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Completely positive mappings and mean matrices

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ABSTRACT

Some functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ induce mean of positive numbers and the matrix monotonicity gives a possibility for means of positive definite matrices. Moreover, such a function f can define a linear mapping $(\mathbb{J}_D^f)^{-1} : \mathbf{M}_n \to \mathbf{M}_n$ on matrices (which is basic in the constructions of monotone metrics). The present subject is to check the complete positivity of $(\mathbb{J}_D^f)^{-1}$ in the case of a few concrete functions f. This problem has been motivated by applications in quantum information.

1. Introduction

Let $D \in \mathbf{M}_n$ be a positive definite matrix and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function. A linear operator $\mathbb{J}_D^f : \mathbf{M}_n \to \mathbf{M}_n$ is defined as

$$\mathbb{J}_D^f = f(\mathbb{L}_D \mathbb{R}_D^{-1})\mathbb{R}_D, \tag{1}$$

where $\mathbb{L}_D, \mathbb{R}_D : \mathbf{M}_n \to \mathbf{M}_n$,

$$\mathbb{L}_D(X) = DX \quad \text{and} \quad \mathbb{R}_D(X) = XD \quad (D \in \mathbf{M}_n)\,.$$

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(The operator $\mathbb{L}_D\mathbb{R}_D^{-1}$ appeared in the modular theory of von Neumann algebras.) The operator $\mathbb{J}_D^f$ and its inverse

$$(\mathbb{J}_D^f)^{-1} = f^{-1}(\mathbb{L}_D\mathbb{R}_D^{-1})\mathbb{R}_D^{-1}$$

occur in several quantum applications [12,14,17]. There the function f should be operator monotone which means that $0 \leq A \leq B$ implies $f(A) \leq f(B)$ for all matrices $A, B \in \mathbf{M}_n$ for every $n \in \mathbb{N}$. For example,

$$\gamma_D(A, B) := \langle A, (\mathbb{J}_D^f)^{-1}B\rangle$$

is a kind of Riemannian metric, $D > 0$ is a foot-point and the self-adjoint matrices A and B are tangent vectors. This inner product is real-valued if $xf(x^{-1}) = f(x)$. We shall call the matrix monotone function f standard if $f(1) = 1$ and $xf(x^{-1}) = f(x)$. Standard functions are used to define (symmetric) matrix means:

$$M_f(A, B) = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2},$$

see [9]. For numbers $m_f(x, y) = xf(y/x)$.

It is well-known, see [9], that if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a standard matrix monotone function, then

$$\frac{2x}{x+1} \leq f(x) \leq \frac{x+1}{2}.$$

For example,

$$\frac{2x}{x+1} \leq \sqrt{x} \leq \frac{x-1}{\log x} \leq \frac{x+1}{2},$$

they correspond to the harmonic, geometric, logarithmic and arithmetic mean.

The linear mappings $(\mathbb{J}_D^f)^{-1} : \mathbf{M}_n \to \mathbf{M}_n$ have the monotonicity condition

$$\alpha^*(\mathbb{J}_{\alpha(D)}^f)^{-1}\alpha \leq (\mathbb{J}_D^f)^{-1} \tag{2}$$

for every completely positive trace preserving mapping $\alpha : \mathbf{M}_n \to \mathbf{M}_m$, if f is a matrix monotone function. The monotonicity property is important in the construction of monotone metrics and Fisher information [11,14] and the requirement of the matrix monotonicity for f is motivated by these applications.

The linear transformation $(\mathbb{J}_D^f)^{-1}$ appeared also in the paper [17] (in a different notation) and the complete positivity was questioned there. The subject of this paper is to find functions f such that $(\mathbb{J}_D^f)^{-1}$ (or $\mathbb{J}_D^f$) is completely positive for every $D > 0$ matrix and to show examples not being completely positive. Presently we cannot find abstract results, only concrete functions are analyzed here. When the matrix monotonicity is not known for a function discussed here, it is proven as well.

2. Preliminaries

A linear mapping $\beta : \mathbf{M}_n \to \mathbf{M}_n$ is completely positive if $id_n \otimes \beta : \mathbf{M}_n \otimes \mathbf{M}_n \to \mathbf{M}_n \otimes \mathbf{M}_n$ is positive, or equivalently

$$\beta(X) = \sum_i V_iXV_i^* \quad \text{with} \quad V_i \in \mathbf{M}_n.$$

In the first definition the matrix size is increased. Since in our context for $(\mathbb{J}_D^f)^{-1}$ all dimensions n are included, it will turn out below that complete positivity is the same as positivity (for every n).

