The absolute degree and the Nielsen root number of compositions and Cartesian products of maps

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Abstract

Brouwer’s homological degree has the multiplicative property for the composition of maps. That is, if \( f : X \to Y \) and \( g : Y \to Z \) are maps between closed oriented manifolds \( X, Y, Z \) of the same dimension, then \( |\deg(g \circ f)| = |\deg(f)||\deg(g)| \). Hopf’s absolute degree is defined for maps between all \( n \)-manifolds, whether orientable or not, and is equal to the absolute value of the Brouwer degree if the manifolds are orientable. It is shown that the absolute degree does not always have the multiplicative property for compositions, but that it does have this property for orientable maps, i.e., for maps that do not map any orientation-reversing loop to a contractible one. If at least one of \( f \) and \( g \) is not an orientable map, the absolute degree of the composition \( g \circ f \) can still be calculated from the absolute degrees of \( f \) and \( g \) if additional information about these two maps and a “correction term” \( \kappa(f, g) \) that depends on the homomorphisms of the fundamental groups induced by \( f \) and \( g \) are included. Although the Nielsen root number is closely related to the absolute degree, the multiplicative property for compositions can fail to hold for it even if the manifolds are orientable, but it does hold after the insertion of the correction term \( \kappa(f, g) \). Other interpretations of this correction term are presented.

Given maps \( f_i : X_i \to Y_i \) between \( n_i \)-manifolds, for \( i = 1, 2 \), the Brouwer degree of their Cartesian product \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) has the multiplicative property \( |\deg(f_1 \times f_2)| = |\deg(f_1)||\deg(f_2)| \). The results obtained concerning the multiplicative property for the composition of maps are used to investigate the multiplicative property for the Cartesian product of maps. We include an appendix on maps of aspherical spaces: Building on previous results of Brooks and Odenthal we show that if \( f : X \to Y \) is a map of connected compact infrasolvmanifolds of the same dimension, then the Nielsen root number and absolute degree of \( f \) are equal. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

For a map $f : X \to Y$ between closed oriented manifolds of the same dimension, the degree $\deg(f)$ was introduced by Brouwer in 1911 [3]. It was obtained with the help of a simplicial approximation of $f$, essentially by counting the number of times the approximation of $f$ covers a maximal simplex of $Y$ with images of maximal simplexes of $X$ in a positive way and subtracting the number of times it covers it in a negative way. This defines an algebraic degree $\deg(f)$ which is homotopy invariant, and that can easily be computed from a homology homomorphism induced by $f$. But it also provides geometric information, as its absolute value equals the geometric degree $G(f)$ of $f$, that is, the least non-negative integer for which there exists a closed Euclidean neighborhood $B \subset \text{int} Y$ and a map $g : X \to Y$ homotopic to $f$ such that $g^{-1}(B)$ has $G(f)$ components, and each component is mapped by $g$ homeomorphically onto $B$. (See [8] and [4] for details.)

Brouwer’s $\deg(f)$ can still be defined when at least one of the manifolds is non-orientable, by using homology with $\mathbb{Z}/2$ coefficients, but it then provides much less information and no longer equals the geometric degree. To obtain a degree which provides geometric information in the non-orientable case as well, and is equal to the geometric degree for all maps between closed manifolds of the same dimension, orientable or not, Hopf introduced in 1930 an integer-valued non-negative degree which he called the absolute degree (Absolutgrad) $A(f)$. It equals the absolute value of the Brouwer degree in the orientable case and is homotopy invariant, but the computation of $A(f)$ is in general more difficult than the computation of $\deg(f)$.

One of the fundamental facts about Brouwer’s degree is that it has the “multiplicative property” [5, p. 172–3] for composition of maps, that is, if $f : X \to Y$ and $g : Y \to Z$, then $|\deg(g \circ f)| = |\deg(f)||\deg(g)|$. Hopf did not discuss the multiplicative property for the absolute degree of a composition, and we will show that, in general, it fails to be true (see Example 3.1). However, we will demonstrate that, for an important class of maps of not necessarily orientable manifolds called “orientable” maps (see Section 2 for the definition), the multiplicative property $A(g \circ f) = A(f)A(g)$ does hold (Theorem 3.5). Moreover, in some cases in which this multiplicative property fails, it is still possible to calculate $A(g \circ f)$ from information about $f$ and $g$ (see Theorem 5.4).

The absolute degree is closely related to the Nielsen root number (cf. [4]), and so it is natural to inquire about the multiplicative property for this number. We will find that the multiplicative property is usually not true for the Nielsen root number of a composition of maps between closed manifolds of the same dimension, even if the manifolds are orientable (see Example 4.1). However, it becomes true after the insertion of a “correction term”
\(\kappa(f, g)\) which can be computed from the induced homomorphisms of the fundamental groups (Theorem 4.3).

There is also a multiplicative property of the Brouwer degree that holds for the Cartesian product of maps [5, p. 173]. That is if, for \(i = 1, 2\), we have maps \(f_i : X_i \to Y_i\) of closed orientable \(n_i\)-manifolds, then the degree of their Cartesian product \(f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2\) has the property \(|\deg(f_1 \times f_2)| = |\deg(f_1)| |\deg(f_2)|\). We will show that this property extends to the absolute degree of orientable maps on not necessarily orientable manifolds (Theorem 3.6) and, with a correction term in some cases, to other maps as well (Theorems 5.5 and 5.6). Moreover, the multiplicative property for the Cartesian product holds for the Nielsen root number (Theorem 4.7).

The paper is organized as follows: In Section 2 we discuss some background material from root theory and establish some facts that will be needed in later proofs. In particular, we prove in Theorem 2.1 that neither the Nielsen root number nor the absolute degree is changed by taking the Cartesian product of a map with the identity map of a manifold. This result is the key both to obtaining results about the composition of maps of surfaces and to applying results about the composition of maps on manifolds, of all dimensions, to obtain information about the Nielsen root number and the absolute degree of a Cartesian product of maps. The first main result of the paper is given in Section 3. It is Theorem 3.5, and it shows that the absolute degree has the multiplicative property for compositions of orientable maps. We also obtain the corresponding result for the Cartesian product of orientable maps. In Section 4 we study the multiplicative property of the Nielsen root number for the compositions of all maps, whether orientable or not (Theorem 4.3). We describe the algebraically defined correction term \(\kappa(f, g)\) of the formula given in this theorem in a more geometric way, as we prove that it equals the number of root classes of \(f\) contained in a root class of \(g \circ f\) (see Theorem 4.5) and, for orientable maps, we also express it in terms of the multiplicities of the maps \(f\), \(g\) and \(g \circ f\) (see Theorem 4.6). In the setting of the Cartesian product of maps, \(\kappa(f, g) = 1\) and thus the multiplicative property for Cartesian products holds in this case. In Section 5 we consider the absolute degree for the composition of two maps that need not be orientable, and we show in Theorem 5.4 that a correction term may again have to be inserted. There is also a correction term for the absolute degree of the Cartesian product of such maps, but it takes a simpler form, as we show in Theorem 5.5. The Appendix contains an application to maps of aspherical spaces: Building on previous results of Brooks and Odenthal [2], we show that if \(f : X \to Y\) is a map of connected compact infrasolvmanifolds of the same dimension, then the Nielsen root number and absolute degree of \(f\) are equal.

An extensive discussion of the absolute degree, its computation and geometric properties, as well as some additional historical information, can be found in [4], where Hopf’s definitions and results are stated and proved in a more modern form. In the present paper, we not only study degrees of maps between closed manifolds, but, as in [4], we consider boundary-preserving maps of manifolds with boundary and also non-compact manifolds, provided that all maps are assumed to be proper.
2. Preliminaries

We begin this section by presenting material from [4]. Let \( X \) and \( Y \) be connected manifolds of the same dimension, not necessarily compact and possibly with non-empty boundaries \( \partial X \) and \( \partial Y \). Let \( f \) be a proper, boundary-preserving map between them, that is, it is a map of pairs \( f : (X, \partial X) \to (Y, \partial Y) \) such that \( f^{-1}(K) \) is compact for each compact subset \( K \) of \( Y \). All homotopies are understood to be boundary-preserving and proper, and so they are proper maps of the form \( H : (X \times I, \partial X \times I) \to (Y, \partial Y) \).

Maps between not necessarily orientable manifolds are classified into types as follows. A map \( f : X \to Y \) is Type I, also called orientation-true, if it maps the orientation-preserving loops in \( X \) to orientation-preserving loops in \( Y \) and the orientation-reversing loops in \( X \) to orientation-reversing loops in \( Y \). A map \( f \) is Type III, also called non-orientable, if there is an orientation-reversing loop in \( X \) whose image under \( f \) is a contractible loop in \( Y \). Maps that are neither of Type I nor Type III are said to be of Type II and the maps of Types I and II are called orientable maps.

Let \( c \in \text{int}Y \), then points \( x_1, x_2 \in f^{-1}(c) \) are in the same root class \( R \) of \( f \) at \( c \) if there is a path \( w : I \to X \) from \( x_1 \) to \( x_2 \) such that \( f \circ w \) is a contractible loop at \( c \). Choosing \( x_0 \in f^{-1}(c) \) let \( p_f : \tilde{Y}_f \to Y \) be the covering space corresponding to the subgroup \( f_#(\pi_1(X, x_0)) \) of \( \pi_1(Y, c) \). As this covering space was first used by Hopf [8] to study Nielsen root classes, we will call it the Hopf covering of \( f \). The index of the subgroup \( f_#(\pi_1(X, x_0)) \) in \( \pi_1(Y, c) \) is denoted by \( j \); it is equal to the cardinality of the fiber of the Hopf covering of \( f \). The map \( f \) lifts to a map \( \tilde{f} : X \to \tilde{Y}_f \) taking \( x_0 \) to the class of the constant path at \( c \). The root classes of \( f \) at \( c \) are the non-empty subsets \( \tilde{f}^{-1}(\tilde{c}) \) for all the \( \tilde{c} \in p_f^{-1}(c) \). We will call \( \tilde{f} \) the Hopf lift of \( f \).

Let \( V \) be a contractible open subset of \( \text{int}Y \) containing \( c \) and choose an orientation for \( V \). For \( R \) a root class of \( f \) at \( c \), let \( U \) be an open subset of \( f^{-1}(V) \) such that \( U \cap f^{-1}(c) = R \).

If the map \( f \) is orientable, then \( U \) is an orientable manifold and it is oriented by the Orientation Procedure (2.6) of [4] as follows. If \( X \) is an oriented manifold, then that orientation is restricted to orient \( U \). Otherwise, choose some \( x_R \in R \) and orient \( U \) locally at \( x_R \). Let \( x \in R \) be any point, then there is a path \( w \) from \( x_R \) to \( x \) such that \( f \circ w \) is a contractible loop in \( Y \). Orient \( U \) at \( x \) by extending the orientation of \( U \) at \( x_R \) along \( w \). Orienting the components of \( U \) that intersect \( R \) in this manner and the other components arbitrarily, we obtain an orientation of \( U \). If \( f \) is a non-orientable map, then \( U \) and \( V \) are oriented with respect to \( \mathbb{Z}/2 \) coefficients. Now, in all cases, the restriction \( f|U : U \to V \) is a map of oriented manifolds so the local degree \( \deg_x(f|U) \) [6, Definition 4.2, p. 267] is defined and its absolute value is called the multiplicity of the root class \( R \), written \( |m(R)| \), that is, \( |m(R)| = |\deg_x(f|U)| \). In the manifold setting of this paper, the multiplicity of a root class is independent not only of the choice of orientations but also of the choice of the root class of \( f \) at \( c \) and of \( c \in \text{int}Y \) itself. Therefore, we will simplify some of the notation used in [4]. We refer to \( |m(R)| \) as the multiplicity of \( f \) and denote it by \( |m(f)| \).

