \mathbb{B}_s$. Then Σ_s is a basis of V_n^s.

Proof. Recall that $\{\tau_\pi\boldsymbol{t}^b\colon \pi \in \mathfrak{S}_s \text{ and } 0 \leqslant b_i \leqslant r-1\}$ is a basis of $H_{s,S,r}$, and that $g_\pi\boldsymbol{x}^b = t(\tau_\pi\boldsymbol{t}^b)$. It follows from this and from Lemma 4.1 that V_n^s is the direct sum over (α, β, x, y) of

$$V_n^s(\alpha, \beta, x, y) = \big\{g_\alpha\big(xy^* \odot t(u)\big)(g_\beta)^*\colon u \in H_{s,S,r}\big\}$$

and that $u \mapsto g_\alpha(T \odot t(u))(g_\beta)^*$ is injective. This implies the result. □

For each s ($s \leqslant n$ and $n-s$ even), let $(\mathcal{C}_s, \Lambda_s)$ be a cellular basis of the cyclotomic Hecke algebra $H_{s,S,r}$. Let $\Lambda = \{(s,\lambda)\colon \lambda \in \Lambda_s\}$ with partial order $(s,\lambda) \geqslant (t,\mu)$ if $s < t$ or if $s = t$ and $\lambda \geqslant \mu$ in Λ_s. For each pair $(s,\lambda) \in \Lambda$, we take $\mathcal{T}(s,\lambda)$ to be the set of triples (α, x, u), where α is an $(n-s, s)$-shuffle; x is a stratified, flagpole descending affine $(0, n-s)$-tangle without closed loops; and $u \in \mathcal{T}(\lambda)$. Define

$$c^{(s,\lambda)}_{(\alpha,x,u),(\beta,y,v)} = g_\alpha\big(xy^* \odot t\big(c^\lambda_{u,v}\big)\big)(g_\beta)^*,$$

and $\mathcal{C}$ to be the set of all $c^{(s,\lambda)}_{(\alpha,x,u),(\beta,y,v)}$.

Lemma 4.3. $(c^{(s,\lambda)}_{(\alpha,x,u),(\beta,y,v)})^* \equiv c^{(s,\lambda)}_{(\beta,y,v),(\alpha,x,u)} \mod \breve{W}^{(s,\lambda)}_{n,S,r}$.

Proof. $(g_\alpha(xy^* \odot t(c^\lambda_{u,v}))(g_\beta)^*)^* = g_\beta(y^*x \odot t(c^\lambda_{u,v})^*)(g_\alpha)^*$, and $t(c^\lambda_{u,v})^* \equiv t(c^\lambda_{v,u})$ modulo the span of diagrams of rank $< s$. Hence $(c^{(s,\lambda)}_{(\alpha,x,u),(\beta,y,v)})^* \equiv c^{(s,\lambda)}_{(\beta,y,v),(\alpha,x,u)}$ modulo the span of diagrams of rank $< s$. □

Lemma 4.4. *For any affine $(n-s, n-s)$-tangle diagram A and affine (s,s)-tangle diagram B, $(A \odot B)(xy^* \odot t(c^\lambda_{u,v}))$ can be written as a linear combination of elements $(x'y^* \odot t(c^\lambda_{u',v}))$, modulo $\breve{W}^{(s,\lambda)}_{n,S,r}$, with coefficients independent of y and v.*

Proof. We have $(A \odot B)(xy^* \odot t(\tau_\pi \boldsymbol{x}^b)) = (A\,xy^* \odot B\,t(\tau_\pi \boldsymbol{x}^b))$, because $t(\tau_\pi \boldsymbol{x}^b)$ has only vertical strands. Therefore, also $(A \odot B)(xy^* \odot t(c^\lambda_{u,v})) = (A\,xy^* \odot B\,t(c^\lambda_{u,v}))$.