If $D = \mathrm{Diag}\,(\lambda_1, \ldots, \lambda_n)$, then

$$(\mathbb{J}_D^f A)_{ij} = A_{ij} m_f(\lambda_i, \lambda_j) \quad (A \in \mathbf{M}_n)$$

and

$$((\mathbb{J}_D^f)^{-1} A)_{ij} = A_{ij} \frac{1}{m_f(\lambda_i, \lambda_j)} \quad (A \in \mathbf{M}_n).$$

Both $\mathbb{J}_D^f$ and $(\mathbb{J}_D^f)^{-1}$ have the form of a Hadamard product $A \mapsto A \circ T$. Note that for $id_n \otimes \mathbb{J}_D^f$ and $id_n \otimes (\mathbb{J}_D^f)^{-1}$ we have similar situation, but the diagonal matrix has n^2 positive parameters.

Lemma 1. *The linear mapping $\beta : \mathbf{M}_n \to \mathbf{M}_n$, $\beta(A) = A \circ T$ is completely positive if and only if the matrix $T \in \mathbf{M}_n$ is positive.*

Proof. If β is completely positive, then $A \circ T \geq 0$ for every positive A. This implies the positivity of T.

The mapping β linearly depends on T. Therefore, it is enough to prove the complete positivity when $T_{ij} = \overline{\lambda_i} \lambda_j$. Then

$$\beta(A) = \mathrm{Diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)^* A \mathrm{Diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

and the complete positivity is clear. $\square$

Assume that a standard matrix monotone function f is given. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be strictly positive numbers. The positivity of the matrix $X_f \in \mathbf{M}_n$ defined as

$$(X_f)_{ij} = m_f(\lambda_i, \lambda_j) \tag{3}$$

is an interesting question. We call X_f *mean matrix*. (A stronger property than positivity is the so-called *infinite divisibility* [2], this is not studied here, but some results are used.)

The choice $\lambda_1 = 1$ and $\lambda_2 = x$ shows that

$$f(x) \leq \sqrt{x}$$

is a necessary condition for the positivity of the mean matrix, in other words m_f should be smaller than the geometric mean. If $f(x) \geq \sqrt{x}$, then the matrix

$$(Y_f)_{ij} = \frac{1}{m_f(\lambda_i, \lambda_j)} \tag{4}$$

can be positive. The matrix (4) was important in the paper [11] for the characterization of monotone metrics, see also [6–8,12–14].

If $f(x) = \sqrt{x}$, then both $\mathbb{J}_D^f$ and $(\mathbb{J}_D^f)^{-1}$ are completely positive, since $(X_f)_{ij} = \sqrt{\lambda_i}\sqrt{\lambda_j}$ and $(Y_f)_{ij} = 1/(\sqrt{\lambda_i}\sqrt{\lambda_j})$ are positive matrices. We show some other simple examples.

Example 1. If $f(x) = (1 + x)/2$, the arithmetic mean, then Y_f is the so-called Cauchy matrix,

$$(Y_f)_{ij} = \frac{2}{\lambda_i + \lambda_j} = 2 \int_0^\infty e^{-s\lambda_i} e^{-s\lambda_j}\, ds,$$

which is positive. Therefore, the mapping $A \mapsto A \circ Y_f$ is completely positive. This can be seen also from the formula

$$(\mathbb{J}_D^f)^{-1}(A) = 2 \int_0^\infty \exp(-sD) A \exp(-sD)\, ds.$$

Example 2. The *logarithmic mean* corresponds to the function $f(x) = (x-1)/\log x$.

The mapping

$$(\mathbb{J}_D^f)^{-1}(A) = \int_0^\infty (D+t)^{-1}A(D+t)^{-1}\, dt$$

is completely positive.

Let $D = \mathrm{Diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be positive definite. Since $(\mathbb{J}_D^f)^{-1}(A) = A \circ Y_f$ is a Hadamard product with

$$(Y_f)_{ij} = \frac{\log\lambda_i - \log\lambda_j}{\lambda_i - \lambda_j},$$

the complete positivity of the mapping $(\mathbb{J}_D^f)^{-1}$ implies the positivity of Y_f.

Another proof comes from the formula

$$\frac{\log\lambda_i - \log\lambda_j}{\lambda_i - \lambda_j} = \int_0^\infty \frac{1}{s+\lambda_i}\,\frac{1}{s+\lambda_j}\, ds$$

due to a Hadamard product.

