Spectral factorization of non-symmetric polynomial matrices

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Abstract

The topic of the paper is spectral factorization of rectangular and possibly non-full-rank polynomial matrices. To each polynomial matrix we associate a matrix pencil by direct assignment of the coefficients. The associated matrix pencil has its finite generalized eigenvalues equal to the zeros of the polynomial matrix. The matrix dimensions of the pencil we obtain by solving an integer linear programming (ILP) minimization problem. Then by extracting a deflating subspace of the pencil we come to the required spectral factorization. We apply the algorithm to most general-case of inner–outer factorization, regardless continuous or discrete time case, and to finding the greatest common divisor of polynomial matrices.
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1. Introduction

Consider $n \times m$-dimensional polynomial matrix $P(\lambda)$,

$$P(\lambda) = P_0 + \lambda P_1 + \cdots + \lambda^d P_d$$

(1.1)
for some real matrices $P_0, P_1, \ldots, P_d$, we state the problem of obtaining a spectral factorization

$$P(\lambda) = P_\beta(\lambda) P_\alpha(\lambda),$$

(1.2)

such that the zeros of the polynomial matrix $P_\alpha(\lambda)$ lie in $C_\alpha$, the zeros of the polynomial matrix $P_\beta(\lambda)$ lie in $C_\beta$, where $C_\alpha \cup C_\beta = C$. For continuous-time systems, $C_\alpha = \{ \lambda \in C : \Re[\lambda] \leq 0 \}$ and $C_\beta = \{ \lambda \in C : \Re[\lambda] \geq 0 \}$, and for discrete-time systems $C_\alpha = \{ \lambda \in C : |\lambda| \leq 1 \}$ and $C_\beta = \{ \lambda \in C : |\lambda| \geq 1 \}$. This problem appears in optimal control [10], where, besides spectral factorization of para-hermitian polynomial matrices, spectral factorization of non-symmetric matrices requires. In Section 6 we present two another applications.

The above problem may have many solutions. Indeed, let us apply Smith form [7]. It means existence of unimodular polynomial matrices $U(\lambda)$ and $V(\lambda)$, and invariant polynomials $i_1(\lambda), \ldots, i_r(\lambda)$ with the property that each $i_k(\lambda)$ divides $i_{k-1}(\lambda)$, such that

$$P = U \begin{bmatrix} 0, \text{diag} \{i_1, \ldots, i_r, 0\} \end{bmatrix} V, \quad \text{or}$$

(1.3a)

$$P = U \text{diag} \{i_1, \ldots, i_r, 0\} V, \quad \text{or}$$

(1.3b)

$$P = U \begin{bmatrix} 0 \text{diag} \{i_1, \ldots, i_r, 0\} \end{bmatrix} V.$$  

(1.3c)

Then a factorization (1.2) can be achieved by factoring the invariant polynomials. It is well known [9] that the algorithm for reduction to Smith form has poor numerical properties.

To narrow the class of possible solutions of the factorization (1.2), we restrict on the solutions that satisfy (i) at least one of $P_\alpha$ and $P_\beta$ is square, (ii) at least one of $P_\alpha$ and $P_\beta$ has full rank, in this paper the square matrix is of full rank; (iii) $P_\alpha$ is generalized column reduced (see Definition 1 below) and $P_\beta$ is generalized row reduced.

Note that if one obtains spectral factorization (1.2) by (1.3), the spectral factors $P_\alpha$ and $P_\beta$ are not column- and row-reduced, respectively.

There are algorithms for $J$-spectral factorization of para-hermitian matrices in the literature (see [1,2,5,13,14,19–22], and references therein), based on various approaches: successive factor extraction, Newton–Raphson method, state-space method. The work [4] elaborates on factorization of scalar polynomials. Before we proceed, remarks on the notation are in order.

**Remarks on the notation.** By the superscript $T$ we denote matrix transposition. The identity matrix is denoted by $I$, or $I_n$ if the matrix dimension requires. The set of complex numbers is denoted by $C$. The complex zero we refer to as origin. By $\Phi^\sim$ the matrix function $\Phi^T(-\lambda)$ is denoted, and $\Phi^{-\sim} = (\Phi^\sim)^{-1}$. Unless otherwise specified, by rank of a matrix function we mean its generic rank on $C$. 
Zeros (and poles) of $n \times m$-dimensional rational matrix $G$ are defined through its McMillan form \cite{17}:

$$G = U \text{diag} \left\{ \frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}, 0 \right\} V,$$

(1.4)

where $\psi_i$ and $\epsilon_i$ are coprime polynomials, $i = 1, \ldots, r$, and $U$ and $V$ are unimodular polynomial matrices. The zeros of $G$ are actually the zeros of $\epsilon = \prod_{i=1}^r \epsilon_i$. Multiplicity of a zero of $G$ is the multiplicity of the same zero of the polynomial $\epsilon$. If we refer to number of zeros, we count multiplicities of the zeros.

The following definition of column and row reduced polynomial matrices is a generalization of the definition \cite{25} that applies to full rank matrices.

**Definition 1.** A polynomial matrix is generalized column reduced if its column leading coefficient matrix \cite{25} has rank equal to the generic rank of the polynomial matrix. Analogously we define the notion of generalized row reduced polynomial matrix.

To solve the problem of spectral factorization of possibly non-full rank polynomial matrix $P(\lambda)$ (1.1), in this paper we use the fact that not all degrees $0, 1, \ldots, d$ in the entries of $P(\lambda)$ are present. The column degrees of the factor $P_\alpha$ and the row degrees of $P_\beta$ in (1.2), we obtain by solving an integer linear programming (ILP) problem. The proposed numerical algorithm is based on invariant or deflating subspaces, which are extracted by orthogonal or near-orthogonal transformation matrices, respectively.

### 2. Two useful spectral factorization formulas

In this section we formulate two generalizations of the optimal LQ return difference equality \cite{3,12} stated in Propositions 1 and 2.

Given square matrices $A_\alpha$ and $A_\beta$ and matrices $B_\alpha$ and $B_\beta$, and matrix $\Sigma$ with compatible partition

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

and let

$$B_\alpha D_\alpha = (\lambda I - A_\alpha)^\top, \quad B_\beta D_\beta = (\lambda I - A_\beta)^\top,$$

(2.1)

for some right coprime polynomial matrices $D_\alpha$ with $T_\alpha$ and $D_\beta$ with $T_\beta$, where $D_\alpha$ and $D_\beta$ are square. Such polynomial matrices always exist \cite{6,17}. For our purposes, in Section 3 we explicitly construct such polynomial matrices.

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1 Only the form like (1.3b) is displayed, for conciseness.
Consider the following algebraic Riccati system (ARS)

\[
A^T \beta X + XA_\alpha + \Sigma_{11} - \bar{F}_\beta^T F_\alpha = 0, \\
\Sigma_{22} F_\alpha = B^T_\beta X + \Sigma_{21}, \\
\bar{F}_\beta^T = XB_\alpha + \Sigma_{12}
\]  \tag{2.2}

for unknown matrices \(X, F_\alpha\) and \(\bar{F}_\beta\). Since quadratic, the ARS may have many solutions. We are interested in special solutions. By \(C_\alpha\)-solution of ARS (2.2)–(2.4) we mean a solution for which the eigenvalues of \(A_\alpha - B_\alpha F_\alpha\) are in \(C_\alpha\).