Note that $A\,x$ is an affine $(0, n-s)$-tangle, and can be reduced using the algorithm of the proof of Propositions 2.18 and 2.19 in [6] to a linear combination of stratified, flagpole descending $(0, n-s)$-tangles x' without closed loops. The process does not affect y^*.

If B has rank strictly less than s, then the product $(A \odot B)(xy^* \odot t(c^\lambda_{u,v}))$ is a linear combination of basis elements T_D with fewer than s vertical strands, so belongs to $\breve{W}^{(s,\lambda)}_{n,S,r}$.

Otherwise, we can suppose that $B = g_\sigma \boldsymbol{x}^b$. Then $B\,t(c^\lambda_{u,v}) = t(\tau_\sigma \boldsymbol{t}^b)t(c^\lambda_{u,v}) \equiv t(\tau_\sigma \boldsymbol{t}^b\, c^\lambda_{u,v})$ modulo the span of basis diagrams with fewer than s vertical strands. Moreover, $t(\tau_\sigma \boldsymbol{t}^b\, c^\lambda_{u,v})$ is a linear combination of elements $t(c^\lambda_{u',v})$, modulo $t(\breve{H}^\lambda_{s,S,r})$, with coefficients independent of v, by the cellularity of the basis $\mathcal{C}_s$ of $H_{s,S,r}$.

The conclusion follows from these observations. □

Theorem 4.5. *Let S be an admissible integral domain. $(\mathcal{C}, \Lambda)$ is a cellular basis of the cyclotomic BMW algebra $W_{n,S,r}$.*

Proof. Theorem 3.16 and Lemma 4.2 implies that $\mathcal{C}$ is a basis of $W_{n,S,r}$, and property (3) of cellular bases holds by Lemma 4.3. It remains to verify axiom (2) for cellular bases. Thus we have to show that for $w \in W_{n,S,r}$, and for a basis element $c^{(s,\lambda)}_{(\alpha,x,u),(\beta,y,v)} = g_\alpha(xy^* \odot t(c^\lambda_{u,v}))(g_\beta)^*$, the product

$$w\,g_\alpha\big(xy^* \odot t\big(c^\lambda_{u,v}\big)\big)(g_\beta)^* \tag{4.2}$$

can be written as a linear combination of elements

$$g_{\alpha'}\big(x'y^* \odot t\big(c^\lambda_{u',v}\big)\big)(g_\beta)^*,$$

modulo $\breve{W}^{(s,\lambda)}_{n,S,r}$ (with coefficients independent of (β, y, v)).

It suffices to consider products as in Eq. (4.2) with w equal to e_i or to g_i for some i, or $w = x_1$. We consider first $w = e_i$ or $w = g_i$. Here there are several cases, depending on the relative position of $\alpha^{-1}(i)$ and $\alpha^{-1}(i+1)$.

Suppose that $\alpha^{-1}(i) > \alpha^{-1}(i+1)$. Then $g_\alpha = g_i g_{\alpha_1}$, where ${\alpha_1}^{-1}(i) < \alpha_1^{-1}(i+1)$, and α_1 is also an $(n-s,s)$-shuffle. Thus $e_i g_\alpha = e_i g_i g_{\alpha_1} = \rho^{-1} e_i g_{\alpha_1}$. Likewise, $g_i g_\alpha = (g_i)^2 g_{\alpha_1} = g_{\alpha_1} + (q-q^{-1}) g_\alpha + (q^{-1}-q)\rho^{-1} e_i g_{\alpha_1}$. We are therefore reduced to considering the case that $\alpha^{-1}(i) < \alpha^{-1}(i+1)$.

Suppose that $\alpha^{-1}(i+1) \leqslant n-s$ or $n-s+1 \leqslant \alpha^{-1}(i)$. Then $\alpha^{-1}(i+1) = \alpha^{-1}(i)+1$, because α is an $(n-s,s)$-shuffle. Write χ_i for g_i or e_i. We have $\chi_i g_\alpha = g_\alpha \chi_{\alpha^{-1}(i)}$, as one can verify with pictures. But $\chi_{\alpha^{-1}(i)} \in W_{n-s} \otimes W_s$, so the conclusion follows from Lemma 4.4.