If the standard matrix monotone function $f(x)$ is between $\sqrt{x}$ and $(x+1)/2$, then $g(x) := 1/f(x^{-1})$ is a standard matrix monotone function as well and $2x/(x+1) \leq g(x) \leq \sqrt{x}$. It follows that the positivity of (4) is equivalent to the positivity of the mean matrix

$$(X_g)_{ij} = m_g(\lambda_i, \lambda_j) = \frac{\lambda_i\lambda_j}{m_f(\lambda_i, \lambda_j)}.$$

3. Results

Example 3. Consider the standard matrix monotone function

$$f(x) := \frac{1}{2}(x^t + x^{1-t}) \geq \sqrt{x} \quad (0 < t < 1).$$

(The corresponding mean is sometimes called *Heinz mean*).

To find the inverse of the mapping

$$\mathbb{J}_D^f(A) = \frac{1}{2}\left(D^tAD^{1-t} + D^{1-t}AD^t\right)$$

we should solve the equation

$$2A = D^tYD^{1-t} + D^{1-t}YD^t,$$

when $Y = (\mathbb{J}_D^f)^{-1}(A)$ is unknown. This has the form

$$2D^{-t}AD^{-t} = YD^{1-2t} + D^{1-2t}Y$$

which is a Sylvester equation. The solution is

$$(\mathbb{J}_D^f)^{-1}(A) = Y = \int_0^\infty \exp(-sD^{1-2t})(2D^{-t}AD^{-t})\exp(-sD^{1-2t})\, ds.$$

This mapping is positive.

The function

$$f_t(x) = 2^{2t-1}x^t(1+x)^{1-2t} = \left(\frac{2x}{1+x}\right)^t \left(\frac{x+1}{2}\right)^{1-t} \tag{5}$$

is a kind of interpolation between the arithmetic mean ($t = 0$) and the harmonic mean ($t = 1$). This function appeared in the paper [4] and it is proven there that it is a standard matrix monotone function.

Theorem 1. *If $t \in (0, 1/2)$, then*

$$f_t(x) = 2^{2t-1}x^t(1+x)^{1-2t} \geq \sqrt{x}$$

and the mapping $(\mathbb{J}_D^{f_t})^{-1}$ is completely positive.

Proof. We shall use the mean matrix approach and show that

$$(Y_{f_t})_{ij} = \frac{1}{m_{f_t}(\lambda_i, \lambda_j)} = \frac{2^{1-2t}}{(\lambda_i + \lambda_j)^{1-2t}}(\lambda_i\lambda_j)^{-t}$$

is positive.

For $|x| < 1$ and $1 - 2t = \alpha > 0$ the binomial expansion yields

$$(1-x)^{-\alpha} = \sum_{k=0}^{\infty} a_k x^k,$$

where

$$a_k = (-1)^k \binom{-\alpha}{k} = (-1)^k \frac{(-\alpha-1)(-\alpha-2)\cdot\dots\cdot(-\alpha-k+1)}{k!} > 0.$$

So that

$$\begin{aligned}
(\lambda_i + \lambda_j)^{-(1-2t)} &= \left(\left(\lambda_i + \frac{1}{2}\right)\left(\lambda_j + \frac{1}{2}\right)\left(1 - \frac{\left(\lambda_i - \frac{1}{2}\right)\left(\lambda_j - \frac{1}{2}\right)}{\left(\lambda_i + \frac{1}{2}\right)\left(\lambda_j + \frac{1}{2}\right)}\right)\right)^{-(1-2t)} \\
&= \left(\lambda_i + \frac{1}{2}\right)^{-(1-2t)}\left(\lambda_j + \frac{1}{2}\right)^{-(1-2t)} \sum_{k=0}^{\infty} a_k \left(\frac{\left(\lambda_i - \frac{1}{2}\right)\left(\lambda_j - \frac{1}{2}\right)}{\left(\lambda_i + \frac{1}{2}\right)\left(\lambda_j + \frac{1}{2}\right)}\right)^k \\
&= \sum_{k=0}^{\infty} a_k \frac{\left(\lambda_i - \frac{1}{2}\right)^k \left(\lambda_j - \frac{1}{2}\right)^k}{\left(\lambda_i + \frac{1}{2}\right)^{k+(1-2t)}\left(\lambda_j + \frac{1}{2}\right)^{k+(1-2t)}}.
\end{aligned}$$

Hence we have

$$(Y_{f_t})_{ij} = 2^{1-2t} \sum_{k=0}^{\infty} a_k \frac{\left(\lambda_i - \frac{1}{2}\right)^k}{\left(\lambda_i + \frac{1}{2}\right)^{k+(1-2t)} \lambda_i^t} \frac{\left(\lambda_j - \frac{1}{2}\right)^k}{\left(\lambda_j + \frac{1}{2}\right)^{k+(1-2t)} \lambda_j^t}$$

and Y_{f_t} is the sum of positive matrices of rank one. $\square$

If $t \in (1/2, 1)$ in (5), then

$$f_t(x) \leq \sqrt{x}$$

and the positivity of the matrix

$$(X_{f_t})_{ij} = m_{f_t}(\lambda_i, \lambda_j)$$

can be shown similarly to the above argument. Therefore $J_D^{f_t}$ is completely positive.