If \( |m(f)| \neq 0 \), then the (finite) number of root classes of \( f \) at \( c \) is called the Nielsen root number of \( f \) and, since it is independent of \( c \in \text{int}Y \), we just write that number as \( NR(f) \). Furthermore, the Nielsen root number \( NR(f) \) is defined to be zero if \( |m(f)| = 0 \).
Let Theorem 2.1. Cartesian product of maps from our results concerning compositions of maps. permit us to obtain information about the Nielsen root number and absolute degree of a maps formulas that are first established in higher dimensions. Moreover, this result will Section 3], but, with the aid of the following result, we will be able to extend to surface maps formulas that are first established in higher dimensions. Moreover, this result will permit us to obtain information about the Nielsen root number and absolute degree of a Cartesian product of maps from our results concerning compositions of maps.

**Theorem 2.1.** Let \( f : (X, \partial X) \to (Y, \partial Y) \) be a proper map and let \( M \) be a manifold that is not necessarily compact and may have non-empty boundary. Let \( id : M \to M \) denote the identity map, then

(a) the map \( f \times id : X \times M \to Y \times M \) is of the same type as the map \( f \),

(b) \( NR(f \times id) = NR(f) \) and \( A(f \times id) = A(f) \).

**Proof.** (a) We will say regarding loops in manifolds that two loops have the same orientability if either both are orientation-preserving or both are orientation-reversing. Thus a map is Type I if and only if, for every loop \( w \), the loops \( w \) and \( f \circ w \) have the same orientability. Choose a base point \( e \in \text{int} M \). A loop \( \Omega \in X \times M \) based at \((x_0, e)\) can be written as \( \Omega = (w, u) \) where \( w \) is a loop in \( X \) based at \( x_0 \) and \( u \) is a loop in \( M \) based at \( e \). The loop \( \Omega \) is orientation-preserving if and only if \( w \) and \( u \) have the same orientability. Therefore, \( (f \times id) \circ \Omega = (f \circ w, u) \) has the same orientability as \( \Omega = (w, u) \) if and only if \( f \circ w \) has the same orientability as \( w \). We conclude that \( f \times id \) is Type I if and only if \( f \) is Type I. Now suppose \( f \) is a Type III map, so there is an orientation-reversing loop \( w \) in \( X \) based at \( x_0 \) such that \( f \circ w \) is contractible to the constant loop at \( c \). Define a loop in \( X \times M \) based at \((x_0, e)\) by \( \Omega = (w, \bar{e}) \), where \( \bar{e} \) denotes the constant path at \( e \), then \( \Omega \) is orientation-reversing. Since \( f \circ w \) is contractible to the constant loop at \( c \) we conclude that \( (f \times id) \circ \Omega = (f \circ w, \bar{e}) \) is contractible to \((c, e)\) in \( Y \times M \) and therefore \( f \times id \) is also Type III. On the other hand, if \( f \times id \) is Type III, then there is an orientation-reversing loop \( \Omega = (w, u) \) based at \((x_0, e)\) such that \((f \times id) \circ \Omega = (f \circ w, u) \) is contractible to the constant loop at \((c, e)\). Consequently, the loop \( u \) is contractible to the constant loop at \( e \) and thus \( u \) is orientation-preserving. It follows that \( w \) is an orientation-reversing loop and since \( f \circ w \) is contractible to \( c \), we see that \( f \) is Type III. We have shown that \( f \times id \) is Type III if and only if \( f \) is Type III, and that the same holds for Type I, so it follows as well that \( f \times id \) is Type II if and only if \( f \) is Type II.

(b) For the embedding \( i_X : X \to X \times M \) defined by \( i_X(x) = (x, e) \), we have \((f \times id)^{-1}(c, e) = i_X(f^{-1}(c))\), and clearly the restriction of \( i_X \) to \( f^{-1}(c) \) is a homeomorphism. Moreover, if \( x, x' \in f^{-1}(c) \) are in the same root class of \( f \) at \( c \) because there is a path \( w \) from \( x_1 \) to \( x_2 \) such that \( f \circ w \) is a contractible loop in \( Y \), then \( i_X(x_1) \) and \( i_X(x_2) \) are in the same root class of \( f \times id \) at \((c, e)\) by means of the path \( i_X \circ w \). Now suppose \((x_1, e)\) and \((x_2, e)\) are in the same root class of \( f \times id \) at \((c, e)\) by means of a path \( w' \) in \( X \times M \) from \((x_1, e)\) to \((x_2, e)\), so \((f \times id) \circ w' \) is a contractible loop in \( Y \times M \) at \((c, e)\).
$w = p_X \circ w'$ be the path in $X$ which is the image under the projection $p_X : X \times M \to X$ of
$w'$. As $f \circ p_X = p_Y \circ (f \times id)$ for $p_Y : Y \times M \to Y$ the projection, we have for the identity
element $1 \in \pi_1(Y, c)$ that

$$1 = p_Y(1) = \left( f \times id \right) \circ w' = \left( p_Y \circ (f \times id) \circ w' \right)$$

Consequently, $x_1 = p_X(x_1, e)$ and $x_2 = p_X(x_2, e)$ are in the same root class of $f$. We
conclude that $R$ is a root class of $f$ at $c$ if and only if $i_X(R)$ is a root class of $f \times id$ at
$(c, e)$, that is, $i_X$ determines a one-to-one correspondence between the root classes of $f$
and the root classes of $f \times id$.

We will now prove that $|m(R)| = |m(i_X(R))|$. We orient a contractible neighborhood $V$
of $c$ in int $Y$. Let $J$ be a Euclidean neighborhood of $e$ in int $M$, then $V \times J$ is a contractible
neighborhood of $(c, e)$ in int $(Y \times M)$. We choose an orientation for $J$ and the product
orientation for $V \times J$. For $i_Y : Y \to Y \times M$ the embedding defined by $i_Y(y) = (y, e)$,
the restriction $i_Y V : V \to V \times J$ is a homotopy equivalence that preserves orientations.
We may assume that $f$ is transverse to $c$ and thus $R$ is finite (see [4, Section 4]), so we
write $R = \{x_1, x_2, \ldots, x_k\}$. Let $U_i$, for $\ell = 1, \ldots, k$, be disjoint Euclidean neighborhoods
of the $x_\ell$ in $f^{-1}(Y)$ that contain no other roots of $f$ at $c$. Setting $s_R = x_1$, orient
$U = \sum_{i = 1}^k U_i$ as in the Orientation Procedure (2.6) of [4] that we described earlier in
this section. The sets $U_i \times J$ are disjoint Euclidean neighborhoods of the points of the root
class $i_X(R) = \{(x_1, e), (x_2, e), \ldots, (x_k, e)\}$ of $f \times id$ at $(c, e)$. As an orientation for $J$ has
been chosen, we can orient each $U_i \times J$ with the product orientation. It is clear that, with
respect to these orientations, $\deg_c(f|U_i) = \deg_{(c,e)}(f \times id|U_i \times J)$ for all $\ell = 1, \ldots, k$.
The orientation of $U \times J$ obtained in this manner is the same as the orientation obtained
by means of the Orientation Procedure (2.6) of [4] if we extend the orientation of $U_1 \times J$
obtained from the restriction of $i_X$ to $U_1$ along paths $i_X(w_\ell)$, where $w_\ell$ is the path from $x_1$
to $x_\ell$ used to orient $U_\ell$. Consequently, we may apply the formula of Example 2.8 of [4] to
conclude that

$$|m(R)| = \left| \sum \left( \deg_c(f|U_\ell): 1 \leq \ell \leq k \right) \right|$$

Thus $|m(f)| = |m(R)| = 0$ if and only if $|m(f \times id)| = |m(i_X(R))| = 0$ and so $NR(f) = NR(f \times id) = 0$ and $\mathcal{A}(f) = \mathcal{A}(f \times id) = 0$ in such a case. Otherwise, the fact that
$|m(f)| \neq 0$ if and only if $|m(f \times id)| \neq 0$ and the one-to-one correspondence between
the root classes of $f$ and the root classes of $f \times id$ determined by $i_X$ imply both that
$NR(f \times id) = NR(f)$ and that $\mathcal{A}(f \times id) = \mathcal{A}(f)$. □

We note that the order of the maps in Theorem 2.1 does not matter. That is, for
$id \times f : M \times X \to M \times Y$, the same argument shows that $id \times f$ is the same type as $f$, $NR(id \times f) = NR(f)$ and $\mathcal{A}(id \times f) = \mathcal{A}(f)$.

The following result was proved in a slightly different setting in [8] (see [8, Satz XIId, pp. 602–603]). We include a brief proof for the reader’s convenience.
Theorem 2.2. Let $f : (X, \partial X) \to (Y, \partial Y)$ be a proper map of $n$-manifolds, $n \neq 2$, and let $A = \{y_1, \ldots, y_k\}$ be a finite subset of $\text{int} Y$. If $\text{NR}(f) = 0$, then there is a proper map $f' : (X, \partial X) \to (Y, \partial Y)$ homotopic to $f$ such that $f'(X) \cap A = \emptyset$.

Proof. We can assume that $n \geq 3$, as $n = 1$ can easily be checked directly. Let the root sets $\{S_j | j = 1, \ldots, k\} \subset \text{int} X$ be defined by $S_j = f^{-1}(y_j)$. We can assume, without loss of generality, that all sets $S_j$ consist of finitely many points. We eliminate these root sets of $f$ successively. Define $X_1 = X - \bigcup\{S_j | j = 2, \ldots, k\}$, $Y_1 = Y - \{y_2, \ldots, y_k\}$ and $f_1 = f|_{X_1} : X_1 \to Y_1$. Then $f_1$ is a map between $n$-manifolds. As $n \geq 3$, the root classes of $f_1$ at $y_1$ equal the root classes of $f$, and so must all have zero multiplicity. Thus $\text{NR}(f_1) = 0$ implies $\text{NR}(f) = 0$, and so it follows from [4, Theorem 4.3] that $f_1$ is homotopic to a map $f''_1 : X_1 \to Y_1$ that has no roots at $y_1$. From the proof of that theorem we observe that, as $n \geq 3$, it is not necessary to alter $f_1$ in a neighborhood of $\bigcup\{S_j | j = 2, \ldots, k\}$. Thus $f_1$ defines a map $f''_1 : X \to Y$ homotopic to $f$ which has no roots at $y_1$, with $f''_1(x) = f(x)$ for $x \in \{S_j | j = 2, \ldots, k\}$. Now let $X_2 = X - \bigcup\{f''_1^{-1}(y_j) | j = 1, 3, \ldots, k\}$, let $Y_2 = Y - \{y_1, y_3, \ldots, y_k\}$ and let $f_2 : X_2 \to Y_2$ be obtained by the restriction of $f''_1$. As before, we can homotope $f_2$ to a map with no roots at $y_2$, and this construction will not pick up any roots at $y_1$. Thus we can construct a map $f''_2 : X \to Y$ homotopic to $f$ which has no roots at $\{y_1, y_2\}$. Then we continue until we obtain a map with no roots at any $y_j$. All maps and homotopies in this proof can be obtained to be proper and boundary-preserving. \qed

From now on, we shall assume that not only $X$ and $Y$, but also $Z$, is a connected manifold, and that all three manifolds are of the same dimension.