It remains to examine the case that $\alpha^{-1}(i) \leqslant n-s$ and $\alpha^{-1}(i+1) \geqslant n-s+1$. In this case, $g_i g_\alpha$ is an $(n-s,s)$-shuffle so $g_i\, g_\alpha(xy^* \odot t(c^\lambda_{u,v}))(g_\beta)^*$ is another basis element.

Next, we have to consider the product $e_i\, g_\alpha(xy^* \odot t(c^\lambda_{u,v}))(g_\beta)^*$. Define a permutation ϱ by

$$\varrho(j) = \begin{cases} j & \text{if } j < \alpha^{-1}(i), \\ j+1 & \text{if } \alpha^{-1}(i) \leqslant j < n-s, \\ \alpha^{-1}(i) & \text{if } j = n-s, \\ \alpha^{-1}(i+1) & \text{if } j = n-s+1, \\ j-1 & \text{if } n-s+1 < j \leqslant \alpha^{-1}(i+1), \\ j & \text{if } j > \alpha^{-1}(i+1). \end{cases} \tag{4.3}$$

Since $\varrho \in \mathfrak{S}_{n-s} \times \mathfrak{S}_r$, we have $\ell(\alpha\varrho) = \ell(\alpha) + \ell(\varrho)$ and $g_{\alpha\varrho} = g_\alpha g_\varrho$. The permutation $\alpha\varrho$ has the following properties: $\alpha\varrho(n-s) = i$; $\alpha\varrho(n-s+1) = i+1$; if $1 \leqslant a < b \leqslant n-s-1$ or $n-s+2 \leqslant a < b \leqslant n$, then $\alpha\varrho(a) < \alpha\varrho(b)$. We have

$$e_i g_\alpha = e_i g_\alpha g_\varrho g_\varrho^{-1} = e_i g_{\alpha\varrho} g_\varrho^{-1}. \tag{4.4}$$

The tangle $e_i g_{\alpha\varrho}$ is stratified and has a horizontal strand connecting the top vertices $\boldsymbol{n-s}$ and $\boldsymbol{n-s+1}$. Contracting that strand, we get

$$e_i g_{\alpha\varrho} = \cap_i g_\sigma \cup_{n-s}, \tag{4.5}$$

for a certain $(n-s-1, s-1)$-shuffle σ. Therefore,

$$e_i g_\alpha\big(xy^* \odot t\big(c^\lambda_{u,v}\big)\big)(g_\beta)^* = \cap_i g_\sigma \cup_{n-s} g_\varrho^{-1}\big(xy^* \odot t\big(c^\lambda_{u,v}\big)\big)(g_\beta)^*. \tag{4.6}$$

Moreover, $g_\varrho^{-1} \in W_{n-s} \otimes W_s \subseteq W_{n,S,r}$, so $g_\varrho^{-1}(xy^* \odot t(c^\lambda_{u,v}))(g_\beta)^*$ is congruent modulo $\breve{W}^{(s,\lambda)}_{n,S,r}$ to a linear combination of elements $(x'y^* \odot t(c^\lambda_{u',v}))(g_\beta)^*$, with coefficients independent of β, y and v, by Lemma 4.4. Thus we have to consider the products

$$\cap_i g_\sigma \cup_{n-s} \big(x'y^* \odot t\big(c^\lambda_{u',v}\big)\big)(g_\beta)^*. \tag{4.7}$$