Example 4. The mean

$$m(x, y) = \frac{1}{2}\left(\frac{x+y}{2} + \frac{2xy}{x+y}\right)$$

induced by the function

$$f(x) = \frac{1}{2}\left(\frac{1+x}{2} + \frac{2x}{1+x}\right)$$

is larger than the geometric mean. Indeed,

$$\frac{1}{2}\left(\frac{x+y}{2} + \frac{2xy}{x+y}\right) \geq \sqrt{\frac{x+y}{2}\frac{2xy}{x+y}} = \sqrt{xy}.$$

The numerical computation shows that in this case already the determinant of a 3×3 matrix Y_f can be negative. This example shows that the corresponding mapping $(\mathbb{J}_D^f)^{-1}$ is not completely positive.

Next we consider the function

$$f_t(x) = t(1-t)\frac{(x-1)^2}{(x^t-1)(x^{1-t}-1)} \tag{6}$$

which was first studied in the paper [5]. If $0 < t < 1$, then the integral representation

$$\frac{1}{f_t(x)} = \frac{\sin t\pi}{\pi}\int_0^\infty d\lambda\, \lambda^{t-1}\int_0^1 ds \int_0^1 dr \frac{1}{x((1-r)\lambda + (1-s)) + (r\lambda + s)} \tag{7}$$

shows that $f_t(x)$ is operator monotone. (Note that in the paper [16] the operator monotonicity was obtained for $-1 \leq t \leq 2$.) The property $xf(x^{-1}) = f(x)$ is obvious.

If $t = 1/2$, then

$$f(x) = \left(\frac{1+\sqrt{x}}{2}\right)^2 \geq \sqrt{x}$$

and the corresponding mean is called binomial mean or power mean. In this case we have

$$(Y_f)_{ij} = \frac{4}{(\sqrt{\lambda_i} + \sqrt{\lambda_j})^2}.$$

The matrix

$$U_{ij} = \frac{1}{\sqrt{\lambda_i} + \sqrt{\lambda_j}}$$

is a kind of Cauchy matrix, so it is positive. Since $Y_{f_t} = 4U \circ U$, Y_{f_t} is positive as well.

If $\gamma(A) = A \circ U$, then $(\mathbb{J}_D^{f_t})^{-1} = 4\gamma^2$. Since

$$\gamma(A) = \int_0^\infty \exp(-s\sqrt{D})A\exp(-s\sqrt{D})\,ds,$$

we have

$$(\mathbb{J}_D^{f_t})^{-1}(A) = 4\int_0^\infty\int_0^\infty \exp(-(s+r)\sqrt{D})A\exp(-(s+r)\sqrt{D})\,ds\,dr. \tag{8}$$

The complete positivity of $(\mathbb{J}_D^{f_t})^{-1}$ is clear from this formula.

For the other values of t in $(0, 1)$ the proof is a bit more sophisticated.

Lemma 2. *If* $0 < t < 1$, *then* $f_t(x) \geqslant \sqrt{x}$ *for* $x > 0$.

Proof. It is enough to show that for $0 < t < 1$ and $x > 0$

$$t\frac{x-1}{x^t-1} \geqslant x^{\frac{1-t}{2}}, \tag{9}$$

since this implies

$$t\frac{x-1}{x^t-1}(1-t)\frac{x-1}{x^{1-t}-1} \geqslant x^{\frac{1-t}{2}}x^{\frac{t}{2}} = \sqrt{x}.$$

Denote

$$g(x) := t(x-1) + x^{\frac{1-t}{2}} - x^{\frac{1+t}{2}}.$$

Then inequality (9) reduces to $g(x) \geqslant 0$ for $x \geqslant 1$ and to $g(x) \leqslant 0$ for $0 < x \leqslant 1$. Since $g(1) = 0$ it suffices to verify that g is monotone increasing, in other words $g' \geqslant 0$. By simple calculation one obtains

$$g'(x) = t + \frac{1-t}{2}x^{\frac{-t-1}{2}} - \frac{1+t}{2}x^{\frac{t-1}{2}}$$

and

$$g''(x) = \frac{1-t^2}{4}x^{\frac{t-3}{2}} - \frac{1-t^2}{4}x^{\frac{-t-3}{2}},$$

which yields $g''(x) \leqslant 0$ for $0 < x < 1$ and $g''(x) \geqslant 0$ for $x \geqslant 1$. Thus, due to $g'(1) = 0$, $g' \geqslant 0$, the statement follows. $\square$