**Proposition 1.** If there exists a \(C_\alpha\)-solution \(X, F_\alpha\) and \(\bar{F}_\beta\) of ARS (2.2), (2.3) and (2.4), the following polynomial factorization identity holds true

\[
\begin{bmatrix}
\bar{F}_\beta^T \\
D_\beta
\end{bmatrix}
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix}
\begin{bmatrix}
\bar{T}_\alpha \\
D_\alpha
\end{bmatrix} = \left( \begin{bmatrix}
D_\beta \Sigma_{22} + \bar{F}_\beta^T F_\alpha \\
D_\beta \Sigma_{12}
\end{bmatrix}
\right)
\left( D_\alpha + F_\alpha \bar{T}_\alpha \right), \tag{2.5}
\]

and the zeros of the right spectral factor \(D_\alpha + F_\alpha \bar{T}_\alpha\) lie in \(C_\alpha\).

**Proof.** Eq. (2.2) can be written as

\[
\bar{F}_\beta^T F_\alpha = \Sigma_{11} - (\lambda I - A_\beta)^\sim X - X(\lambda I - A_\alpha).
\]

Using (2.1), (2.3) and (2.4), we obtain

\[
\begin{align*}
\bar{F}_\beta^T F_\beta^T F_\alpha \bar{T}_\alpha &= \bar{F}_\beta^T \left[ \Sigma_{11} - (\lambda I - A_\beta)^\sim X - X(\lambda I - A_\alpha) \right] \bar{T}_\alpha \\
&= \bar{F}_\beta^T \Sigma_{11} \bar{T}_\alpha - D_\beta \Sigma_{12} \bar{T}_\alpha^T X \bar{T}_\alpha - \bar{T}_\beta X B_\alpha D_\alpha \\
&= \bar{F}_\beta^T \Sigma_{11} \bar{T}_\alpha - D_\beta \left( \Sigma_{22} F_\alpha - \Sigma_{21} \right) \bar{T}_\alpha - \bar{T}_\beta \left( \bar{F}_\beta^T - \Sigma_{12} \right) D_\alpha \\
&= \bar{F}_\beta^T \Sigma_{11} \bar{T}_\alpha - D_\beta \Sigma_{12} F_\alpha \bar{T}_\alpha + D_\beta \Sigma_{22} \bar{T}_\alpha \\
&\quad - \bar{T}_\beta \bar{F}_\beta^T D_\alpha + \bar{T}_\beta \Sigma_{12} D_\alpha.
\end{align*}
\]

We have

\[
\begin{align*}
\left( D_\beta \Sigma_{22} + \bar{F}_\beta^T F_\alpha \right)
&\left( D_\alpha + F_\alpha \bar{T}_\alpha \right) \\
&= D_\beta \Sigma_{22} D_\alpha + D_\beta \Sigma_{22} F_\alpha \bar{T}_\alpha + \bar{F}_\beta^T \bar{T}_\alpha \bar{F}_\beta D_\alpha + \bar{F}_\beta^T \bar{T}_\alpha D_\alpha \\
&= D_\beta \Sigma_{22} D_\alpha + D_\beta \Sigma_{22} F_\alpha \bar{T}_\alpha + \bar{F}_\beta^T \bar{T}_\alpha \bar{F}_\beta D_\alpha + \bar{F}_\beta^T \bar{T}_\alpha \Sigma_{11} \bar{T}_\alpha \\
&\quad - D_\beta \Sigma_{22} F_\alpha \bar{T}_\alpha + D_\beta \Sigma_{22} \bar{T}_\alpha - \bar{F}_\beta^T \bar{T}_\alpha \bar{F}_\beta D_\alpha + \bar{F}_\beta^T \Sigma_{12} D_\alpha \\
&= \left[ \bar{F}_\beta^T, D_\beta \right]
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix}
\begin{bmatrix}
\bar{T}_\alpha \\
D_\alpha
\end{bmatrix}.
\]
To prove the second part of the proposition, write
\[
\det \left( D_\alpha + F_\alpha T_\alpha \right) = \det(D_\alpha) \det \left( I + F_\alpha T_\alpha D_\alpha^{-1} \right)
\]
\[
= \det(D_\alpha) \det \left( I + T_\alpha D_\alpha^{-1} F_\alpha \right)
\]
\[
= \det(D_\alpha) \det \left[ I + (\lambda I - A_\alpha)^{-1} B_\alpha F_\alpha \right]
\]
\[
= \frac{\det(D_\alpha)}{\det(\lambda I - A_\alpha)} \det(\lambda I - A_\alpha + B_\alpha F_\alpha).
\]
Since \( \det(D_\alpha)/\det(\lambda I - A_\alpha) \) is constant, the zeros of \( D_\alpha + F_\alpha T_\alpha \) equal the eigenvalues of \( A_\alpha - B_\alpha F_\alpha \), and by definition, are located in \( C_\alpha \). \( \square \)

Analogously, given the following ARS:
\[
A^T_\beta X + XA_\alpha + \Sigma_{11} - F^T_\beta F_\alpha = 0,
\]
\[ (2.6) \]
\[
F_\alpha = B^T_\beta X + \Sigma_{21},
\]
\[ (2.7) \]
\[
F^T_\beta \Sigma_{22} = XB_\alpha + \Sigma_{12}.
\]
\[ (2.8) \]
for the unknown matrices \( X, F_\alpha \) and \( F_\beta \). By \( C_\beta \)-solution we mean the solution for which the eigenvalues of \( A_\beta - B_\beta F_\beta \) are in \( C_\beta \). The proof of the following Proposition 2 is analogous to the proof of Proposition 1.

**Proposition 2.** If there exists a \( C_\beta \)-solution \( X, F_\alpha \) and \( F_\beta \) of ARS \((2.6), (2.7)\) and \( (2.8) \), the following polynomial factorization identity holds true:
\[
\begin{bmatrix}
T^\sim_\beta & D^\sim_\beta
\end{bmatrix}
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix}
\begin{bmatrix}
T_\alpha \\
D_\alpha
\end{bmatrix}
= \left( D^\sim_\beta + T^\sim_\beta F^T_\beta \right) \left( \Sigma_{22} D_\alpha + F_\alpha T_\alpha \right)
\]
\[ (2.9) \]
and the zeros of the left spectral factor \( D^\sim_\beta + T^\sim_\beta F^T_\beta \) lie in \( C_\beta \).