Theorem 2.3. Let $f : (X, \partial X) \to (Y, \partial Y)$ and $g : (Y, \partial Y) \to (Z, \partial Z)$ be proper maps between $n$-manifolds. If $\text{NR}(g \circ f) > 0$, then both $\text{NR}(f) > 0$ and $\text{NR}(g) > 0$.

Proof. We will prove that if either $\text{NR}(f)$ or $\text{NR}(g)$ is zero, so also is $\text{NR}(g \circ f)$. The case $n = 1$ can be checked directly. We next assume that $n \geq 3$. Choose $c \in \text{int} Z$. Suppose first that $\text{NR}(g) = 0$. Then, by [4, Theorem 4.3], there exists a map $g' : (Y, \partial Y) \to (Z, \partial Z)$ homotopic to $g$ which has no roots at $c$, and hence $g' \circ f$ has no roots at $c$ so $\text{NR}(g \circ f) = \text{NR}(g' \circ f) = 0$. Now assume that $\text{NR}(f) = 0$. Then we can assume, by transversality, that $g^{-1}(c) = \{y_1, \ldots, y_k\} = A$ is a finite set of points in $Y$ and thus there exists, by Theorem 2.2, a map $f' : (X, \partial X) \to (Y, \partial Y)$ homotopic to $f$ with $f'^{-1}(A) = \emptyset$. Since $g \circ f'$ is a map homotopic to $g \circ f$ that has no roots at $c$, we see that $\text{NR}(g \circ f) = 0$. It remains to establish the result in the case $n = 2$. Suppose $\text{NR}(f) = 0$, then for $id : S^1 \to S^1$ the identity map we have $\text{NR}(f \times id) = 0$ by Theorem 2.1(b) and therefore $\text{NR}((g \circ f) \circ (f \times id)) = 0$ by the first part of the proof so, applying 2.1(b) again, we have $\text{NR}(g \circ f) = 0$. In the same way, $\text{NR}(g) = 0$ also implies that $\text{NR}(g \circ f) = 0$. \qed

3. Multiplicative properties of the absolute degree for orientable maps

As we mentioned in the introduction, the multiplicative property $A(g \circ f) = A(g)A(f)$ does not hold in general. The following example illustrates this.
Example 3.1. Let $S$ denote the 2-dimensional sphere, $P$ the projective plane and $f : S \to P$ the orientable covering. The covering space is itself the Hopf covering of $f$ and the Hopf lift of $f$ is the identity map. Since $S$ is an orientable manifold and $f$ is a map of Type I, Theorem 3.12 of [4] implies that $A(f) = j \cdot \deg(id) = (2)(1) = 2$. Representing $P$ as the disc with antipodal points of the boundary identified, define $g : P \to S$ by collapsing the boundary of the disc to a point $s_0$. Let $c$ be any point in $S$ other than $s_0$. Then $g^{-1}(c)$ is a single point in the interior of the disc, which is a Euclidean neighborhood of it that is mapped homeomorphically onto the Euclidean neighborhood $S - s_0$ of $c$. Thus $|m(g)| = 1$ and we see that $A(g) = 1$. Now $g \circ f : S \to S$ is a map of oriented manifolds so $A(g \circ f) = |\deg(g \circ f)|$. Since the homomorphism of 2-dimensional integer homology $(g \circ f)_* : H_2(S; \mathbb{Z}) \to H_2(S; \mathbb{Z})$ induced by $g \circ f$ factors through $H_2(P; \mathbb{Z}) = 0$, then $\deg(g \circ f) = 0$ and hence $A(g \circ f) = 0$ even though $A(f)$ and $A(g)$ are nonzero.

Example 3.1 also shows that the converse of Theorem 2.3 is false since, in the example, $NR(f)$ and $NR(g)$ are non-zero but $NR(g \circ f) = 0$.

The multiplicative property for compositions can fail even when $A(g \circ f)$ is non-zero, as the following slightly more complicated example demonstrates.

Example 3.2. Let $f = f_1 \times f_2 : P \times S \to P \times S$, where $f_1 : P \to P$ is the identity map of the projective plane and $f_2 : S \to S$ is a degree $d$ map of the 2-sphere, with $d > 1$ and odd. Then, according to Example 3.15 of [4], we have $NR(f) = 1$ and $A(f) = d$. Now let $g_1 : P \to S$ be the map $g$ of the previous example. Define $g = g_1 \times g_2 : P \times S \to S \times S$ by letting $g_2$ be the identity, then $g$ is clearly of Type III. Choosing $c = (c_1, c_2) \in S \times S$ with $c_1 \neq s_0$ (cf. the previous example), then $g^{-1}(c)$ is a single point $y \in P \times S$ and there is a Euclidean neighborhood of $y$ that is mapped homeomorphically onto a Euclidean neighborhood of $c$. It follows from the definition that $A(g) = 1$. We may choose $c \in S \times S$ so that $(g \circ f)^{-1}(c)$ consists of $d$ points and $g \circ f$ is transverse to $c$. We apply Example 2.8 in [4] with coefficients in $\mathbb{Z}/2$ and obtain from the fact that $d$ is odd that $|m(g \circ f)| = 1$. Since $S \times S$ is simply-connected, there is only one root class for $g \circ f$ and we conclude that $A(g \circ f) = 1$ whereas $A(f)A(g) = d > 1$.

However, the multiplicative property for compositions is valid for all compositions of orientable (i.e., Type I or Type II) maps. To prove it, we require two preliminary results.

Lemma 3.3. Let $f : (X, \partial X) \to (Y, \partial Y)$ be a proper Type I map of $n$-manifolds where $X$ is orientable, $Y$ is non-orientable and $n > 2$. Let $q^o : Y^o \to Y$ be the orientable covering of $Y$. Then there is a lift $f^o : X \to Y^o$ of $f$ through $q^o$ and $A(f) = |2 \deg(f^o)| = 2A(f^o)$.

Proof. Since $f : X \to Y$ is Type I, and $X$ is orientable, the image under $f$ of every loop in $X$ is an orientable loop in $Y$. It follows that $f$ may be lifted through $q^o$. This gives us the following diagram, in which all maps are proper, and boundary-preserving if the manifolds have non-empty boundaries. The composition $q^o \circ \hat{q} : \overline{Y} \to Y$ is the Hopf covering for
Let \( c \in \text{int} \, Y \). Since \( q^\circ \) is a double covering, \( (q^\circ)^{-1}(c) = \{c_i^0, c_i^1\} \) for precisely two points \( c_i^0, c_i^1 \in Y^\circ \). Thus \( (q^\circ \circ \tilde{\varphi})^{-1}(c) = \tilde{\varphi}^{-1}(c_i^0) \cup \tilde{\varphi}^{-1}(c_i^1) \), so

\[
\{\tilde{\varphi}^{-1}(\tilde{c}) \mid \tilde{c} \in (q^\circ \circ \tilde{\varphi})^{-1}(c)\} = \{\tilde{\varphi}^{-1}(\tilde{c}) \mid \tilde{c} \in \tilde{\varphi}^{-1}(c_i^0)\} \cup \{\tilde{\varphi}^{-1}(\tilde{c}) \mid \tilde{c} \in \tilde{\varphi}^{-1}(c_i^1)\}.
\]

That is, the set of root classes of \( f \) at \( c \) is the disjoint union of the set of root classes of \( f^\circ \) at \( c_i^0 \) and the set of root classes of \( f^\circ \) at \( c_i^1 \). Hence, in order to prove the lemma, it suffices to show that each root class has the same multiplicity whether viewed as a root class of \( f^\circ \) or viewed as a root class of \( f \). Since \( n > 2 \), we may assume that every root class consists of a single point. Let \( \{x\} \) be a root class of \( f \). Let \( V \) be an elementary Euclidean neighborhood of \( c \) and let \( U \) be an Euclidean neighborhood of \( x \) small enough so that \( f(U) \subset V \) and \( f^{-1}(c) \cap U = \{x\} \). Since \( V \) is elementary, we may write \( q^\circ \circ \tilde{\varphi}^{-1}(V) = V_i^0 \cup V_i^1 \), where \( V_i^0 \) is mapped homeomorphically onto \( V \) by \( q^\circ \) and \( c_i^0 \in V_i^0 \) for \( i = 0, 1 \). Either \( f^\circ(x) = c_i^0 \) or \( f^\circ(x) = c_i^1 \) and we suppose, without loss of generality, that \( f^\circ(x) = c_i^0 \). We then can write \( f|U \) in the form \( f|U = q^\circ|V_i^0 \circ f^\circ|U \) where \( f|U, f^\circ|U, \) and \( q^\circ|V_i^0 \) are restrictions of \( f, f^\circ \) and \( q^\circ \) respectively. Then the multiplicity of \( \{x\} \) as a root class of \( f^\circ \) is \( |\deg_{c_i^0}(f^\circ|U)| \), and its multiplicity as a root class of \( f \) is \( |\deg_c(f|U)| \). But since \( V_i^0 \) is connected, \([6, \text{Corollary 4.6, p. 268}]\) implies that

\[
|\deg_c(f|U)| = |\deg_c((q^\circ|V_i^0) \circ (f^\circ|U))| = |\deg_c(q^\circ|V_i^0) \circ (f^\circ|U)| = |\deg_{c_i^0}(f^\circ|U)|.
\]

Since \( q^\circ|V_i^0 \) is a homeomorphism, \(|\deg_c(q^\circ|V_i^0)| = 1 \). Therefore, \(|\deg_c(f|U)| = |\deg_{c_i^0}(f^\circ|U)| \) and we conclude that the two multiplicities are the same. \( \square \)

**Lemma 3.4.** Suppose that \( f : (X, \partial X) \to (Y, \partial Y) \) is a proper orientable map of two non-orientable \( n \)-manifolds and that \( n > 2 \). Let \( p^\circ : X^\circ \to X \) be the orientable covering of \( X \), then \( \mathcal{A}(f \circ p^\circ) = 2 \mathcal{A}(f) \).