It follows from Lemma 2 that the matrix

$$(Y_{f_t})_{ij} = t(1-t) \times \frac{\lambda_i^t - \lambda_j^t}{\lambda_i - \lambda_j} \times \frac{\lambda_i^{1-t} - \lambda_j^{1-t}}{\lambda_i - \lambda_j} \quad (1 \le i, j \le m)$$

can be positive. It is a Hadamard product, so it is enough to see that

$$U_{ij}^{(t)} = \frac{\lambda_i^t - \lambda_j^t}{\lambda_i - \lambda_j} \quad (1 \le i, j \le m)$$

is positive for $0 < t < 1$. It is a well-known fact (see [1]) that the function $g : \mathbb{R}^+ \to \mathbb{R}$ is matrix monotone if and only if the Löwner matrices

$$L_{ij} = \frac{g(\lambda_i) - g(\lambda_j)}{\lambda_i - \lambda_j} \qquad (1 \le i, j \le m)$$

are positive. The function $g(x) = x^t$ is matrix monotone for $0 < t < 1$ and the positivity of $U^{(t)}$ and Y_{f_t} follows. So we have:

Theorem 2. *For the function (6) the mapping $(\mathbb{J}_D^{f_t})^{-1}$ is completely positive if* $0 < t < 1$.

To see the explicit complete positivity of $(\mathbb{J}_D^{f_t})^{-1}$, the mappings $\gamma_t(A) = A \circ U^{(t)}$ are useful, we have

$$(\mathbb{J}_D^{f_t})^{-1}(A) = t(1-t)\gamma_t(\gamma_{1-t}(A)).$$

Instead of the Hadamard product, which needs the diagonality of D, we can use

$$\gamma_t(A) = \frac{\partial}{\partial x}(D + xA)^t\Big|_{x=0}.$$

We compute γ_t from

$$(D + xA)^t = \frac{\sin \pi t}{\pi} \int_0^\infty \left(I - s(D + xA + sI)^{-1}\right) s^{t-1}\, ds.$$

So we obtain

$$\gamma_t(A) = \frac{\sin \pi t}{\pi} \int_0^\infty s^t (D + sI)^{-1} A (D + sI)^{-1}\, ds$$

and

$$(\mathbb{J}_D^{f_t})^{-1}(A) = t(1-t)\frac{\sin \pi t\ \sin \pi(1-t)}{\pi^2} \int_0^\infty \int_0^\infty r^{1-t} s^t (D + rI)^{-1}(D + sI)^{-1} A (D + sI)^{-1}(D + rI)^{-1}\, ds\, dr.$$

Example 5. The *power difference means* are determined by the functions

$$f_t(x) = \frac{t-1}{t}\frac{x^t - 1}{x^{t-1} - 1} \quad (-1 \le t \le 2), \tag{10}$$

where the values $t = -1,\ 1/2,\ 1,\ 2$ correspond to the well-known means: harmonic, geometric, logarithmic and arithmetic mean. The functions (10) are operator monotone [3] and we show that for fixed $x > 0$ the value $f_t(x)$ is increasing function of t.

By substituting $x = e^{2\lambda}$ one has

$$f_t(e^{2\lambda}) = \frac{t-1}{t}\frac{e^{\lambda t}\frac{e^{\lambda t} - e^{-\lambda t}}{2}}{e^{\lambda(t-1)}\frac{e^{\lambda(t-1)} - e^{-\lambda(t-1)}}{2}} = e^{\lambda}\frac{t-1}{t}\frac{\sinh(\lambda t)}{\sinh(\lambda(t-1))}.$$

Since

$$\frac{d}{dt}\left(\frac{t-1}{t}\frac{\sinh(\lambda t)}{\sinh(\lambda(t-1))}\right) = \frac{\sinh(\lambda t)\sinh(\lambda(t-1)) - \lambda t(t-1)\sinh(\lambda)}{t^2 \sinh^2(\lambda(t-1))},$$

it suffices to show that

$$g(t) = \sinh(\lambda t)\sinh(\lambda(t-1)) - \lambda t(t-1)\sinh(\lambda) \geqslant 0.$$

Observe that $\lim_{\pm\infty} g = +\infty$ thus g has a global minimum. By simple calculations one obtains

$$g'(t) = \lambda(\sinh(\lambda(2t-1)) - (2t-1)\sinh(\lambda)).$$

It is easily seen that the zeros of g' are $t = 0$, $t = 1/2$ and $t = 1$ hence $g(0) = g(1) = 0$ and $g(\frac{1}{2}) = \sinh^2(\frac{\lambda}{2}) + \frac{\lambda}{4}\sinh(\lambda) \geqslant 0$ implies that $g \geqslant 0$.