**3. Transformation of polynomial spectral factorization to solving algebraic Riccati system**

Denote the degree of the \((i, j)\) entry of \( P(\lambda) \) by \( \delta_{ij} \). Let the indices \( \alpha_1, \alpha_2, \ldots, \alpha_m \) and \( \beta_1, \beta_2, \ldots, \beta_n \) are such that
\[
\delta_{ij} \leq \beta_i + \alpha_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m.
\]
\[ (3.1) \]

We start elaboration of the spectral factorization algorithm. The first step is a pre-factorization of \( P \), explained in the following proposition.
Proposition 3. Given polynomial matrix \( P(\lambda) \), the indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \), which satisfy (3.1), there exists a matrix \( \mathcal{F} \) such that

\[
P(\lambda) = \Gamma_\beta \mathcal{F} \Gamma_\alpha,
\]

where

\[
\Gamma_\alpha(\lambda) = \begin{bmatrix}
\chi_{\alpha_1}(\lambda) & 0 & \cdots & 0 \\
0 & \chi_{\alpha_2}(\lambda) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \chi_{\alpha_m}(\lambda)
\end{bmatrix}, \quad \chi_{\alpha_i}(\lambda) = \begin{bmatrix}
1 \\
\lambda \\
\vdots \\
\lambda^{\alpha_i}
\end{bmatrix},
\]

\[
\Gamma_\beta(\lambda) = \begin{bmatrix}
\chi_{\beta_1}(\lambda) & 0 & \cdots & 0 \\
0 & \chi_{\beta_2}(\lambda) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \chi_{\beta_n}(\lambda)
\end{bmatrix}, \quad \chi_{\beta_i}(\lambda) = \begin{bmatrix}
1 \\
\lambda \\
\vdots \\
\lambda^{\beta_i}
\end{bmatrix}.
\]

Proof. Denote by \( p_{ij}(\lambda) \) the \((i, j)\) entry of \( P(\lambda) \). We shall prove that each \( p_{ij}(\lambda) \) can be expressed as

\[
p_{ij}(\lambda) = \tilde{\chi}_{\beta_i} F_{ij} \chi_{\alpha_j}, \quad i = 1, \ldots, n, \ j = 1, \ldots, m
\]

for some matrices \( F_{ij} \), which will be used as sub-matrices in \( \mathcal{F} \). For simplicity, we shall omit the indices \( i, j \) of the polynomial \( p_{ij}(\lambda) \) and its coefficients in the following representation:

\[
p(\lambda) = p_0 + p_1 \lambda + \cdots + p_{\delta_{ij}} \lambda^{\delta_{ij}}, \quad p_{\delta_{ij}} \neq 0.
\]

Our construction of the sub-matrices \( F_{ij} \) is based on the inequality (3.1).

If \( \beta_i > \alpha_j \) and \( \delta_{ij} > 2\alpha_j \), we define the following matrix:

\[
F_{ij} = \frac{1}{2} \begin{bmatrix}
2p_0 & p_1 & 0 & 0 & \cdots & 0 & 0 \\
-p_1 & -2p_2 & -p_3 & 0 & \cdots & 0 & 0 \\
0 & p_3 & 2p_4 & p_5 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & (-1)^{\alpha_j - 1}p_{2\alpha_j - 1} & 2(-1)^{\alpha_j}p_{2\alpha_j} \\
0 & 0 & 0 & 0 & \cdots & 0 & 2(-1)^{\alpha_j + 1}p_{2\alpha_j + 1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & 0 & 2(-1)^{\alpha_j - 1}p_{2\alpha_j - 1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]
If $\delta_{ij} \leq 2\alpha_j$ we have simpler matrix $\mathcal{F}_{ij}$ with zero lower (below the dashed line) block.\footnote{Note that the sub-matrix above the dashed line is square and of dimension $\alpha_j + 1$.}

For the case $\beta_i \leq \alpha_j$ we define the matrix $\mathcal{F}_{ij}$ analogously:

\[
\mathcal{F}_{ij} = \frac{1}{2} \begin{bmatrix}
2p_0 & p_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
-p_1 & -2p_2 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & p_3 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 2(-1)^{\alpha_i}p_{2\alpha_i} & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 2(-1)^{\alpha_i}p_{2\alpha_i+1} & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{bmatrix}.
\]

We shall use the pre-factorization (3.2) as a starting point in the development of our algorithm. We shall choose the matrices $A_\alpha, B_\alpha, A_\beta, B_\beta$ and $\Sigma$ so that the equality (2.5) or (2.9) reduces to the required spectral factorization. For strictly positive integers $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$, define

\[
A_\alpha = \begin{bmatrix}
A_{\alpha 1} & 0 & \cdots & 0 \\
0 & A_{\alpha 2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{\alpha m}
\end{bmatrix}, \quad A_{\alpha i} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix} \quad \alpha_i, i = 1, \ldots, m,
\]

(3.6)

where

\[
B_\alpha = \begin{bmatrix}
b_{\alpha 1} & 0 & \cdots & 0 \\
0 & b_{\alpha 2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b_{\alpha m}
\end{bmatrix}, \quad b_{\alpha i} = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix} \quad \alpha_i, \quad i = 1, \ldots, m,
\]

(3.7)
and

\[
A_\beta = \begin{bmatrix} A_{\beta 1} & 0 & \cdots & 0 \\ 0 & A_{\beta 2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{\beta n} \end{bmatrix}, \quad A_{\beta i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{\beta_i},
\]

where

\[
B_\beta = \begin{bmatrix} b_{\beta 1} & 0 & \cdots & 0 \\ 0 & b_{\beta 2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{\beta n} \end{bmatrix}, \quad b_{\beta i} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\beta_i}, \quad i = 1, \ldots, n.
\]

(3.8)

If some index \( \alpha_i \) (or \( \beta_i \)) is zero, then the corresponding \( A_{\alpha i} \) and \( b_{\alpha i} \) (or \( A_{\beta i} \) and \( b_{\beta i} \)) are void.

It is easy to check that the following polynomial matrices \( \bar{T}_\alpha, D_\alpha, T_\beta \) and \( D_\beta \), satisfy (2.1) and the requirement to be coprime:

\[
\bar{T}_\alpha(\lambda) = \begin{bmatrix} \bar{\chi}_{\alpha 1}(\lambda) & 0 & \cdots & 0 \\ 0 & \bar{\chi}_{\alpha 2}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\chi}_{\alpha m}(\lambda) \end{bmatrix}, \quad \bar{\chi}_{\alpha i}(\lambda) = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ \lambda^{\alpha_i - 1} \end{bmatrix},
\]

(3.10)

\[
D_\alpha(\lambda) = \begin{bmatrix} \lambda^{\alpha 1} & 0 & \cdots & 0 \\ 0 & \lambda^{\alpha 2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{\alpha m} \end{bmatrix},
\]

(3.11)

\[
\bar{T}_\beta(\lambda) = \begin{bmatrix} \bar{\chi}_{\beta 1}(\lambda) & 0 & \cdots & 0 \\ 0 & \bar{\chi}_{\beta 2}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\chi}_{\beta n}(\lambda) \end{bmatrix}, \quad \bar{\chi}_{\beta i}(\lambda) = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ \lambda^{\beta_i - 1} \end{bmatrix},
\]

(3.12)
\[
D_\beta(\lambda) = \begin{bmatrix}
\lambda^{\beta_1} & 0 & \cdots & 0 \\
0 & \lambda^{\beta_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda^{\beta_n}
\end{bmatrix}.
\] (3.13)

Let permute the rows of \( D_\alpha \) and rows of \( D_\beta \) so that

\[
\Pi_\alpha \begin{bmatrix} T_\alpha \\ D_\alpha \end{bmatrix} = \Gamma_\alpha, \quad \Pi_\beta \begin{bmatrix} T_\beta \\ D_\beta \end{bmatrix} = \Gamma_\beta,
\] (3.14)

where \( \Pi_\alpha \) and \( \Pi_\beta \) are permutation matrices, and \( \Gamma_\alpha \) and \( \Gamma_\beta \) are the polynomial matrices defined in Proposition 3.