Focus for a moment on the product $\cap_i g_\sigma \cup_{n-s}(x' \odot 1)$, and write x' in the form

$$g_{\alpha_2} x_1^{a_1} x_3^{a_3} \cdots x_{n-s-1}^{a_{n-s-1}} (\cap_{n-s-1} \cdots \cap_3 \cap_1),$$

with $\alpha_2 \in \mathcal{D}_{f,f}$ ($f = (n-s)/2$). Fig. 4.2 provides a guide to the computations. Write $a = a_{n-s-1}$. We have

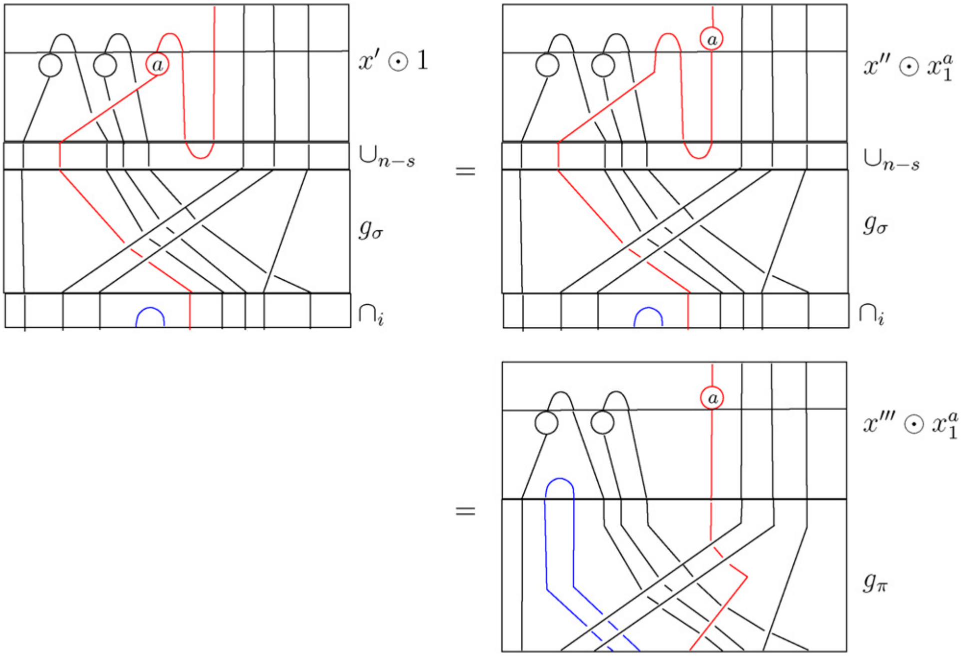

Fig. 4.2.

$$\cap_i g_\sigma \cup_{n-s} (x' \odot 1) = \cap_i g_\sigma \cup_{n-s} \bigl(g_{\alpha_2} x_1^{a_1} x_3^{a_3} \cdots x_{n-s-1}^{a}\bigr)(\cap_{n-s-1} \cdots \cap_3 \cap_1)$$
$$= \cap_i g_\sigma \bigl(g_{\alpha_2} x_1^{a_1} x_3^{a_3} \cdots x_{n-s-1}^{a}\bigr) \cup_{n-s} \cap_{n-s-1} (\cap_{n-s-3} \cdots \cap_3 \cap_1). \qquad (4.8)$$

Note that

$$x_{n-s-1}^{a} \cup_{n-s} \cap_{n-s-1} = \rho^{-a} \cup_{n-s} x_{n-s}^{-a} \cap_{n-s-1}$$
$$= \cup_{n-s} \cap_{n-s-1} x_{n-s+1}^{a},$$

by [5], Lemma 6.8 and Remark 6.9. Thus,

$$\cap_i g_\sigma \cup_{n-s} (x' \odot 1) = \cap_i g_\sigma \cup_{n-s} \bigl(g_{\alpha_2} x_1^{a_1} x_3^{a_3} \cdots x_{n-s-3}^{a_{n-s-3}}\bigr)(\cap_{n-s-1} \cdots \cap_3 \cap_1) x_{n-s+1}^{a}$$
$$= \cap_i g_\sigma \cup_{n-s} \bigl(x'' \odot x_1^{a}\bigr) \qquad (4.9)$$