It follows that

$$\sqrt{x} \le f_t(x) \le \frac{1+x}{2} \quad (1/2 \le t \le 2) \quad \text{and} \quad \frac{2x}{x+1} \le f_t(x) \le \sqrt{x} \quad (-1 \le t \le 1/2).$$

For the values $1/2 \le t \le 2$ the complete positivity holds. This follows from the next lemma which contains a bigger interval for t.

Lemma 3. *The matrix*

$$(Y_{f_t})_{ij} := \frac{t}{t-1}\frac{\lambda_i^{t-1} - \lambda_j^{t-1}}{\lambda_i^t - \lambda_j^t}$$

is positive if $\frac{1}{2} \leqslant t$.

Proof. For $t > 1$ the statement follows from the proof of Theorem 2, since

$$\frac{t}{t-1}\frac{\lambda_i^{t-1} - \lambda_j^{t-1}}{\lambda_i^t - \lambda_j^t} = \frac{t}{t-1}\frac{(\lambda_i^t)^{\frac{t-1}{t}} - (\lambda_j^t)^{\frac{t-1}{t}}}{\lambda_i^t - \lambda_j^t},$$

where $0 < \frac{t-1}{t} < 1$, further, for $t = 1$ as limit $(Y_{f_1})_{ij} = (\log\lambda_i - \log\lambda_j)/(\lambda_i - \lambda_j)$ the statement follows from Example 2. If $\frac{1}{2} \leqslant t < 1$ let $s := 1 - t$ where $0 < s \leqslant \frac{1}{2}$. Then

$$(Y_{f_t})_{ij} = \frac{t}{t-1}\frac{\lambda_i^{t-1} - \lambda_j^{t-1}}{\lambda_i^t - \lambda_j^t} = \frac{1-s}{-s}\frac{\lambda_i^{-s} - \lambda_j^{-s}}{\lambda_i^t - \lambda_j^t} = \frac{1-s}{s}\frac{(\lambda_i^t)^{\frac{s}{t}} - (\lambda_j^t)^{\frac{s}{t}}}{\lambda_i^t - \lambda_j^t}\frac{1}{\lambda_i^s\lambda_j^s}$$

so that $(\mathbb{J}_D^{f_t})^{-1}$ is the Hadamard product of U and V, where

$$U_{ij} = \frac{(\lambda_i^t)^{\frac{s}{t}} - (\lambda_j^t)^{\frac{s}{t}}}{\lambda_i^t - \lambda_j^t}$$

is positive due to $0 < \frac{s}{t} \leqslant 1$ and

$$V_{ij} = \frac{1-s}{s}\frac{1}{\lambda_i^s\lambda_j^s}$$

is positive, too. $\square$

Example 6. Another interpolation between the arithmetic mean ($t = 1$) and the harmonic mean ($t = 0$) is the following:

$$f_t(x) = \frac{2(tx+1)(t+x)}{(1+t)^2(x+1)} \qquad (0 \le t \le 1).$$

First we compare this mean with the geometric mean:

$$f_t(x^2) - x = \frac{(x-1)^2(2tx^2 - (1-t)^2x + 2t)}{(1+t)^2(x^2+1)}$$

and the sign depends on

$$x^2 - \frac{(1-t)^2}{2t}x + 1 = \left(x - \frac{(1-t)^2}{4t}\right)^2 + 1 - \left(\frac{(1-t)^2}{4t}\right)^2.$$

So the positivity condition is $(1-t)^2 \le 4t$ which gives $3-2\sqrt{2} \le t \le 3+2\sqrt{2}$. For these parameters $f_t(x) \ge \sqrt{x}$ and for $0 < t < 3-2\sqrt{2}$ the two means are not comparable.

For $3-2\sqrt{2} \le t \le 1$ the matrix monotonicity is rather straightforward:

$$f_t(x) = \frac{2}{(1+t)^2}\left(tx + t^2 - t + 1 - \frac{(t-1)^2}{x+1}\right).$$

However, the numerical computations show that $Y_{f_t} \ge 0$ is not true.

In the rest we concentrate on the matrix monotonicity of some functions. First the *Stolarsky mean* is investigated in [10,15].

Theorem 3. *Let*

$$f_p(x) := \left(\frac{p(x-1)}{x^p - 1}\right)^{\frac{1}{1-p}}, \tag{11}$$

where $p \neq 1$. Then f_p is matrix monotone if $-2 \leqslant p \leqslant 2$.