Now define the matrix \( \Sigma \) as permuted matrix \( \mathcal{F} \), i.e.

\[
\Sigma = \Pi_\beta^T \mathcal{F} \Pi_\alpha.
\] (3.15)

Then, by (3.15), (3.14) and (3.2), we have

\[
\begin{bmatrix} \mathcal{F}_\beta^\sim, D_\beta^\sim \end{bmatrix} \Sigma \begin{bmatrix} T_\alpha \\ D_\alpha \end{bmatrix} = \begin{bmatrix} \mathcal{F}_\beta^\sim, D_\beta^\sim \end{bmatrix} \Pi_\beta^T \mathcal{F} \Pi_\alpha \begin{bmatrix} T_\alpha \\ D_\alpha \end{bmatrix} = \Gamma_\beta^\sim \mathcal{F} \Gamma_\alpha = P.
\] (3.16)

By (3.16) and Proposition 1, we obtain

\[
P = \left( D_\beta^\sim \Sigma_{22} + \mathcal{F}_\beta^\sim \mathcal{F}_\beta^T \right) \left( D_\alpha + \mathcal{F}_\alpha \mathcal{F}_\alpha^T \right) \overset{\text{def}}{=} P_\beta P_\alpha.
\] (3.17)

The obtained result is summarized in the following proposition.

**Proposition 4.** Given polynomial matrix \( P \) and indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) which satisfy (3.1). Let the matrices \( A_\alpha, B_\alpha, A_\beta, B_\beta \) and \( \Sigma \) be given by (3.6)–(3.9) and (3.15), respectively. Suppose there exists a \( C_\alpha \)-solution \( X, F_\alpha, \mathcal{F}_\beta \) of ARS (2.2), (2.3) and (2.4). Then the polynomial spectral factorization (3.17) holds, where the zeros of the right spectral factor \( P_\alpha \) lie in \( C_\alpha \).

Analogously with (3.17), using Proposition 2, we obtain

\[
P = \left( D_\beta^\sim + \mathcal{F}_\beta^\sim \mathcal{F}_\beta^T \right) \left( \Sigma_{22} D_\alpha + \mathcal{F}_\alpha \mathcal{F}_\alpha^T \right) \overset{\text{def}}{=} P_\beta P_\alpha.
\] (3.18)

Therefore, we have the following analogous to Proposition 4.

**Proposition 5.** Given polynomial matrix \( P \) and indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) which satisfy (3.1). Let the matrices \( A_\alpha, B_\alpha, A_\beta, B_\beta \) and \( \Sigma \) be given by (3.6)–(3.9) and (3.15), respectively. Suppose there exists a \( C_\beta \)-solution \( X, \mathcal{F}_\alpha, F_\beta \) of ARS (2.6), (2.7) and (2.8). Then the polynomial spectral factorization (3.18) holds, where the zeros of the left spectral factor \( P_\beta \) lie in \( C_\beta \).
4. Solving ARS

Definition 2. Let $H$ be a square matrix. A subspace $V$ is called invariant for this matrix, if for an arbitrary basis matrix $V$ of $V$, there exists a square matrix $S$ such that

$$HV = VS.$$ 

If the eigenvalues of $S$ are in $C$, the invariant subspace will be called $C$-invariant. If the top square sub-matrix of $V$ is nonsingular the invariant subspace will be called disconjugate.

Definition 3. Let $\lambda M - N$ be a matrix pencil. A subspace $V$ is called deflating for this matrix pencil, if for an arbitrary basis matrix $V$ of $V$, there exists a square matrix $S$ such that

$$NV = MVS.$$ 

If the eigenvalues of $S$ are in $C$, the deflating subspace will be called $C$-deflating. If $MV$ has full column rank, the deflation subspace will be called proper. A special proper deflating subspace in which the top square sub-matrix of $V$ is nonsingular will be called disconjugate.

Let $P$ be a general polynomial matrix. Consider the matrix pencil, associated to ARS (2.2)–(2.4), for indices $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$,

$$\lambda M_\alpha - N_\alpha = \lambda \begin{bmatrix} I_{n_\alpha} & 0 & 0 \\ 0 & I_{n_\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A_\alpha & 0 & B_\alpha \\ -\Sigma_{11} & -A^T_{\beta} & -\Sigma_{12} \\ \Sigma_{21} & B^T_{\beta} & \Sigma_{22} \end{bmatrix},$$

(4.1)

where $n_\alpha = \sum_{i=1}^m \alpha_i$ and $n_\beta = \sum_{i=1}^n \beta_i$.

The following rational factorization, that is easy to check, explains why we consider such a matrix pencil. Namely, for all complex numbers $\lambda$ except the origin (i.e. $\lambda = 0$), we have

$$\lambda M_\alpha - N_\alpha = F(\lambda)G(\lambda)H(\lambda),$$

(4.2)

where

$$F(\lambda) = \begin{bmatrix} I & 0 & 0 \\ \Sigma_{11}(\lambda I - A_\alpha)^{-1} & I & 0 \\ -\Sigma_{21}(\lambda I - A_\alpha)^{-1} & -B^T_{\beta}(\lambda I + A^T_{\beta})^{-1} & I \end{bmatrix},$$

$$G(\lambda) = \begin{bmatrix} \lambda I - A_\alpha & 0 & 0 \\ 0 & \lambda I + A^T_{\beta} & 0 \\ 0 & 0 & \Phi(\lambda) \end{bmatrix}, \quad \Phi = -D^{-1}_{\beta}PD_{\alpha}^{-1},$$
\[ H(\lambda) = \begin{bmatrix}
    I & 0 & - (\lambda I - A_\alpha)^{-1} B_\alpha \\
    0 & I & (\lambda I + A_\beta^T)^{-1} \left[ \Sigma_{12} + \Sigma_{11} (\lambda I - A_\alpha)^{-1} B_\alpha \right] \\
    0 & 0 & I
\end{bmatrix}, \]

Since \( F \) and \( H \) are unimodular matrices, from (4.2) follows:

**Proposition 6.** If \( \lambda \neq 0 \) is a zero of the polynomial matrix \( P \), then it is a zero of the matrix pencil \( \lambda M_\alpha - N_\alpha \). Conversely, if \( \lambda \neq 0 \) is a zero of the matrix pencil \( \lambda M_\alpha - N_\alpha \), then it is a zero of the polynomial matrix \( P \).

A consequence of Proposition 6 is that a different choice of indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) does not change the zeros of the pencil \( \lambda M_\alpha - N_\alpha \), except maybe adding or subtracting zeros at the origin.

To present an interpretation of Proposition 6 from control-system-theoretic viewpoint, group the matrix pencil \( \lambda M_\alpha - N_\alpha \) as

\[ \lambda M_\alpha - N_\alpha = \begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix}, \quad (4.3) \]

where

\[ A = \begin{bmatrix} A_\alpha & 0 \\ -\Sigma_{11} & -A_\beta^T \end{bmatrix}, \quad B = \begin{bmatrix} -B_\alpha \\ \Sigma_{12} \end{bmatrix}, \quad C = \begin{bmatrix} \Sigma_{21}, B_\beta^T \end{bmatrix}, \quad D = -\Sigma_{22}. \quad (4.4) \]

It is easy to check that the associated transfer matrix [17] to this matrix pencil is \( D + C (\lambda I - A)^{-1} B = \Phi = -D_\beta \sim P D_\alpha^{-1} \). It is well known that if the realization \((A, B, C, D)\) is minimal [17], the zeros of the matrix pencil (4.3) and of its associated transfer matrix are the same, including the multiplicities. However, our realization of the transfer matrix \( -D_\beta \sim P D_\alpha^{-1} \) may be non-minimal, thus nothing we can say about the zero at the origin.