**Proof.** Let \( c \in \text{int} \, Y \). It suffices to show that each root class of \( f \) at \( c \) corresponds to exactly two roots of \( f \circ p^\circ \) at \( c \), and that the root classes of \( f \) and of \( f \circ p^\circ \) have the same multiplicity. Because \( n > 2 \), we may assume that each root class of \( f \) consists of a single point. Let \( \{x\} \) be a root class of \( f \). Then since \( p^\circ \) is a double covering, there are exactly two points, \( x_i^0 \) and \( x_i^1 \) in the fiber \( p^\circ^{-1}(x) \), each of which is a root of \( f \circ p^\circ \). Let \( w \) be a path in \( X^\circ \) from \( x_i^0 \) to \( x_i^1 \). Then \( p^\circ \circ w \) is an orientation-reversing loop in \( X \) so, since \( f \) is orientable, \(|f \circ p^\circ \circ w| \neq [c] \). Since this holds for every path in \( X^\circ \) from \( x_i^0 \) to \( x_i^1 \), we conclude that \( \{x_i^0\} \) and \( \{x_i^1\} \) are distinct root classes of \( f \circ p^\circ \). Further, it is easy
to see that two points \(x^o\) and \(x^{o'}\) which lie over two different roots of \(f\) cannot be in the same root class of \(f \circ p^o\). For suppose they did, then there would be a path \(w\) joining them such that \([f \circ p^o \circ w] = [c]\), and then \(p^o \circ w\) would be a path joining \(x\) to \(x'\) such that \([f \circ p^o \circ w] = [c]\), contradicting our assumption that each root class of \(f\) consists of a single point.

It remains to show that the local degrees of \(f\) restricted to neighborhoods of \([x]\), \([x^o]\), and \([x^{o'}]\) are all the same, up to sign. Choose a Euclidean neighborhood \(V\) of \(c\) in \(Y\). Let \(U\) be an elementary Euclidean neighborhood of \(x\) in \(X\) such that \(f(U) \subset V\) and \(U \cap f^{-1}(c) = \{x\}\). Then \((p^o)^{-1}(U)\) is the disjoint union of two Euclidean sets one of which, call it \(U^o\), is a neighborhood of \(x^o\), and the other, \(U^{o'}\), is a neighborhood of \(x^{o'}\). Each of these is mapped homeomorphically onto \(U\) by \(p^o\). The multiplicity of \([x]\), up to sign, is \(\text{deg}_c(f|U)\). The multiplicity of \([x^o]\), up to sign, is \(\text{deg}_c(f|U) \circ (p^o|U^o]\). But since \(p^o|U^o\) is a homeomorphism, \(\text{deg}_c(f|U) \circ (p^o|U^o]\) = \(\text{deg}_c(f|U)\) (cf. the proof of Lemma 3.3). Thus the restriction of \(f\) to neighborhoods of \([x]\) and \([x^o]\) and similarly of \([x]\) and \([x^{o'}]\) have, up to sign, the same local degrees. Therefore, we have demonstrated that \(A(f \circ p^o) = 2A(f)\).

**Theorem 3.5.** Suppose \(f : (X, \partial X) \to (Y, \partial Y)\) and \(g : (Y, \partial Y) \to (Z, \partial Z)\) are proper, orientable maps of \(n\)-manifolds. Then the absolute degree has the multiplicative property for compositions

\[
A(g \circ f) = A(f)A(g).
\]

**Proof.** We first assume that both \(f\) and \(g\) are Type I maps. Since it is impossible to have an orientable map with a non-orientable manifold as domain and an orientable manifold as range, there are only four cases to consider:

1. \(X, Y, \text{and } Z\) are all orientable manifolds,
2. \(X\) and \(Y\) are orientable but \(Z\) is non-orientable,
3. \(X\) is orientable but \(Y\) and \(Z\) are non-orientable, and
4. \(X, Y, \text{and } Z\) are all non-orientable manifolds.

In case (1), that is for maps of orientable manifolds, the absolute degree is the absolute value of the Brouwer degree and the result is immediate from [6, Corollary 4.6, p. 268]. Since all 1-manifolds are orientable, we can now exclude that dimension. In order to apply Lemmas 3.3 and 3.4, we assume for now that the dimension of the manifolds is greater than 2.

For case (2) in which \(X\) and \(Y\) are orientable manifolds but \(Z\) is non-orientable, let \(r^o : Z^o \to Z\) be the orientable covering and let \(g^o : Y \to Z^o\) be a lift of \(g\) through \(r^o\), so \(g^o \circ f\) is a lift of \(g \circ f\) and therefore \(A(g \circ f) = 2A(g^o \circ f)\) by Lemma 3.3. Since \(g^o\) and \(f\) are maps of orientable manifolds, the previous case implies that \(2A(g^o \circ f) = 2A(f)A(g^o)\) and applying Lemma 3.3, this time to \(g^o\), completes this case.

For case (3) in which \(X\) is an orientable manifold but \(Y\) and \(Z\) are non-orientable manifolds, let \(q^o : Y^o \to Y\) be the orientable covering of \(Y\), and let \(f^o : X \to Y^o\) be a lift of \(f\) through \(q^o\). Since \(f = q^o \circ f^o\), we can write \(g \circ f\) as \((g \circ q^o) \circ f^o\) and, since \(f^o : X \to Y^o\) is a map of orientable manifolds, case (2) applies and we conclude that
\( A(g \circ f) = A(f^o)A(g) \). Lemma 3.4 implies that \( A(g \circ g^o) = 2A(g) \), so applying Lemma 3.3 to \( f^o \) gives us \( A(g \circ f) = 2A(f^o)A(g) = A(f)A(g) \).

For case (4) in which all manifolds are non-orientable, let \( p^o : X^o \to X \) be the orientable covering of \( X \). Lemma 3.4 gives us \( 2A(g \circ f) = A((g \circ f) \circ p^o) = A(g \circ (f \circ p^o)) \). Case (3) applies to the last composition, so we conclude that \( A(g \circ (f \circ p^o)) = A(f \circ p^o)A(g) \). Applying Lemma 3.4 to \( f \circ p^o \) we have \( A(f \circ p^o) = 2A(f) \) so \( 2A(g \circ f) = 2A(f)A(g) \) and this completes the verification of the multiplicative property for compositions for Type I maps on manifolds of dimension other than 2.

Now suppose \( f : (X, \partial X) \to (Y, \partial Y) \) and \( g : (Y, \partial Y) \to (Z, \partial Z) \) are proper Type I maps of surfaces. Letting \( \text{id} \) denote the identity map of the circle, we consider the maps of 3-manifolds \( f \times \text{id} : (X \times S^1, \partial X \times S^1) \to (Y \times S^1, \partial Y \times S^1) \) and \( g \times \text{id} : (Y \times S^1, \partial Y \times S^1) \to (Z \times S^1, \partial Z \times S^1) \). These maps are also Type I by Theorem 2.1(a) so, by what we have just proved, \( A((g \circ f) \times \text{id}) = A(f \times \text{id})A(g \times \text{id}) \) and therefore \( A(g \circ f) = A(f)A(g) \) by Theorem 2.1(b). Thus we see that the multiplicative property for compositions holds for Type I maps of manifolds of all dimensions.

If either of \( f : (X, \partial X) \to (Y, \partial Y) \) or \( g : (Y, \partial Y) \to (Z, \partial Z) \) (or both) is a Type II map, then it follows from [4], Theorem 3.11 that the Nielsen root number of that maps is zero. Therefore \( NR(g \circ f) = 0 \) by Theorem 2.3. Consequently \( A(g \circ f) \) and at least one of \( A(f) \) and \( A(g) \) are zero, so the multiplicative property for compositions holds when \( f \) or \( g \) is Type II because both sides of the equation equal zero. \( \square \)

A Cartesian product of two maps can be written as a composition of Cartesian products such that, in each factor of the composition, one of the two maps is an identity map. We shall use this, as well as Theorem 2.1, in the proof of the next theorem, and again in the proof of Theorem 4.7 below, in order to obtain results concerning the multiplicative property for Cartesian products from our results concerning the multiplicative property for compositions.

**Theorem 3.6.** For \( i = 1, 2 \), let \( f_i : (X_i, \partial X_i) \to (Y_i, \partial Y_i) \) be proper, orientable maps of \( n_i \)-manifolds and let \( f_1 \times f_2 : (X_1 \times X_2, \partial (X_1 \times X_2)) \to (Y_1 \times Y_2, \partial (Y_1 \times Y_2)) \) be the Cartesian product map. Then the absolute degree has the multiplicative property for Cartesian products

\[ A(f_1 \times f_2) = A(f_1)A(f_2). \]

**Proof.** Set \( f = f_1 \times id_2 \) where \( id_2 \) is the identity map of \( X_2 \) and \( g = id_1 \times f_2 \) where \( id_1 \) is the identity map of \( Y_1 \), then \( f_1 \times f_2 = g \circ f \). The maps \( f \) and \( g \) are proper and, by Theorem 2.1(a), they are orientable since \( f_1 \) and \( f_2 \) are. Therefore, by Theorem 3.5,

\[ A(f_1 \times f_2) = A(f)A(g) = A(f_1 \times id_2)A(id_1 \times f_2) \]

and the result follows by Theorem 2.1(b). \( \square \)
4. Multiplicative properties of the Nielsen root number

As the definitions indicate, the Nielsen root number \( NR(f) \) of a map is closely related to its absolute degree \( \mathcal{A}(f) \). This relationship is explored in some detail in [4]. Thus, since we were able to establish the multiplicative property for compositions and for Cartesian products in the previous section for the absolute degree, we are led to investigate the same issue with regard to the Nielsen root number. In Example 3.1, \( NR(g \circ f) = 0 \) whereas \( NR(f) \) and \( NR(g) \) are non-zero, so we cannot expect a multiplicative property for compositions to hold in general, but even for maps of orientable manifolds, the expected property is not valid, as the following example demonstrates.

Example 4.1. Let \( T = S^1 \times S^1 \) be the torus and \( S \) the 2-sphere. Let \( f = f_1 \times f_2: T \to T \) where \( f_1 \) is the identity map of the circle and \( f_2 \) is the complex squaring map, then \( NR(f) = 2 \) by [4], Theorem 3.13. Let \( g: T \to S \) be the map of degree 1 obtained by viewing \( T \) as the square with identifications on the boundary and \( S \) as the square with the entire boundary identified to a point and sending the boundary in the representation of \( T \) (i.e., a figure-eight in the torus) to that point. Then \( NR(g) = NR(g \circ f) = 1 \) since \( S \) is simply-connected and the maps have non-zero degree. Thus we have \( NR(g \circ f) = 1 \neq (1)(2) = NR(f)NR(g) \).

Although Example 4.1 demonstrates that the equation \( NR(g \circ f) = NR(f)NR(g) \) fails even in very favorable circumstances, we will next demonstrate that a modified version of that formula is true whenever \( NR(g \circ f) > 0 \). Thus we have a modified multiplicative property for compositions for the Nielsen root number even in circumstances in which the multiplicative property for compositions for the absolute degree fails to hold, for instance in the setting of Example 3.2, where \( \mathcal{A}(f) = 1 \) clearly implies that \( NR(f) = 1 \), and hence \( NR(f) = NR(g) = NR(g \circ f) = 1 \). From Example 3.1, where \( NR(f) \) and \( NR(g) \) are non-zero but \( NR(g \circ f) = 0 \), we know that such a result for the Nielsen root number will be the best possible.

The modified multiplicative property for compositions for the Nielsen root number depends on an algebraic lemma. For \( G \) a group and \( A \) a subgroup, the notation \([G : A]\) stands for the index of the subgroup \( A \) in the group \( G \).