Proof. First note that $f_2(x) = (x+1)/2$ is the arithmetic mean, the limiting case $f_0(x) = (x-1)/\mathrm{log}x$ is the logarithmic mean and $f_{-1}(x) = \sqrt{x}$ is the geometric mean, their matrix monotonicity is well-known. If $p = -2$ then

$$f_{-2}(x) = \left(\frac{2x^2}{x+1}\right)^{\frac{1}{3}}$$

which will be shown to be matrix monotone at the end of the proof.

Now let us suppose that $p \neq -2, -1, 0, 1, 2$. By Löwner's theorem [1] f_p is matrix monotone if and only if it has a holomorphic continuation mapping the upper half plane into itself []. We define $\mathrm{log}z$ as $\log 1 := 0$ then in case $-2 < p < 2$, since $z^p - 1 \neq 0$ in the upper half plane, the real function $p(x-1)/(x^p-1)$ has a holomorphic continuation to the upper half plane, moreover it is continuous in the closed upper half plane, further, $p(z-1)/(z^p-1) \neq 0$ $(z \neq 1)$ so f_p also has a holomorphic continuation to the upper half plane and it is also continuous in the closed upper half plane.

Assume $-2 < p < 2$ then it suffices to show that f_p maps the upper half plane into itself. We show that for every $\varepsilon > 0$ there is $R > 0$ such that the set $\{z : |z| \geqslant R, \mathrm{Im}\, z > 0\}$ is mapped into

$\{z : 0 \leqslant \arg z \leqslant \pi + \varepsilon\}$, further, the boundary $(-\infty, +\infty)$ is mapped into the closed upper half plane. By the open mapping theorem the image of a connected open set by a holomorphic function is either a connected open set or a single point thus it follows that the upper half plane is mapped into itself by f_p.

Clearly, $[0, \infty)$ is mapped into $[0, \infty)$ by f_p.

Now first suppose $0 < p < 2$. Let $\varepsilon > 0$ be sufficiently small and $z \in \{z : |z| = R,\ \mathrm{Im}\, z > 0\}$ where $R > 0$ is sufficiently large. Then

$$\arg(z^p - 1) = \arg z^p \pm \varepsilon = p \arg z \pm \varepsilon,$$

and similarly $\arg z - 1 = \arg z \pm \varepsilon$ so that

$$\arg \frac{z-1}{z^p-1} = (1-p) \arg z \pm 2\varepsilon.$$

Further,

$$\left| \frac{z-1}{z^p-1} \right| \geqslant \frac{|z|-1}{|z|^p+1} = \frac{R-1}{R^p+1},$$

which is large for $0 < p < 1$ and small for $1 < p < 2$ if R is sufficiently large, hence

$$\arg \left(\frac{z-1}{z^p-1} \right)^{\frac{1}{1-p}} = \frac{1}{1-p} \arg \left(\frac{z-1}{z^p-1} \right) \pm 2\varepsilon = \arg z \pm 2\varepsilon \frac{2-p}{1-p}.$$

Since $\varepsilon > 0$ was arbitrary it follows that $\{z : |z| = R,\ \mathrm{Im}\, z > 0\}$ is mapped into the upper half plane by f_p if $R > 0$ is sufficiently large.

Now, if $z \in [-R, 0)$ then $\arg(z-1) = \pi$, further, $p\pi \leqslant \arg(z^p - 1) \leqslant \pi$ for $0 < p < 1$ and $\pi \leqslant \arg(z^p - 1) \leqslant p\pi$ for $1 < p < 2$ whence

$$0 \leqslant \arg \left(\frac{z-1}{z^p-1} \right) \leqslant (1-p)\pi \quad \text{for} \quad 0 < p < 1,$$

and

$$(1-p)\pi \leqslant \arg \left(\frac{z-1}{z^p-1} \right) \leqslant 0 \quad \text{for} \quad 1 < p < 2.$$

Thus by

$$\pi \arg \left(\frac{z-1}{z^p-1} \right)^{\frac{1}{1-p}} = \frac{1}{1-p} \arg \left(\frac{z-1}{z^p-1} \right)$$

it follows that

$$0 \leqslant \arg \left(\frac{z-1}{z^p-1} \right)^{\frac{1}{1-p}} \leqslant \pi$$

so z is mapped into the closed upper half plane.