Suppose there exists a matrix \( \begin{bmatrix} V_1^T & V_2^T & V_3^T \end{bmatrix}^T \) that is a basis of a \( C_\alpha \)-deflating subspace of the pencil \( \lambda M_\alpha - N_\alpha \), disconjugate of dimension \( n_\alpha \). By Definition 3, it means that \( X, F_\alpha \), given by \( X = V_2 V_1^{-1}, F_\alpha = -V_3 V_1^{-1} \), and \( F_\beta \) (2.4) are solutions of ARS (2.2)–(2.4). Also, since the eigenvalues of \( S = V_1 (A_\alpha - B_\alpha F_\alpha) V_1^{-1} \) are in \( C_\alpha \) by the definition of \( C_\alpha \)-deflating subspace, the eigenvalues of \( A_\alpha - B_\alpha F_\alpha \) are in \( C_\alpha \). It means that \( X, F_\alpha, F_\beta \) is a \( C_\alpha \)-solution, and as such, it results in the spectral factorization (3.17). It remains to find a \( C_\alpha \)-deflating subspace of the matrix pencil \( \lambda M_\alpha - N_\alpha \) of dimension \( n_\alpha \).

The matrix pencil \( \lambda M_\alpha - N_\alpha \), by application of strictly equivalent transformation matrices [7], can be brought to Kronecker canonical form

\[ \text{diag} \left\{ \left[ \lambda I_{n_r} - A_r, -B_r \right], \lambda I_{k_\alpha} - N_\beta, \lambda I_{k_\beta} + N_b, \lambda M_\infty - I_{n_{\infty}}, \left[ \lambda I_{n_f} - A_f \right], -C_f \right\}. \]
where the matrix pair \((A_r, B_r)\) has a form like \((A_\alpha, B_\alpha)\) in (3.6) and (3.7), where instead of \(\alpha_i\), there are right minimal indices \(\epsilon_i\). The matrix pair \((C_l, A_l)\) has a form like \((B^T_{T\alpha}, A^T_{T\alpha})\), where instead of \(\alpha_i\), there are left minimal indices \(\eta_i\). The eigenvalues of the Jordan matrix \(N_g\) are in \(C_\alpha\), the eigenvalues of the Jordan matrix \(N_b\) are in \(C_\beta\), and \(M_\infty\) is a nilpotent matrix. The eigenvalues of \(N_g\) and \(N_b\) are called finite generalized eigenvalues of the matrix pencil \(\lambda M_\alpha - N_\alpha\).

Invariant indices of the matrix pencil \(\lambda M_\alpha - N_\alpha\) are \(n_r = \sum_i \epsilon_i\), \(k_\alpha, k_\beta, n_\infty\) and \(n_l = \sum_i \eta_i\). Further we shall find some connections between these invariant indices without actual finding the canonical form. By the transformation of the argument \(\lambda \to \lambda^{-1}\) in (4.2), we obtain the following rational identity:

\[
M_\alpha - \lambda N_\alpha = F_1(\lambda)G_1(\lambda)H_1(\lambda),
\]

where

\[
F_1(\lambda) = \begin{bmatrix}
I & 0 & 0 \\
\lambda \Sigma_{11}(I - \lambda A_\alpha)^{-1} & I & 0 \\
-\lambda \Sigma_{21}(I - \lambda A_\alpha)^{-1} & -\lambda B^T_\beta \left( I + \lambda A^T_\beta \right)^{-1} & I
\end{bmatrix},
\]

\[
G_1(\lambda) = \begin{bmatrix}
I - \lambda A_\alpha & 0 & 0 \\
0 & I + \lambda A^T_\beta & 0 \\
0 & 0 & \lambda \Phi(\lambda^{-1})
\end{bmatrix},
\]

\[
H_1(\lambda) = \begin{bmatrix}
I & 0 & -\lambda(I - \lambda A_\alpha)^{-1} B_\alpha \\
0 & I & \left( I + \lambda A^T_\beta \right)^{-1} \left[ \lambda \Sigma_{12} + \lambda^2 \Sigma_{11}(I - \lambda A_\alpha)^{-1} B_\alpha \right] \\
0 & 0 & I
\end{bmatrix}.
\]

that holds for all \(\lambda \neq 0\). From the fact that \(F_1\) and \(H_1\) are unimodular rational matrices, we deduce

**Proposition 7.** (i) The rank of the matrix pencil \(\lambda M_\alpha - N_\alpha\) is \(n_\alpha + n_\beta + \rho\), where \(\rho\) is the rank of \(P\).

(ii) \(n_\alpha + n_\beta + \rho = n_r + n_l + k_\alpha + k_\beta + n_\infty\).

(iii) \(n_\infty = \rho + \pi_0\), where \(\pi_0\) is the multiplicity of the zero at the origin of the matrix function \(\Phi(\lambda^{-1})\).

(iv) \(n_l \leq (n - \rho) \eta_{\max}^3\) and \(n_r \leq (m - \rho) \epsilon_{\max}\), where \(\eta_{\max}\) and \(\epsilon_{\max}\) are the maximal of the left and right minimal indices of \(\lambda M_\alpha - N_\alpha\).

**Proof.** Point (i) is obvious. Point (ii) can be obtained by summing ranks of diagonal matrices. To prove point (iii), note that the multiplicity \(n_\infty\) of the zero at the origin in \(M_\alpha - \lambda N_\alpha\) coincides with the multiplicity of the zero at the origin in \(\lambda \Phi(\lambda^{-1})\). But the matrix function \(\lambda \Phi(\lambda^{-1})\) has \(\rho + \pi_0\) zeros at the origin. The last fact is obvious

\[\text{Equality holds iff } \eta_i = 1 \text{ for all } i, \text{ where } \eta_i \text{ are the left minimal indices of } \lambda M_\alpha - N_\alpha.\]
if we apply McMillan form (1.4) to the rational function \(\Phi(\lambda)\). Point (iv) holds because the number of left minimal indices of \(\lambda M_\alpha - N_\alpha\) is \(n - \rho\), and the number of right minimal indices is \(m - \rho\). \(\square\)

It is proved in [12], Theorem VI.1, that there exists a proper \(C_\alpha\)-deflating subspace of the matrix pencil \(\lambda M_\alpha - N_\alpha\) if and only if \(n_r + k_\alpha > 0\). The maximal dimension of proper \(C_\alpha\)-deflating subspace is \(n_r + k_\alpha\).

The previous argumentation is sufficient for the proof of the following theorem.

**Theorem 1.** Given polynomial matrix \(P\), and indices \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\), satisfying (3.1). Let \(\begin{bmatrix} V_1^T & V_2^T & V_3^T \end{bmatrix}^T\) be a basis of a maximal proper \(C_\alpha\)-deflating subspace of the pencil \(\lambda M_\alpha - N_\alpha\) (4.1). Suppose that it is disconjugate. Then the matrices \(X, F_\alpha\), given by \(X = V_2 V_1^{-1}\), \(F_\alpha = -V_3 V_1^{-1}\), and \(\overline{F}_\beta\) (2.4), define a spectral factorization (3.17) such that the zeros of the right spectral factor \(P_\alpha\) are in \(C_\alpha\).

A proper \(C_\alpha\)-deflating subspace of the pencil \(\lambda M_\alpha - N_\alpha\) can be found by the deflation algorithm in Section VIII of [12] for extracting the subspace that corresponds to the finite generalized eigenvalues and assigning in \(C_\alpha\) zeros of the right spectral factor in (3.17). If the deflating subspace corresponds to the finite generalized eigenvalues of \(\lambda M_\alpha - N_\alpha\) only (\(n_r = 0\)), then only orthogonal transformations matrices require.