Lemma 4.2. Suppose \( \alpha : G \to H \) and \( \beta : H \to K \) are group homomorphisms, then

\[
[ker \beta : ker \beta \cap im \alpha] [K : im(\beta \circ \alpha)] = [H : im \alpha][K : im \beta].
\]

Proof. If \( A \supseteq B \supseteq C \) are groups, then \([A : C] = [A : B][B : C]\) (see [9, middle of p. 45]) and therefore

\[
[K : im(\beta \circ \alpha)] = [K : im \beta] [im \beta : im(\beta \circ \alpha)].
\]

By the first isomorphism theorem, \( \beta \) induces an isomorphism \( H/ker \beta \to im \beta \). The isomorphism takes \((ker \beta)(im \alpha)/ker \beta \) onto \( im(\beta \circ \alpha) \). Therefore

\[
[im \beta : im(\beta \circ \alpha)] = \left[ H/ker \beta : (ker \beta)(im \alpha)/ker \beta \right].
\]
Next, suppose $A \supset B \supset C$ are groups and that $C$ is normal in $A$, then $[A : B] = [A/C : B/C]$. (This fact is part of the so-called “Correspondence Theorem”; see [12, Theorem 2.28, p. 38] or [9, pp. 35–36].) Consequently we have,

$$H/\ker \beta : (\ker \beta)(\im \alpha)/\ker \beta = [H : (\ker \beta)(\im \alpha)].$$

We apply this equality to (2) and find that $[\im \beta : \im (\beta \circ \alpha)] = [H : (\ker \beta)(\im \alpha)]$. Using this on the right side of (1), we obtain

$$[K : \im (\beta \circ \alpha)] = [K : \im \beta] [H : (\ker \beta)(\im \alpha)].$$

(3)

Multiplying both sides of (3) by $(\ker \beta)(\im \alpha) : \im \alpha$, we have

$$[(\ker \beta)(\im \alpha) : \im \alpha] [K : \im (\beta \circ \alpha)] = [K : \im \beta] [H : (\ker \beta)(\im \alpha)] [(\ker \beta)(\im \alpha) : \im \alpha].$$

(4)

Another application of the formula $[A : C] = [A : B][B : C]$ gives us

$$[H : (\ker \beta)(\im \alpha)] [(\ker \beta)(\im \alpha) : \im \alpha] = [H : \im \alpha].$$

Hence, using this equation in (4), we obtain

$$[(\ker \beta)(\im \alpha) : \im \alpha] [K : \im (\beta \circ \alpha)] = [K : \im \beta] [H : \im \alpha].$$

(5)

If $B$ and $C$ and $BC$ are subgroups of some group $A$, then $[B : B \cap C] = [BC : C]$ (see [9, Exercise 2.8]). Thus we have

$$[(\ker \beta)(\im \alpha) : \im \alpha] = [\ker \beta : \ker \beta \cap \im \alpha].$$

and the lemma follows by using this equation on the left side of (5). □

We have proper maps $f : (X, \partial X) \to (Y, \partial Y)$ and $g : (Y, \partial Y) \to (Z, \partial Z)$. We choose $c \in \im Z$, then $y_0 \in g^{-1}(c)$, and finally $x_0 \in f^{-1}(y_0)$, so we have fundamental group homomorphisms $f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_\# : \pi_1(Y, y_0) \to \pi_1(Z, c)$. To state the modified multiplicative property for compositions of the Nielsen root number, we introduce the correction term

$$\kappa(f, g) := [\ker g_\# : \ker g_\# \cap \im f_\#].$$

**Theorem 4.3.** Suppose $f : (X, \partial X) \to (Y, \partial Y)$ and $g : (Y, \partial Y) \to (Z, \partial Z)$ are proper maps such that $\NR(g \circ f) > 0$, then

$$\kappa(f, g) \NR(g \circ f) = \NR(f) \NR(g).$$

**Proof.** If a map has a non-zero Nielsen root number, then there is a one-to-one correspondence between the root classes of the map and the points in a fiber of its Hopf covering ([8, Satz III, p. 576], see also [1, Lemma 2]). Thus the hypothesis and Theorem 2.3 imply that $\NR(f) = [\pi_1(Y, y_0) : \im f_\#]$, $\NR(g) = [\pi_1(Z, c) : \im g_\#]$ and $\NR(g \circ f) = [\pi_1(Z, c) : \im (g_\# \circ f_\#)]$. Applying Lemma 4.2, we obtain the equation. □

Since Theorem 4.3 shows us that the Nielsen root number behaves well with respect to compositions of maps, we might expect that the classical Nielsen (fixed point) number of a
composition would also have attractive properties, but the following example demonstrates
that, for selfmaps \( f \) and \( g \) of a manifold \( X \), there is no relationship among the Nielsen
numbers \( N(f) \), \( N(g) \) and \( N(g \circ f) \).

**Example 4.4.** Let \( f, g \) and hence also \( g \circ f \) be selfmaps of the circle, of degrees \( r, s \),
and therefore \( rs \), respectively, and assume all the degrees are greater than one. Then
\( N(f) = r - 1, N(g) = s - 1 \) and \( N(g \circ f) = rs - 1 \) by [10, p. 23], Proposition 8.4, and thus
\[
N(g \circ f) - N(f)N(g) = rs - 1 - (r - 1)(s - 1) = r + s - 2
\]
which can be made arbitrarily large.

Although the algebraic form of Lemma 4.2 makes it natural to define the correction
term \( \kappa(f, g) \) of the modified multiplicative property for compositions of the Nielsen root
number in terms of induced fundamental group homomorphisms, it is also possible to
describe \( \kappa(f, g) \) in terms of root classes, a basic concept for both the Nielsen root number
and the absolute degree, as follows.

**Theorem 4.5.** Let \( f : (X, \partial X) \to (Y, \partial Y) \) and \( g : (Y, \partial Y) \to (Z, \partial Z) \) be proper maps such
that \( NR(g \circ f) > 0 \). Let \( R \subset X \) be a root class of \( g \circ f \) at \( c \in \text{int} Z \) and choose \( y \in f(R) \).
Theorem \( 4.5 \) then \( \kappa(f, g) \) is the number of root classes of \( f \) at \( y \) that are contained in \( R \).

**Proof.** Recall that we have chosen \( y_0 \in g^{-1}(c) \) and \( x_0 \in f^{-1}(y_0) \). Let \( p_f : \tilde{Y}_f \to Y \) be
the Hopf covering for \( f \). The points of \( \tilde{Y}_f \) are equivalence classes (\( \bar{w} \)) of paths in \( Y \)
that start at \( y_0 \). Similarly, let \( p : \tilde{Z}_{g \circ f} \to Z \) be the Hopf covering for \( g \circ f \), then the points of
\( \tilde{Z}_{g \circ f} \) are equivalence classes of paths in \( Z \) that start at \( c \). Let \( \tilde{g} : \tilde{Y}_f \to \tilde{Z}_{g \circ f} \) be a lift of
the map \( g \circ p_f : \tilde{Y}_f \to \tilde{Z}_{g \circ f} \), i.e., let \( \tilde{g} \) be a map with \( p \circ \tilde{g} = g \circ p_f \). Hence \( \tilde{g} \)
is defined by \( \tilde{g}(\bar{w}) \equiv (g \circ \bar{w}) \) (see [11, Theorem 5.1, p. 156]). Let \( \tilde{f} : X \to \tilde{Y}_f \) be the Hopf
lift of \( f \), then \( \tilde{g} \circ f \) is the Hopf lift of \( g \circ f \). The hypothesis that \( NR(g \circ f) > 0 \) implies,
by Theorem 2.3, that there is a one-to-one correspondence between the root classes of \( f \)
at \( y \) and the points of \( p_f^{-1}(y) \), and between the root classes of \( g \circ f \) at \( c \) and the points of
\( p^{-1}(c) \). Let \( \tilde{c} \) be the point of \( p^{-1}(c) \) corresponding to \( R \), then a root class \( F \) of \( f \) at \( y \) is a
subset of \( R \) if and only if for \( \tilde{y} = \tilde{f}(F) \) we have \( \tilde{g}(\tilde{y}) = \tilde{c} \). Thus the number of root classes
of \( f \) at \( y \) that are contained in \( R \) is equal to \( \#(p_f^{-1}(y) \cap \tilde{g}^{-1}(\tilde{c})) \), where \( \#(A) \) denotes the
cardinality of a set \( A \). It remains to prove that \( \#(p_f^{-1}(y) \cap \tilde{g}^{-1}(\tilde{c})) \equiv \kappa(f, g) = [\ker g_\# : \ker g_\# \cap \im f_\#] \).

Choose a path \( v \) in \( Y \) from \( y \) to \( y_0 \), and define \( \Delta : p_f^{-1}(y) \to p_f^{-1}(y_0) \) by \( \Delta(\bar{w}) \equiv (w \cdot v) \) for \( w \) a path in \( Y \) from \( y_0 \) to \( y_1 \). Then \( \Delta \) is a one-to-one function such that
\( \tilde{g}(\Delta(\bar{w})) = \tilde{c} \) if and only if \( \tilde{g}(\Delta(\bar{w})) = \tilde{c}' \) where \( \tilde{c}' = \tilde{c} \cdot (g \circ v) \). We conclude that
\[
\#(p_f^{-1}(y) \cap \tilde{g}^{-1}(\tilde{c})) = \#(p_f^{-1}(y_0) \cap \tilde{g}^{-1}(\tilde{c}')).
\]
The restriction of \( \tilde{g} \) to \( p_f^{-1}(y_0) \) is equivalent to the mapping of coset spaces
\[
g_\# : \pi_1(Y, y_0)/f_\#(X, x_0) \to \pi_1(Z, c)/(g \circ f)_\#(X, x_0)
\]
induced by the homomorphism $g_\# : \pi_1(Y, y_0) \to \pi_1(Z, c)$. Thus we can identify $P_f^{-1}(y_0) \cap \tilde{g}^{-1}(\tilde{c}')$ with $g_\#^{-1}(\tilde{c}')/(g_\#^{-1}(\tilde{c}') \cap f_\# \pi_1(X, x_0))$. Let $\phi : G \to H$ be a group homomorphism, let $K \subset G$ be a subgroup and let $h \in \phi(G)$, then the mapping of coset spaces $\phi : G/K \to H/\phi(K)$ induced by $\phi$ has the property

$$\#(\phi^{-1}(h)/(\phi^{-1}(h) \cap K)) = \#(\ker \phi/(\ker \phi \cap K)).$$

Taking $\phi = g_\# : \pi_1(Y, y_0) \to \pi_1(Z, c), h = \tilde{c}'$ and $K = f_\#(\pi_1(X, x_0))$, we conclude that

$$\#(P_f^{-1}(y_0) \cap \tilde{g}^{-1}(\tilde{c}')) = \#(\ker g_\#/(\ker g_\# \cap \text{im } f_\#)) = \kappa(f, g)$$

which completes the proof. $\square$

For orientable maps, we may also characterise the correction term $\kappa(f, g)$ in terms of the multiplicities of the maps $f$, $g$ and $g \circ f$, for we have

**Theorem 4.6.** Let $f : (X, \partial X) \to (Y, \partial Y)$ and $g : (Y, \partial Y) \to (Z, \partial Z)$ be proper orientable maps between $n$-manifolds such that $NR(g \circ f) > 0$, then the correction term $\kappa(f, g)$ and the multiplicities of $f$, $g$, and $g \circ f$ are related as follows:

$$\kappa(f, g) = \frac{|m(g \circ f)|}{|m(f)| \cdot |m(g)|}.$$

**Proof.** Theorem 3.5 tells us that $A(g \circ f) = A(f)A(g)$ in this case so by definition we have

$$|m(g \circ f)|NR(g \circ f) = |m(f)|NR(f)|m(g)|NR(g).$$

By Theorem 4.3,

$$\kappa(f, g)NR(g \circ f) = NR(f)NR(g),$$

and when we substitute into the previous equation we obtain

$$|m(g \circ f)|NR(g \circ f) = |m(f)| |m(g)| \kappa(f, g)NR(g \circ f).$$

Dividing by $NR(g \circ f) > 0$, and noting that Theorem 2.3 implies $|m(f)| > 0$ and $|m(g)| > 0$ we obtain the conclusion of the theorem. $\square$

We now use our results concerning the multiplicative property of the Nielsen root number for compositions to obtain information about the multiplicative property of this number for Cartesian products.