The case $-2 < p < 0$ can be treated similarly by studying the arguments and noting that

$$f_p(x) = \left(\frac{p(x-1)}{x^p-1} \right)^{\frac{1}{1-p}} = \left(\frac{|p| x^{|p|}(x-1)}{x^{|p|}-1} \right)^{\frac{1}{1+|p|}}.$$

Finally, we show that $f_{-2}(x)$ is matrix monotone. Clearly f_{-2} has a holomorphic continuation to the upper half plane (which is not continuous in the closed upper half plane). If $0 < \arg z < \pi$ then $\arg z^{\frac{2}{3}} = \frac{2}{3} \arg z$ and $0 < \arg(z+1) < \arg z$ so

$$0 < \arg \left(\frac{z^{\frac{2}{3}}}{(z+1)^{\frac{1}{3}}} \right) < \pi$$

thus the upper half plane is mapped into itself by f_{-2}. $\square$

The limiting case $p = 1$ is the so-called *identric mean*:

$$f_1(x) = \frac{1}{e} x^{\frac{x}{x-1}} = \exp\left(\frac{x \log x}{x-1} - 1 \right).$$

It is not so difficult to show that f_1 is matrix monotone.

The inequality

$$\sqrt{x} \leq f_p(x) \leq \frac{1+x}{2}$$

holds if $p \in [-1, 2]$. It is proved in [2] that the matrix

$$(Y_{f_p})_{ij} = \left(\frac{\lambda_i^p - \lambda_j^p}{p(\lambda_i - \lambda_j)} \right)^{\frac{1}{1-p}}$$

is positive.

Corollary 1. *The Stolarsky mean function is matrix monotone for $p \in [-1, 2]$ and the induced mapping $(\mathbb{J}_D^{f_p})^{-1}$ is completely positive.*

The *power mean* or *binomial mean*

$$m(a, b) = \left(\frac{a^p + b^p}{2} \right)^{\frac{1}{p}}$$

is induced by

$$f_p(x) = \left(\frac{x^p + 1}{2} \right)^{\frac{1}{p}}$$

can be also a matrix monotone function:

Theorem 4. *The function*

$$f_p(x) = \left(\frac{x^p + 1}{2} \right)^{\frac{1}{p}} \tag{12}$$

is matrix monotone if and only if $-1 \leqslant p \leqslant 1$.

Proof. Observe that $f_{-1}(x) = 2x/(x+1)$ and $f_1(x) = (x+1)/2$, so f_p could be matrix monotone only if $-1 \leqslant p \leqslant 1$. We show that it is indeed matrix monotone. The case $p = 0$ as limit $f_0(x) = \sqrt{x}$ is well-known. Further, note that if f_p is matrix monotone for $0 < p < 1$ then $f_{-p}(x) = 1/f_p(x^{-1})$ is also matrix monotone since $x \mapsto x^{-1}$ is matrix monotone decreasing.

So let us assume that $0 < p < 1$. Then, since $z^p + 1 \neq 0$ in the upper half plane, f_p has a holomorphic continuation to the upper half plane (by defining logz as $\log 1 = 0$). By Löwner's theorem it suffices to show that f_p maps the upper half plane into itself. If $0 < \arg z < \pi$ then $0 < \arg(z^p + 1) < \arg z^p = p \arg z$ so

$$0 < \arg\left(\frac{z^p + 1}{2}\right)^{\frac{1}{p}} = \frac{1}{p} \arg\left(\frac{z^p + 1}{2}\right) < \arg z < \pi$$

thus z is mapped into the upper half plane. $\square$

In the special case $p = \frac{1}{n}$,

$$f_{1/n}(x) = \left(\frac{x^{\frac{1}{n}} + 1}{2}\right)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} x^{\frac{k}{n}},$$

and it is well-known that x^α is matrix monotone for $0 < \alpha < 1$ thus $f_{1/n}$ is also matrix monotone.

Since the power mean is infinitely divisible [2] and $f_{-p}(x) = 1/f_p(x^{-1})$, we have:

Corollary 2. *The function of the power mean is matrix monotone for* $-1 \leqslant p \leqslant 1$. *The mapping* $(\mathbb{J}_D^{f_p})^{-1}$ *induced by the power mean is completely positive for* $p \in [0, 1]$ *and* $\mathbb{J}_D^{f_p}$ *is completely positive for* $p \in [-1, 0]$.

4. Discussion and conclusion

The complete positivity of some linear mappings $(\mathbb{J}_D^f)^{-1} : \mathbf{M}_n \to \mathbf{M}_n$ has been a question in physical applications when the mapping is determined by a standard matrix monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$. The mean induced by the function f is larger than the geometric mean. In the present paper several concrete functions are studied, for example, Heinz mean, power difference means, Stolarsky mean and interpolations between some means. The complete positivity of $(\mathbb{J}_D^f)^{-1}$ is equivalent to the positivity of a mean matrix. The analysis of the functions studied here is very concrete, general statement is not known.

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