Disadvantages of Theorem 1 are that we have not a guaranty that the zeros of the left spectral factor \(P_\beta\) lie in \(C_\beta\), and that the origin can be a zero of one of the spectral factors, even it is not a zero of \(P\). Connected with the above disadvantages is that the (two) dimensions of the matrix \(\lambda M_\alpha - N_\alpha\) are not minimal, hence of the indices \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\).

Analogously to Theorem 1, consider the following matrix pencil, associated to ARS (2.6)-(2.8),

\[
\lambda M_\beta - N_\beta = \lambda \begin{bmatrix} I_{\beta} & 0 & 0 \\ 0 & I_{\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A_\beta & 0 & B_\beta \\ -\Sigma_{11}^T & -A_\alpha^T & -\Sigma_{21}^T \\ \Sigma_{12}^T & B_\alpha^T & \Sigma_{22}^T \end{bmatrix}. \tag{4.6}
\]

It can be verified that this matrix pencil is strictly equivalent to the pencil \((\lambda M_\alpha - N_\alpha)^\sim\). The following theorem is analogous to Theorem 1.

**Theorem 2.** Given polynomial matrix \(P\), and indices \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\) satisfying (3.1). Let \(\begin{bmatrix} V_1^T & V_2^T & V_3^T \end{bmatrix}^T\) be a basis of a maximal proper \(C_\beta\)-deflating subspace of the pencil \(\lambda M_\beta - N_\beta\) (4.6). If it is disconjugate, the matrices \(X, F_\beta\), given by \(X = V_1^{-T} V_2^T\), \(F_\beta = -V_3 V_1^{-1}\), and \(\overline{F}_\alpha\) (2.7), define a spectral factorization (3.18) such that the zeros of the left spectral factor \(P_\beta\) are in \(C_\beta\).

Theorem 2 has the same disadvantages as Theorem 1.
In the next Theorems 3 and 4 we overcome the two disadvantages of Theorems 1 and 2, corresponding to correct location of zeros, and the origin not to be a zero of the spectral factors. The algorithm remains the same, we only narrow the class of polynomial matrices $P$ and the type of spectral factorization.

**Theorem 3** ($P$ full column rank and $P_{\alpha}$ square). Given polynomial matrix $P$ that is of full column rank, and indices $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$, satisfying (3.1). Let $\begin{bmatrix} V_1^T & V_2^T & V_3^T \end{bmatrix}^T$ be a basis of a maximal proper $C_{\alpha}$-deflating subspace of the pencil $\lambda M_{\alpha} - N_{\alpha}$ (4.1). Suppose that it is disconjugate. Then the matrices $X, F_{\alpha}$, given by $X = V_2 V_1^{-1}$, $F_{\alpha} = -V_3 V_1^{-1}$, and $\overline{F}_{\beta}$ (2.4), define a spectral factorization (3.17) such that the zeros of the right spectral factor $P_{\alpha}$ are in $C_{\alpha}$, and the zeros of the left spectral factor $P_{\beta}$ are in $C_{\beta}$. If the origin is not a zero of $P$, then it is not a zero of the spectral factors $P_{\alpha}$ and $P_{\beta}$.

**Proof.** With regard to Theorem 1, we have to prove only that the zeros of the left spectral factor $P_{\beta}$ are in $C_{\beta}$. Indeed, since $P$ has full column rank, the zeros of $P$ are the zeros of $P_{\alpha}$ together with the zeros of $P_{\beta}$. Since the zeros of $P_{\alpha}$ are in $C_{\alpha}$, it follows that the zeros of $P_{\beta}$ have to be in $C_{\beta}$. □

**Remark 4.1.** If the origin is not a zero of $P$, then it is not a zero of the spectral factors $P_{\alpha}$ and $P_{\beta}$, although it may be a zero of the matrix pencil $\lambda M_{\alpha} - N_{\alpha}$. If the origin is a zero of $P$ of some multiplicity, then the matrix pencil $\lambda M_{\alpha} - N_{\alpha}$ may have a zero at the origin with different multiplicity.

The following theorem is analogous to Theorem 3.

**Theorem 4** ($P$ full row rank and $P_{\beta}$ square). Given polynomial matrix $P$ that is of full row rank, and indices $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ satisfying (3.1). Let $\begin{bmatrix} V_1^T & V_2^T & V_3^T \end{bmatrix}^T$ be a basis of a maximal proper $C_{\beta}$-deflating subspace of the pencil $\lambda M_{\beta} - N_{\beta}$ (4.6). If it is disconjugate, the matrices $X, F_{\beta}$, given by $X = V_1^{-1} V_2^T$, $F_{\beta} = -V_3 V_1^{-1}$, and $\overline{F}_{\alpha}$ (2.7), define a spectral factorization (3.18) such that the zeros of the left spectral factor $P_{\beta}$ are in $C_{\beta}$, and the zeros of the right spectral factor $P_{\alpha}$ are in $C_{\alpha}$. If the origin is not a zero of $P$, then it is not a zero of the spectral factors $P_{\alpha}$ and $P_{\beta}$.

5. Nonderogatory matrices, minimal realization, and minimal indices in spectral factorization

In this section at first we discuss on solving the problem of correct location of zeros of possibly non-full-rank rectangular matrices $P$. For that purpose we introduce the notion of nonderogatory matrices. Secondly, with an additional assumption of
minimality of realization, we solve the problem of absence of zeros at the origin in the spectral factors.

For given indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) satisfying (3.1), define the matrix
\[
P_L = \lim_{|\lambda| \to \infty} D_\beta \sim P D_\alpha^{-1},
\]
(5.1) where the matrices \( D_\alpha \) and \( D_\beta \) are given in (3.11) and (3.13). It is easy to see that the matrix \( P_L \) is finite if and only if the conditions (3.1) hold true.

There is a connection between the matrix \( \Sigma \) defined in (3.15) and matrix \( P_L \) defined in (5.1). Namely,

Proposition 8. \( \Sigma_{22} = P_L \), where same indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) are used in the definition of \( \Sigma_{22} \) and \( P_L \).

Proof. Pre- and post-multiplying (3.16) by \( D_\sim \beta \) and \( D_\alpha \), having in mind that
\[
\lim_{|\lambda| \to \infty} T_\alpha D_\alpha^{-1} = 0 \quad \text{and} \quad \lim_{|\lambda| \to \infty} T_\beta D_\beta^{-1} = 0,
\]
we have \( \Sigma_{22} = P_L \). \( \square \)

Definition 4. The polynomial matrix \( P \) is nonderogatory for indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) if the matrix \( P_L \) (5.1) is finite and has rank equal to the rank of \( P \).

There are nonderogatory matrices. For example, if \( P(\lambda) \) is generalized column reduced (Definition 1), we take the indices \( \alpha_i, \ i=1, \ldots, m \) equal to column degrees of \( P \) and \( \beta_1 = \cdots = \beta_n = 0 \). If \( P(\lambda) \) is generalized row reduced, we take \( \beta_j, \ j=1, \ldots, n \) equal to row degrees of \( P \) and \( \alpha_1 = \cdots = \alpha_m = 0 \). If \( P(\lambda) \) is diagonally reduced para-hermitian matrix \([14] \), we take \( \alpha_i \) equal to half diagonal degrees of \( P \) and \( \beta_i = \alpha_i, \ i=1, \ldots, m \).