**Theorem 4.7.** For $i = 1, 2$, let $f_i : (X_i, \partial X_i) \to (Y_i, \partial Y_i)$ be proper maps of $n_i$-manifolds and let $f_1 \times f_2 : (X_1 \times X_2, \partial (X_1 \times X_2)) \to (Y_1 \times Y_2, \partial (Y_1 \times Y_2))$ be the Cartesian product map. If $NR(f_1 \times f_2) > 0$ then the Nielsen root number has the multiplicative property for Cartesian products

$$NR(f_1 \times f_2) = NR(f_1)NR(f_2).$$
Proof. Just as in the proof of Theorem 3.6, we set \( f = f_1 \times f_2 \) where \( id_2 \) is the identity map of \( X_2 \) and \( g = id_1 \times f_2 \) where \( id_1 \) is the identity map of \( Y_1 \), so then \( f_1 \times f_2 = g \circ f \). Theorems 4.3 and 2.1(b) then imply that
\[
\kappa(f, g) NR(f_1 \times f_2) = NR(f_1) NR(f_2)
\]
where \( \kappa(f, g) = [\ker g_\# : \ker g_\# \cap \text{im} f_\#] \). In this case,
\[
\text{im} f_\# = \text{im}(f_1 \times id_2)_\# = f_1#(\pi_1(X_1)) \oplus \pi_1(X_2) \subseteq \pi_1(Y_1) \oplus \pi_1(X_2)
\]
whereas
\[
\ker g_\# = \ker(id_1 \times f_2)_\# = 1 \oplus \ker f_2# \subset f_1#(\pi_1(X_1)) \oplus \pi_1(X_2) = \text{im} f_\#
\]
so \( \ker g_\# = \ker g_\# \cap \text{im} f_\# \) which implies that \( \kappa(f, g) = 1 \) and completes the argument. \( \square \)

Example 4.8. The statement of Theorem 4.7 includes the hypothesis that \( NR(f_1 \times f_2) > 0 \). We will demonstrate that this hypothesis is required for the theorem to hold in general by exhibiting, for \( i = 1, 2 \), maps \( f_i : X_1 \to Y_1 \) such that both \( NR(f_i) > 0 \) but \( NR(f_1 \times f_2) = 0 \).

We define the map \( f_1 \) to be the map of Example 2.4(b) of [4]. That is, let \( T^2 \) denote the torus and \( P^2 \) the projective plane. Let \( D \) be an open disc in \( T^2 \) and define \( f_1 : X_1 = T^2 \# P^2 \to T^2 = Y_1 \) to be the identity map on \( T^2 - D \) and extend the identity map on the boundary of \( D \) in \( P^2 - D \) as a map from \( P^2 - D \to D \subset T^2 \). Taking \( c_1 \in T^2 - D \), we see that \( f_1^{-1}(c_1) = c_1 \) and that \( f_1 \) is the identity map in a Euclidean neighborhood \( U_1 \) of \( c_1 \). The map \( f_1 \) is Type III so the multiplicity of its single root class is the degree of \( f_1|U_1 \) at \( c_1 \) with \( \mathbb{Z}/2 \) coefficients, and that degree equals one, thus we conclude that \( NR(f_1) = 1 \) (see [4, Definition 2.7]). Let \( f : S^1 \to S^1 \) be a map of degree \( 2k \) where \( k \neq 0 \) and let \( f_2 : X_2 = S^2 \to S^2 = Y_2 \) be the suspension of \( f \). Letting \( c_2 \) be one of the vertex points of the suspension, then \( f_2^{-1}(c_2) = c_2 \) and the multiplicity of this single root class is the local integer degree of \( f_2|U_2 \) at \( c_2 \), where \( U_2 \) is a Euclidean neighborhood of \( c_2 \). By excision, that degree equals the (integer) Brouwer degree of the map \( f_2 \), that is, \( 2k \neq 0 \), so \( NR(f_2) = 1 \). Next we note that, since \( f_1 \) is a Type III map, \( f_1 \times f_2 \) is also Type III. Now \( (f_1 \times f_2)^{-1}(c_1, c_2) = (c_1, c_2) \) so there is a single root class and its multiplicity, the multiplicity of \( f_1 \times f_2 \) that is denoted \( m(f_1 \times f_2) \), is the local degree, with \( \mathbb{Z}/2 \) coefficients, of \( f_1 \times f_2|U_1 \times U_2 \) at \((c_1, c_2)\). Since all manifolds are orientable with respect to \( \mathbb{Z}/2 \) coefficients, excision implies that \( m(f_1 \times f_2) = \deg(f_1 \times f_2, 2) \), the Brouwer degree of \( f_1 \times f_2 \) with respect to \( \mathbb{Z}/2 \) coefficients. It follows from the Künneth theorem (see [14, 5.3.11, p. 235]) that \( \deg(f_1 \times f_2, 2) = \deg(f_1, 2) \cdot \deg(f_2, 2) \) and clearly \( \deg(f_2, 2) = 0 \) since it is the congruence class of \( 2k \) modulo \( 2 \), so we conclude that \( m(f_1 \times f_2) = 0 \) and therefore \( NR(f_1 \times f_2) = 0 \).

5. The absolute degree of compositions that include non-orientable maps

The principal aim of this section is to prove a multiplicative formula for \( A(g \circ f) \) in the case where at least one of the maps \( f, g \) is non-orientable (Theorem 5.4), and use it to
present a final summary of multiplicative formulae for $A(g \circ f)$ for all types of maps $f, g$ (Theorem 5.7).

**Lemma 5.1.** Let $g : (Y, \partial Y) \to (Z, \partial Z)$ be a proper Type III map of $n$-manifolds with $n > 2$. If $q^o : Y^o \to Y$ is the orientable covering of $Y$, then $\text{NR}(g \circ q^o) = A(g \circ q^o) = 0$.

**Proof.** Let $c \in \text{int } Z$. As $n > 2$ we may assume that each root class of $g$ at $c$ is a single point. Let $y \in Y$ be such a root class. Since $g$ is Type III, there is an orientation-reversing loop $\ell$ at $y$ in $Y$ such that $g \circ \alpha$ is contractible. Lift this loop to a path $\ell^o$ in $Y^o$. It follows from the fact that $\ell$ is orientation reversing that the two ends $a^o_0 = \ell^o(0)$ and $a^o_1 = \ell^o(1)$ must be distinct. The loop $g \circ q^o \circ \ell^o = g \circ \ell$ is contractible, and so the points $a^o_0$ and $a^o_1$ are in the same root class of $g \circ q^o$ and these two roots constitute the entire root class. We will show that $|m((a^o_0, a^o_1))] = 0$ and thus every root class of $g \circ q^o$ has multiplicity zero.

Let $V \subset Z$ be an elementary Euclidean neighborhood of $c$, and let $U \subset Y$ be a Euclidean neighborhood of $y$ small enough that $g(U) \subset V$ and $y$ is the only root of $g$ at $c$ that is in $U$. Let $U^o_0$ and $U^o_1$ be the two components of $U^o = q^o^{-1}(U)$, with $a^o_0 \in U^o_0$ and $a^o_1 \in U^o_1$. Orient $Y^o$ and then restrict this orientation to $U^o = U^o_0 \cup U^o_1$. Then, since $\ell$ is orientation reversing, $\deg_e(q^o|U^o_0) = -\deg_e(q^o|U^o_1) = \pm 1$, so, applying the additivity property of the local degree [6, Proposition 4.7, p. 269] and also its multiplicative property for compositions [6, Corollary 4.6. p. 268], we find that

$$|m((a^o_0, a^o_1))] = |\deg_e(g \circ q^o|U^o_0)|$$
$$= |\deg_e(g \circ q^o|U^o_0) + \deg_e(g \circ q^o|U^o_1)|$$
$$= |\deg_e(g(U) \deg_y(q^o|U^o_0) + \deg_e(g(U) \deg_y(q^o|U^o_1)|$$
$$= |\deg_e(g(U)) \cdot |\deg_y(q^o|U^o_0) + \deg_y(q^o|U^o_1)| = 0.$$  \(\square\)

**Lemma 5.2.** Suppose $f : (X, \partial X) \to (Y, \partial Y)$ and $g : (Y, \partial Y) \to (Z, \partial Z)$ are proper maps of $n$-manifolds with $n > 2$. Suppose further that $X$ is orientable and that $g$ is Type III, then $\text{NR}(g \circ f) = A(g \circ f) = 0$.

**Proof.** If $f$ is Type II, then by [4, Theorem 3.11], $\text{NR}(f) = 0$, and thus $\text{NR}(g \circ f) = 0$ by Theorem 2.3. Since $X$ is orientable, the only other possibility is that $f$ is Type I and in that case we may lift $f$ to obtain a map $f^o : X \to Y^o$, so $g \circ f = g \circ q^o \circ f^o$. By Lemma 5.1, $\text{NR}(g \circ q^o) = 0$ and therefore, applying Theorem 2.3 again, we conclude that $\text{NR}(g \circ f) = 0$.  \(\square\)

**Theorem 5.3.** Suppose $f : (X, \partial X) \to (Y, \partial Y)$ and $g : (Y, \partial Y) \to (Z, \partial Z)$ are proper maps of $n$-manifolds. Suppose further that $g \circ f$ is an orientable map and $\text{NR}(g \circ f) > 0$. Then $f$ and $g$ are both Type I maps.

**Proof.** We first assume $n > 2$ so that we can apply the previous lemma. Let $p^o : X^o \to X$ be the orientable covering. If $X$ is orientable, then $X^o = X$ and thus $\text{NR}((g \circ f) \circ p^o) = \text{NR}(g \circ f)$. If $X$ is non-orientable, since $g \circ f$ is orientable we may apply Lemma 3.4 to
conclude that $NR((g \circ f) \circ p^o) = 2NR(g \circ f)$. In either case, $NR(g \circ f) > 0$ implies that $NR(g \circ (f \circ p^o)) > 0$. But $X^o$ is orientable so, by Lemma 5.2, $g$ cannot be Type III. Nor can $f$ be Type III because, in that case, $g \circ f$ would also be Type III. Since $NR(g \circ f) > 0$ implies both $NR(f) > 0$ and $NR(g) > 0$ by Theorem 2.3, neither $f$ nor $g$ can be Type II and it follows that they are both Type I.