where x'' is another stratified, flagpole descending affine $(0, n-s)$-tangle diagram (without closed loops) with the property that the strand incident with the vertex $\overline{\boldsymbol{n-s}}$ has winding number 0 with the flagpole. See the second stage in Fig. 4.2. (In the figure, a "bead" on the jth strand is supposed to indicate a power of x_j; a bead labeled by a indicates x_j^a.)

Since this affine tangle diagram is stratified, the strand incident with the top vertex $\mathbf{1}$ can be pulled straight, and the horizontal strand connecting the bottom vertices $\overline{\boldsymbol{i}}$ and $\overline{\boldsymbol{i}+1}$ can be pulled up. The result is an affine tangle diagram with the factorization $g_\pi(x''' \odot x_1^a)$, where x''' is a stratified, flagpole descending affine $(0, n-s)$-tangle diagram and g_π is a positive permutation braid; this is illustrated in the final stage of Fig. 4.2.

Consequently, we have:

$$\cap_i g_\sigma \cup_{n-s} \big(x' y^* \odot t\big(c^\lambda_{u',v}\big)\big)(g_\beta)^* = g_\pi \big(x''' y^* \odot x_1^a t\big(c^\lambda_{u',v}\big)\big)(g_\beta)^*. \tag{4.10}$$

By Lemma 4.4, this is congruent mod $\breve{W}^{(s,\lambda)}_{n,S,r}$ to a linear combination of terms

$$g_\pi \big(x''' y^* \odot t\big(c^\lambda_{u'',v}\big)\big)(g_\beta)^*, \tag{4.11}$$

with coefficients independent of y, β, and v.

Finally, the permutation π can be factored as $\pi = \pi_1\pi_2$, where π_1 is an $(n-s, s)$-shuffle, $\pi_2 \in \mathfrak{S}_{n-s} \times \mathfrak{S}_s$ and $\ell(\pi) = \ell(\pi_1) + \ell(\pi_2)$. Consequently, $g_\pi = g_{\pi_1} g_{\pi_2}$, where $g_{\pi_2} \in W_{n-s} \otimes W_s$. We can now apply Lemma 4.4 again to rewrite the product of Eq. (4.11) as a linear combination of elements $g_{\pi_1}(x'''' y^* \odot t(c^\lambda_{u''',v}))(g_\beta)^*$, modulo $\breve{W}^{(s,\lambda)}_{n,S,r}$, with coefficients independent of β, y, and v. This completes the proof of the case: $w = e_i$, $\alpha^{-1}(i) \leqslant n-s$ and $n-s+1 \leqslant \alpha^{-1}(i+1)$.

It remains to consider the product

$$x_1 \; g_\alpha \big(x y^* \odot t\big(c^\lambda_{u,v}\big)\big)(g_\beta)^*.$$

Since α is an $(n-s, s)$-shuffle, either $\alpha^{-1}(1) = 1$ or $\alpha^{-1}(1) = n-s+1$. In case $\alpha^{-1}(1) = 1$, we have $x_1 g_\alpha = g_\alpha x_1$, and the result follows by applying Lemma 4.4. If $\alpha^{-1}(1) = n-s+1$, then we can write g_α as $g_\alpha = g_{\alpha_2}(g_1 g_2 \cdots g_{n-s-1})$, where g_{α_2} is a word in g_j, $j \geqslant 2$. In this case, $x_1 g_\alpha = g_{\alpha_2}(g_1^{-1} g_2^{-1} \cdots g_{n-s-1}^{-1}) x_{n-s+1}$. Now the result follows by first applying Lemma 4.4, and then expanding each g_j^{-1} in terms of g_j and e_j and appealing to the previous part of the proof. □

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