On the other hand, not each polynomial matrix \( P \) is nonderogatory. Namely,

Proposition 9. If \( P \) has zeros at infinity then it is not nonderogatory for any choice of indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \).

Proof. Assume that \( P \) is nonderogatory for some indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \). By Definition 4, \( \Phi = -D_\beta \sim P D_\alpha^{-1} \) has not zeros at infinity. By \( P = -D_\beta \sim \Phi D_\alpha \), it follows that \( P \) has not zeros at infinity. \( \square \)

Example 5.1. By this example we show that the converse of Proposition 9 does not hold. Let
\[
P(\lambda) = \begin{bmatrix} \lambda + 1 & \lambda \\ \lambda & \lambda \end{bmatrix}.
\]
This polynomial matrix is not nonderogatory for any choice of indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \), but it has not zeros at infinity. Indeed, matrix \( P^{-1} \) is proper rational.
After these preliminary results, we are ready to elaborate on spectral factorization of possibly non-full-rank rectangular matrices.

Suppose that \( P \) is nonderogatory for the indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \). Then \( \Phi(\infty) = -P_L \) and \( \pi_0 = 0 \). By point (iii) of Proposition 7, we have \( n_\infty = \rho \), and by point (ii) we have
\[
n_\alpha + n_\beta = n_r + n_l + k_\alpha + k_\beta.
\]
If we take \( n_\alpha = n_r + k_\alpha \) (corresponds to extracting a maximal dimension proper \( C_\alpha \)-deflating subspace), then \( n_\beta = n_l + k_\beta \leq (n - \rho) \eta_{\max} + k_\beta \) (by point iv) of Proposition 7).

Under the condition (5.2), we shall prove that \( l_\beta \leq k_\beta \), where \( l_\beta \) is the number of zeros of \( P_\beta \) in (3.17). Since the rank of \( P_\beta \) is \( \rho \), there is at least one nonzero \( \rho \)-order minor of \( P_\beta \). The zeros of \( P_\beta \) have to annihilate all its nonzero \( \rho \)-order minors. Choose a nonzero \( \rho \)-order minor of minimal degree. Its degree is \( \leq \beta_{i_1} + \cdots + \beta_{i_\rho} \) for some indices \( i_1, \ldots, i_\rho \in \{1, \ldots, n\} \). Denote \( I = \{i_1, \ldots, i_\rho\} \) and \( J = \{1, \ldots, n\} \setminus I \), and let
\[
\beta_{J_{\min}} = \min_{i \in J} \beta_i \geq \eta_{\max}.
\]

Since the zeros of \( P_\beta \) are zeros of the chosen \( \rho \)-order minor of minimal degree, the number of zeros \( l_\beta \) of \( P_\beta \) satisfies
\[
l_\beta \leq \sum_{i \in I} \beta_i = n_\beta - \sum_{i \in J} \beta_i \leq (n - \rho) \eta_{\max} + k_\beta - \sum_{i \in J} \beta_i
\]
\[
\leq (n - \rho) \eta_{\max} + k_\beta - (n - \rho) \beta_{J_{\min}}
\]
\[
= (n - \rho)( \eta_{\max} - \beta_{J_{\min}} ) + k_\beta \leq k_\beta.
\]

All eigenvalues of \( N_\alpha \) and \( N_\beta \) have to be zeros of \( P \), except maybe the origin. From the fact that the zeros of \( P_\alpha \) can not be others than the eigenvalues of \( N_\alpha \) union the assigned \( n_r \) zeros in \( C_\alpha \), it follows that the factor \( P_\beta \) have to have (at least) the \( k_\beta \) eigenvalues of \( N_\beta \). However, from the proven fact \( l_\beta \leq k_\beta \) it follows that \( P_\beta \), as zeros, has only the eigenvalues of \( N_\beta \).

Still the number of zeros in the spectral factors is not minimal, because the origin may be their zero. To remove it, we state an assumption of minimality of the realization \((A, B, C, D)\) in (4.3) and (4.4). The conditions
\[
\text{rank}
\begin{bmatrix}
\lambda I - A_\alpha & 0 \\
\Sigma_{11} & \lambda I + A_\beta^T & -B_\alpha \\
\Sigma_{12} & & \Sigma_{22}
\end{bmatrix} = n_\alpha + n_\beta, \quad \forall \lambda \in \mathbb{C},
\]
(5.4)

and
\[
\text{rank}
\begin{bmatrix}
\lambda I - A_\alpha \\
\Sigma_{11} & \lambda I + A_\beta^T \\
-\Sigma_{21} & -B_\beta^T
\end{bmatrix} = n_\alpha + n_\beta, \quad \forall \lambda \in \mathbb{C}
\]
(5.5)

\footnote{A strict inequality \(<\) can hold if the rows of \( \Sigma_{22} \) for the indices \( i_1, \ldots, i_\rho \) are linearly dependant.}
are actually necessary and sufficient conditions for controllability and observability, respectively, of the realization \((A, B, C, D)\), given by (4.3) and (4.4), of the transfer matrix \(\Phi\).

We shall prove that the controllability condition (5.4) is equivalent to the linear independence of the rows of the matrix \(P_0\), and the observability condition (5.5) is equivalent to the linear independence of the columns of the matrix \(P_0\). Let us introduce \(n \times m\)-dimensional matrix \(\sigma\) that obtains from the matrix \(\Sigma_{11}\) by retaining the first, \(\alpha_1 + 1\), \(\alpha_1 + \alpha_2 + 1\), \ldots column and the first, \(\beta_1 + 1\), \(\beta_1 + \beta_2 + 1\), \ldots row, and discarding the rest of its columns and rows. It is easy to check that the condition (5.4) is equivalent to the linear independence of the rows of the matrix \(\sigma\). Analogously, the condition (5.5) is equivalent to the linear independence of the columns of the matrix \(\sigma\).

On the other hand, since
\[
P(\lambda) = \begin{bmatrix}
\tilde{T}_\beta, & D_\beta
\end{bmatrix}
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix}
\begin{bmatrix}
T_\alpha \\
D_\alpha
\end{bmatrix},
\]
we have \(\sigma = \tilde{T}_\beta^T(0)\Sigma_{11}\tilde{T}_\alpha(0) = P(0) = P_0\).

**Theorem 5.** I. Assume

1. The polynomial matrix \(P\) is nonderogatory for the indices \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\).
2. \(\beta J_{\min} \geq \eta_{\max}\).
3. \(n_\alpha = n_r + k_\alpha\).
4. The maximal dimension proper \(C_\alpha\)-deflating subspace of the matrix pencil \(\lambda M_\alpha - N_\alpha\) (4.1) is disconjugate.

Let
\[
\begin{bmatrix}
V_1^T & V_2^T & V_3^T
\end{bmatrix}^T
\]
be a basis of that subspace. Then the matrices \(X, F_\alpha\), given by
\[
X = V_2V_1^{-1}, \quad F_\alpha = -V_3V_1^{-1}, \quad \text{and} \quad \tilde{F}_\beta\) (2.4), define a spectral factorization (3.17) such that the zeros of the right spectral factor \(P_\alpha\) are in \(C_\alpha\), and the zeros of the left spectral factor \(P_\beta\) are in \(C_\beta\).