Now suppose that $f$ and $g$ are proper maps of surfaces such that $g \circ f$ is orientable and $NR(g \circ f) > 0$. Letting $id$ denote the identity map of the circle, we consider the proper maps of 3-manifolds $f \times id : (X \times S^1, \partial X \times S^1) \to (Y \times S^1, \partial Y \times S^1)$ and $g \times id : (Y \times S^1, \partial Y \times S^1) \to (Z \times S^1, \partial Z \times S^1)$. By Lemma 2.1(a), $(g \times id) \circ (f \times id) = (g \circ f) \times id$ is orientable and $NR((g \times id) \circ (f \times id)) > 0$ by Theorem 2.1(b). From the first part of the proof we know that $f \times id$ and $g \times id$ are Type I and thus it follows by Theorem 2.1(a) that $f$ and $g$ are also Type I. $\square$

In Example 3.2 we saw that the multiplicative property for compositions of the absolute degree can fail when one of the maps $f$ and $g$ is non-orientable, even when $NR(g \circ f) > 0$. We will now demonstrate that, in such a case, there is still a relationship between the absolute degree of $g \circ f$ and the absolute degrees of $f$ and $g$, though it is a more complicated one.

**Theorem 5.4.** Suppose that $g \circ f : (X, \partial X) \to (Z, \partial Z)$ is a composition of proper maps, at least one of which is non-orientable. If $NR(g \circ f) > 0$, then

$$\lambda(f, g)A(g \circ f) = A(f)A(g)$$

where $\lambda(f, g) = \kappa(f, g) \cdot |m(f)| \cdot |m(g)|$.

**Proof.** Since $NR(g \circ f) > 0$, then $NR(f) > 0$ and $NR(g) > 0$ by Theorem 2.3, and therefore $|m(f)| > 0$ and $|m(g)| > 0$ as well. Moreover, by Theorem 4.3, we have

$$\kappa(f, g)NR(g \circ f) = NR(f)NR(g).$$

It is immediate from the definitions that the absolute degree of a non-orientable map is equal to its Nielsen root number. The map $g \circ f$ is non-orientable by Theorem 5.3, so we have

$$\kappa(f, g)A(g \circ f) = NR(f)NR(g).$$

Since, by definition, the absolute degree is the product of the Nielsen root number and the multiplicity of a map, we may write $NR(f) = A(f)/|m(f)|$, and similarly for $g$, and the result follows. $\square$

As at least one of the maps $f$ and $g$ in Theorem 5.4 is non-orientable, at least one of $A(f) = NR(f)$ and $A(g) = NR(g)$ is true by [4], Theorems 3.11 and 3.12, so either
\[ \lambda(f, g) = \kappa(f, g) \cdot |m(f)| \] or \[ \lambda(f, g) = \kappa(f, g) \cdot |m(g)|. \] In Example 3.2, the map \( g \) is non-orientable and \( \lambda(f, g) = |m(f)| = d. \)

**Theorem 5.5.** Let \( f_i : (X_i, \partial X_i) \to (Y_i, \partial Y_i), \) for \( i = 1, 2, \) be proper maps of \( n_i \)-manifolds such that their cartesian product \( f_1 \times f_2 : (X_1 \times X_2, \partial(X_1 \times X_2)) \to (Y_1 \times Y_2, \partial(Y_1 \times Y_2)) \) has the property \( NR(f_1 \times f_2) > 0. \) If exactly one of the maps \( f_1 \) and \( f_2 \) is orientable, then

\[ |m(f_i)| \mathcal{A}(f_1 \times f_2) = \mathcal{A}(f_1) \mathcal{A}(f_2), \]

for \( f_i \) the orientable map.

**Proof.** We again write \( f_1 \times f_2 \) as the composition \( f_1 \times f_2 = f \circ g = (id_1 \times f_2) \circ (f_1 \times id_2). \) By Theorem 5.4 and Theorem 2.1(b) we have

\[ \lambda(f, g) \mathcal{A}(f_1 \times f_2) = \mathcal{A}(f_1) \mathcal{A}(f_2) \]

where \( \lambda(f, g) = |m(id_1 \times f_2)| \cdot |m(f_1 \times id_2)| \) since we proved in Theorem 4.7 that \( \kappa(f, g) = 1 \) in this case. Then Theorem 2.1(b) implies that \( \lambda(f, g) = |m(f_2)| \cdot |m(f_1)|. \) One of the maps \( f_1 \) and \( f_2 \) is orientable and we call it \( f_i \) so, as we noted in the remark following the proof of Theorem 5.4, we have \( \lambda(f, g) = |m(f_i)|. \)

Example 4.8 demonstrates that Theorem 5.5 requires the hypothesis \( NR(f_1 \times f_2) > 0. \) That hypothesis is not needed to establish the multiplicative property for the cartesian product when both maps are orientable, as we showed in Theorem 3.6, nor is it required for the remaining case of that multiplicative property for compositions, that is, when both maps are non-orientable.

**Theorem 5.6.** If, for \( i = 1, 2, \) proper maps \( f_i : (X_i, \partial X_i) \to (Y_i, \partial Y_i) \) of \( n_i \)-manifolds are both non-orientable, then their cartesian product \( f_1 \times f_2 : (X_1 \times X_2, \partial(X_1 \times X_2)) \to (Y_1 \times Y_2, \partial(Y_1 \times Y_2)) \) has the multiplicative property for cartesian products

\[ \mathcal{A}(f_1 \times f_2) = \mathcal{A}(f_1) \mathcal{A}(f_2). \]

**Proof.** If \( NR(f_1 \times f_2) > 0, \) then Theorem 4.7 states that

\[ NR(f_1 \times f_2) = NR(f_1)NR(f_2), \]

and since \( f_1 \) and \( f_2, \) and therefore \( f_1 \times f_2, \) are all non-orientable, then the multiplicative property for cartesian products follows from the fact that the Nielsen root number equals the absolute degree for such maps. Thus the property

\[ \mathcal{A}(f_1 \times f_2) = \mathcal{A}(f_1) \mathcal{A}(f_2), \]

can fail only if there are proper non-orientable maps \( f_1 \) and \( f_2 \) with non-zero absolute degrees such that \( \mathcal{A}(f_1 \times f_2) = 0. \) We will now show that no such maps exist. We first assume that all the manifolds are of dimension at least three so that, choosing \( c_i \in \text{int } Y_i, \) we may assume that we have root classes that are single points; call them \( x_i^+ \). Let \( V_i \) be euclidean neighborhoods of the \( c_i \) and choose euclidean neighborhoods \( U_i \) of the \( x_i^+ \) such
that \( f_1(U_1) \subset V_1 \). We note that the point \( (x_1^*, x_2^*) \) is a root class of \( f_1 \times f_2 \). Since the map \( f_1 \times f_2 \) is non-orientable, the multiplicity of \( f_1 \times f_2 \) is the local degree of \( f_1 \times f_2 | U_1 \times U_2 \) at \( (c_1, c_2) \) with coefficients in \( \mathbb{Z}/2 \). Writing \( f_1 \times f_2 = (id_1 \times f_2) \circ (f_1 \times id_2) \) once more, the restriction to \( U_1 \times U_2 \) can be written as the composition

\[
f_1 \times f_2 | U_1 \times U_2 = (id_1 | V_1 \times f_2 | U_2) \circ (f_1 | U_1 \times id_2 | U_2).
\]

Since \( V_1 \times U_2 \) is connected, \([6, \text{Corollary 4.6, p. 268}]\) implies that the local degrees with \( \mathbb{Z}/2 \) coefficients have the property

\[
\deg_{(c_1, c_2)}(f_1 \times f_2 | U_1 \times U_2) = \deg_{(c_1, c_2)}(id_1 | V_1 \times f_2 | U_2) \cdot \deg_{(c_1, c_2)}(f_1 | U_1 \times id_2 | U_2).
\]

Now suppose \( A(f_i) > 0 \) for \( i = 1, 2 \), then both \( |m(f_i)| = 1 \in \mathbb{Z}/2 \). Thus, as in the proof of Theorem 2.1(b), we have \( |m(id_1 \times f_2)| = \deg_{(c_1, c_2)}(id_1 | V_1 \times f_2 | U_2) = |m(f_2)| = 1 \) and \( |m(f_1 \times id_2)| = \deg_{(c_1, c_2)}(f_1 | U_1 \times id_2 | U_2) = |m(f_1)| = 1 \). We have demonstrated that \( |m(f_1 \times f_2)| = \deg_{(c_1, c_2)}(f_1 \times f_2 | U_1 \times U_2) = 1 \) and therefore \( A(f_1 \times f_2) > 0 \). Now suppose there are proper non-orientable maps of \( n_i \)-manifolds, where at least one of the \( n_i < 3 \), such that the \( A(f_i) > 0 \) but \( A(f_1 \times f_2) = 0 \). Let \( id : M \to M \) be the identity map of a manifold of dimension at least 2. Set \( f'_1 = id \times f_1 : M \times X_1 \to M \times Y_1 \) and \( f'_2 = f_2 \times id : X_2 \times M \to Y_2 \times M \), then the \( A(f'_i) > 0 \) by Theorem 2.1(b). Further application of Theorem 2.1(b) gives us

\[
A(f'_1 \times f'_2) = A(id \times (f_1 \times f_2 \times id)) = A(f_1 \times f_2 \times id) = A(f_1 \times f_2) = 0.
\]

But then the \( f'_i \) are maps of manifolds of dimension at least 3 such that \( A(f'_i) > 0 \) for which \( A(f'_1 \times f'_2) = 0 \) and we showed that no such maps exist. We conclude that for the absolute degree the multiplicative property for Cartesian products holds for the Cartesian product of proper non-orientable maps on manifolds of any dimension and for any value of the absolute degree of the product. \( \square \)

We summarize the relationship between the absolute degree of \( g \circ f \) and the absolute degrees of \( f \) and \( g \) as follows (part (a) is Theorem 3.5, part (b) is immediate from Theorem 5.4 and the definition of \( A(g \circ f) \)):

**Theorem 5.7.** Let \( f : (X, \partial X) \to (Y, \partial Y) \) and \( g : (Y, \partial Y) \to (Z, \partial Z) \) be proper maps between \( n \)-manifolds.

(a) If both \( f \) and \( g \) are orientable maps, then \( A(g \circ f) = A(f)A(g) \).

(b) If at least one of the maps \( f \) and \( g \) is not orientable and \( NR(g \circ f) > 0 \), then \( \lambda(f, g)A(f \circ g) = A(f)A(g) \) where \( \lambda(f, g) = \kappa(f, g) \cdot |m(f)| \cdot |m(g)| \), but if \( NR(g \circ f) = 0 \), then \( A(g \circ f) = 0 \).

For the Cartesian product we have

**Theorem 5.8.** Let \( f_i : (X_i, \partial X_i) \to (Y_i, \partial Y_i), \) for \( i = 1, 2 \), be proper maps of \( n_i \)-manifolds and let \( f_1 \times f_2 : (X_1 \times X_2, \partial(X_1 \times X_2)) \to (Y_1 \times Y_2, \partial(Y_1 \times Y_2)) \) be their Cartesian product.
(a) If \( f_1 \) and \( f_2 \) are both orientable maps or both are non-orientable maps, then \( \mathcal{A}(f_1 \times f_2) = \mathcal{A}(f_1) \mathcal{A}(f_2) \).

(b) if one of the maps \( f_1 \) and \( f_2 \) is orientable and the other is non-orientable and \( NR(f_1 \times f_2) > 0 \), then \(|m(f_i)|\mathcal{A}(f_1 \times f_2) = \mathcal{A}(f_1) \mathcal{A}(f_2)\) where \( f_i \) denotes the orientable map, but if \( NR(f_1 \times f_2) = 0 \), then \( \mathcal{A}(f_1 \times f_2) = 0 \).

Appendix A. An application to maps of aspherical manifolds (by Robin Brooks)

A space \( X \) is aspherical if it is connected and its higher homotopy groups \( \pi_n(X) \) for \( n > 1 \) are all trivial. Equivalently, its universal covering space is contractible. Such a space is also called an Eilenberg–MacLane space of type \( K(\pi, 1) \), where \( \pi \) is the fundamental group of \( X \). The relation between the Nielsen root number and the degree for maps of acyclic spaces was studied in [2]. At that time they were not aware of the concept of absolute degree, so in order to state results for nonorientable aspherical manifolds, they used the Brouwer degree of lifts to the orientable double covers. In this appendix we restate their principal results much more simply, using the concept of absolute degree.

The following theorem is the simplified restatement of Theorem 1 of [2].

**Theorem A.1.** Suppose \( f: X \to Y \) is a map of closed aspherical manifolds of the same dimension, and the induced fundamental group homomorphism \( f_\#: \pi(X) \to \pi(Y) \) is a monomorphism. Then \( \mathcal{A}(f) = NR(f) = [\pi_1(Y): f_\#(\pi_1(Y))] \).

**Proof.** From [2, Theorem 1], \( NR(f) = [\pi_1(Y): f_\#(\pi_1(Y))] > 0 \), so it suffices to show that \( \mathcal{A}(f) = NR(f) \).

Suppose first that both \( X \) and \( Y \) are orientable, then \( \mathcal{A}(f) = |\deg(f)| \). Also, from [2, Theorem 1], \( NR(f) = |\deg(f)| \), and therefore \( \mathcal{A}(f) = NR(f) \).

For the next part of the proof we rely on Lemmas 3.3 and 3.4, hence we assume for the moment that \( n > 2 \).

First suppose \( X \) is orientable and \( Y \) is not. Let \( q^o: Y^o \to Y \) be the orientable double covering of \( Y \). Then according to [2, Theorem 1], there is a lift \( f^o: X \to Y^o \) of \( f \) through \( q^o \), and \( NR(f) = 2|\deg(f^o)| \). According to Lemma 3.3, \( 2|\deg(f^o)| = \mathcal{A}(f) \). Thus \( NR(f) = \mathcal{A}(f) \).

Next, suppose \( X \) is not orientable. Then according to [2, Theorem 1], \( Y \) is not orientable, there is a lift \( f^o: X^o \to Y^o \) of \( f \circ p^o: X^o \to Y \) through \( q^o \), where \( p^o: X^o \to X \) is the orientable double covering of \( X \), and \( NR(f) = |\deg(f^o)| \). By Lemma 3.3, \( \mathcal{A}(q^o \circ f^o) = 2|\deg(f^o)| \), so \( 2NR(f) = \mathcal{A}(q^o \circ f^o) \). Thus, since \( f \circ p^o = q^o \circ f^o \), we have \( 2NR(f) = \mathcal{A}(f \circ p^o) \). According to Lemma 3.4, \( \mathcal{A}(f \circ p^o) = 2\mathcal{A}(f) \). Thus \( 2NR(f) = 2\mathcal{A}(f) \), and therefore \( NR(f) = \mathcal{A}(f) \). This completes the proof if \( X \) and \( Y \) are orientable or if \( \dim(X) > 2 \).

Now suppose that \( \dim(X) \leq 2 \). If \( \dim X = 1 \), then \( X \) and \( Y \) are circles and therefore orientable, so we have already shown that \( \mathcal{A}(f) = NR(f) \). Suppose \( \dim(X) = 2 \), and let \( S^1 \) be a circle. Then \( X \times S^1 \) and \( Y \times S^1 \) are also aspherical manifolds, and \( f \times \text{id}: X \times S^1 \to Y \times S^1 \) is a map of closed aspherical manifolds of the same dimension, and the induced fundamental group homomorphism \( f_\#: \pi(X) \to \pi(Y) \) is a monomorphism. Then \( \mathcal{A}(f) = NR(f) = [\pi_1(Y): f_\#(\pi_1(Y))] \).

**Proof.** From [2, Theorem 1], \( NR(f) = [\pi_1(Y): f_\#(\pi_1(Y))] > 0 \), so it suffices to show that \( \mathcal{A}(f) = NR(f) \).

Suppose first that both \( X \) and \( Y \) are orientable, then \( \mathcal{A}(f) = |\deg(f)| \). Also, from [2, Theorem 1], \( NR(f) = |\deg(f)| \), and therefore \( \mathcal{A}(f) = NR(f) \).

For the next part of the proof we rely on Lemmas 3.3 and 3.4, hence we assume for the moment that \( n > 2 \).

First suppose \( X \) is orientable and \( Y \) is not. Let \( q^o: Y^o \to Y \) be the orientable double covering of \( Y \). Then according to [2, Theorem 1], there is a lift \( f^o: X \to Y^o \) of \( f \) through \( q^o \), and \( NR(f) = 2|\deg(f^o)| \). According to Lemma 3.3, \( 2|\deg(f^o)| = \mathcal{A}(f) \). Thus \( NR(f) = \mathcal{A}(f) \).

Next, suppose \( X \) is not orientable. Then according to [2, Theorem 1], \( Y \) is not orientable, there is a lift \( f^o: X^o \to Y^o \) of \( f \circ p^o: X^o \to Y \) through \( q^o \), where \( p^o: X^o \to X \) is the orientable double covering of \( X \), and \( NR(f) = |\deg(f^o)| \). By Lemma 3.3, \( \mathcal{A}(q^o \circ f^o) = 2|\deg(f^o)| \), so \( 2NR(f) = \mathcal{A}(q^o \circ f^o) \). Thus, since \( f \circ p^o = q^o \circ f^o \), we have \( 2NR(f) = \mathcal{A}(f \circ p^o) \). According to Lemma 3.4, \( \mathcal{A}(f \circ p^o) = 2\mathcal{A}(f) \). Thus \( 2NR(f) = 2\mathcal{A}(f) \), and therefore \( NR(f) = \mathcal{A}(f) \). This completes the proof if \( X \) and \( Y \) are orientable or if \( \dim(X) > 2 \).

Now suppose that \( \dim(X) \leq 2 \). If \( \dim X = 1 \), then \( X \) and \( Y \) are circles and therefore orientable, so we have already shown that \( \mathcal{A}(f) = NR(f) \). Suppose \( \dim(X) = 2 \), and let \( S^1 \) be a circle. Then \( X \times S^1 \) and \( Y \times S^1 \) are also aspherical manifolds, and \( f \times \text{id}: X \times S^1 \to Y \times S^1 \) is a map of closed aspherical manifolds of the same dimension, and the induced fundamental group homomorphism \( f_\#: \pi(X) \to \pi(Y) \) is a monomorphism. Then \( \mathcal{A}(f) = NR(f) = [\pi_1(Y): f_\#(\pi_1(Y))] \).
$Y \times S^1$ also induces a monomorphism of the fundamental groups, so from the $\dim(X) > 2$ proof we have $A(f \times id) = NR(f \times id)$. From Theorem 2.1, $A(f \times id) = A(f)$ and $NR(f \times id) = NR(f)$. Thus, $A(f) = NR(f)$. \hfill \Box

A group $G$ satisfies the ascending chain condition on normal subgroups if every strictly increasing chain of normal subgroups has finite length. A group is polycyclic if it has a normal series whose factor groups are all cyclic. It is virtually polycyclic if contains a polycyclic subgroup of finite index.

Theorem 4 of [2] may now be rewritten in the following sharper form:

**Theorem A.2.** Suppose $f : X \to Y$ a map of closed aspherical manifolds of the same dimension, and either $X = Y$ and $\pi(X)$ satisfies the ascending chain condition on normal subgroups, or $\pi(X)$ is virtually polycyclic. Then $NR(f) = A(f)$ and the following are equivalent:

(i) $f# : \pi(X) \to \pi(Y)$ is a monomorphism.

(ii) $NR(f) > 0$.

(iii) $[\pi_1(Y) : f#(\pi_1(Y))] < \infty$.

(iv) $NR(f) = [\pi_1(Y) : f#(\pi_1(Y))]$.

**Proof.** The equivalence of (i)-(iv) above is immediate from [2, Theorem 4]. Thus, if $NR(f) > 0$, then $f#$ is a monomorphism, so from Theorem 1, $NR(f) = A(f)$. On the other hand, $A(f) > 0$ implies that $NR(f) > 0$, so if $NR(f) = 0$, then, $A(f) = 0$. In either case, $A(f) = NR(f)$. \hfill \Box

An important class of aspherical manifolds is the class of infrasolvmanifolds. This class includes the solvmanifolds, infranilmanifolds, and nilmanifolds. For a brief description of infrasolvmanifolds see [2, p. 408]. For a fuller description see [7, pp. 15–18].

Every infrasolvmanifold has a virtually polycyclic fundamental group, so Theorem A2 implies

**Theorem A.3.** Suppose $f : X \to Y$ is a map of closed aspherical manifolds of the same dimension and $X$ is an infrasolvmanifold. Then $NR(f) = A(f)$, and the following are equivalent:

(i) $f# : \pi(X) \to \pi(Y)$ is a monomorphism.

(ii) $NR(f) > 0$.

(iii) $[\pi_1(Y) : f#(\pi_1(Y))] < \infty$.

(iv) $NR(f) = [\pi_1(Y) : f#(\pi_1(Y))]$.

**Example A.4** (cf. [4, Example 3.18]). Let $f : K \to K$ be a map of the Klein bottle. Then by [2, Proposition 6.4] there are integers $b, d$, and $e$ such that $f#(\alpha) = \alpha^b \beta^d$ and $f#(\beta) = \beta^e$, for appropriately chosen generators $\alpha$ and $\beta$ of $\pi_1(K)$, and $NR(f) = |be|$. The Klein bottle is a solvmanifold; therefore $A(f) = NR(f) = |be|$.
The question: “When is the Nielsen root number equal to the degree?” is analogous to corresponding questions in fixed point and coincidence theory: “When is the Nielsen fixed point number equal to the Lefschetz number?” and “When is the Nielsen coincidence number equal to the Lefschetz coincidence index?”. For a summary of the literature on these questions as well as related recent results in the more general coincidence setting see [15].

References