II. The condition (2) can be replaced by the condition of full row rank of the matrix \(P\).

III. If \(P\) is square and \(\det P_0 \neq 0\), then the origin is not a zero of the pencil \(\lambda M_\alpha - N_\alpha\) (4.1) and of the spectral factors \(P_\alpha\) and \(P_\beta\).

**Proof.** We have proved the first (I) and the third part (III). Concerning the second (II) part, if the matrix \(P\) is of full row rank, then \(n_1 = 0\) and (5.3) holds without applying iv) of Proposition 7. □

**Remark 5.1.** In [25] there are column- and row-reduction algorithms of polynomial matrices by application of unimodular matrices. If we apply these algorithms preliminary to the spectral factorization algorithm of Theorem 5 to achieve the nonderogacy condition, the overall spectral factors \(P_\alpha\) or \(P_\beta\) may be non-column-reduced.
or non-row-reduced, respectively. Another difficulty with this procedure is that the column- and row-reduction algorithms may be numerically unstable since based on elementary column and row operations.

The following theorem is analogous to Theorem 5.

Theorem 6. I. Assume

1. The polynomial matrix $P$ is nonderogatory for the indices $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$.
2. $\alpha_{J_{\min}} \geq \epsilon_{\max}$, where $\alpha_{J_{\min}}$ is defined analogously with $\beta_{J_{\min}}$.
3. $n_{\beta} = n_1 + k_{\beta}$.
4. The maximal dimension proper $C_\beta$-deflating subspace of the matrix pencil $\lambda M_{\beta} - N_{\beta}$ (4.6) is disconjugate.

Let $[V_1^T V_2^T V_3^T]^T$ be a basis of that subspace. Then the matrices $X, F_{\beta}$, given by $X = V_1^{-T}V_2^T$, $F_{\beta} = -V_3V_1^{-1}$, and $F_{\alpha}$ (2.7), define a spectral factorization (3.18) such that the zeros of the left spectral factor $P_{\beta}$ are in $C_{\beta}$, and the zeros of the right spectral factor $P_{\alpha}$ are in $C_{\alpha}$.

II. The condition (2) can be replaced by the condition of full column rank of the matrix $P$.

III. If $P$ is square and $det P_0 \neq 0$, then the origin is not a zero of the pencil $\lambda M_{\beta} - N_{\beta}$ (4.6) and of the spectral factors $P_{\alpha}$ and $P_{\beta}$.

Remark 5.2. The conditions (2) of Theorems 5 and 6 can be checked without computing the spectral factors and their zeros.

Remark 5.3. If $P$ has zeros at the origin but is nonsingular, minimal realization can be achieved by considering matrices

$$A_{\alpha i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{i1}^{(\alpha)} & a_{i2}^{(\alpha)} & a_{i3}^{(\alpha)} & \cdots & a_{i,\alpha}^{(\alpha)} \end{bmatrix}$$

$$A_{\beta i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{i1}^{(\beta)} & a_{i2}^{(\beta)} & a_{i3}^{(\beta)} & \cdots & a_{i,\beta}^{(\beta)} \end{bmatrix}$$
instead of (3.6) and (3.8), where the eigenvalues of matrices $A_{\alpha i}$ and $A_{\beta i}$ are pairwise distinct of the zeros of $P$.

The only unsolved up-to-now problem is the condition (3) of Theorems 5 and 6. To elaborate it, at first we prove the following proposition.

**Proposition 10.** Let $\alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_n$ be indices that satisfy (3.1). If we choose new indices $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ equal to the indices with prime except $\beta_i = \beta'_i + 1$, then the $i$th row of the new matrix $P_L$ is zero. If we choose $\beta_i = \beta'_i + 2$, instead of $\beta_i = \beta'_i + 1$, then the $i$th row of the new matrix $P_L$ is zero and the realization of the new transfer function $PL^{-1}P \phi_{i9818} = -D_\beta^{-}\sim P D_\alpha^{-1}$ is not minimal.

**Proof.** By (3.5) the $i$th row of the matrix $[\Sigma_{21}, \Sigma_{22}]$ is zero, hence of $P_L = \Sigma_{22}$. To prove the second part of the proposition, by (3.5) the row of $[\Sigma_{11}, \Sigma_{12}]$ that corresponds to the last row of the matrix $A^T_{\beta i}$ (3.8), is zero. Hence by choosing $\lambda = 0$ in (5.4), we annihilate that row of the matrix $[\Sigma_{11}, \lambda I + A^T_{\beta i}, \Sigma_{12}]$. The controllability condition (5.4) fails, hence the realization is not minimal. □

The previous proposition motivate us to introduce the following minimization problem. Given $n_\alpha = n_r + k_\alpha$, choose the indices $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ by solving the following ILP problem

$$\min_{\alpha_1 + \cdots + \alpha_m = n_\alpha \atop \beta_1 + \cdots + \beta_n = n} (\beta_1 + \cdots + \beta_n).$$

The problem (5.6) has a solution always. Namely, for given indices $\alpha_1, \ldots, \alpha_m$, one can find indices $\beta_1, \ldots, \beta_n$ such that (3.1) holds. One possible choice is $\beta_i \geq \max_{j \in \{1, \ldots, m\}} (\delta_{ij} - \alpha_j), \ i = 1, \ldots, n$. The following property of the solutions of (5.6) shows that $P_L(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$ may be of rank equal to the rank of $P$.

**Proposition 11.** If integers $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$, where $\beta_1, \ldots, \beta_n$ are strictly positive, are a solution of the ILP problem (5.6), then the matrix $P_L = P_L(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$ (5.1) is finite and no row of that matrix is zero.

**Proof.** Let $k$th row of $P_L$ be zero. It means that the strict inequalities $\beta_k + \alpha_j > \delta_{kj}, \ j = 1, \ldots, n$ hold. Take new indices $\alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_n$ all equal to $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ except the index $\beta'_k$ which we take equal to $\beta_k - 1$. The indices $\alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_n$ are feasible also and result in a less minimum, so the indices $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ are not solutions of (5.6). □

**Remark 5.4.** If some row of $P$ is constant, for example, if $\delta_{kj} = 0, \ j = 1, \ldots, n$, then from $\beta_k + \alpha_j \geq \delta_{kj} = 0, \ j = 1, \ldots, n$ follows that the optimal solution of (5.6) satisfies $\alpha_k = 0$. Then it is possible some row of $P_L$ to be zero.
If the realization that we obtain by indices found by (5.6) is not observable and/or the matrix \( \Sigma_{22} \) has rank less than the rank of \( P \), we try with the dual to (5.6) problem:

\[
\min_{\beta_1 + \cdots + \beta_n = n \beta, \beta_i + \alpha_j \geq \delta_{ij}} (\alpha_1 + \cdots + \alpha_m). \tag{5.7}
\]

If neither these indices result in a minimal realization and/or matrix \( \Sigma_{22} \) with rank equal to the rank of \( P \), then by Theorems 1 and 2, the spectral factorization (1.2) can exists, but the mentioned disadvantages of Theorems 1 and 2 may appear.

To find zeros of the polynomial matrix \( P \), we solve the following problem.

\[
\min_{\beta_i + \alpha_j \geq \delta_{ij}} \left( \sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{n} \beta_i \right). \tag{5.8}
\]

The problem (5.8) may have many solutions for \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \), although the minimum is unique. For example, let \( k \leq \min_{i \in \{1, \ldots, n\}} \beta_i \neq 0. \) Then the indices \( \alpha'_i = \alpha_i + k, \ i = 1, \ldots, m \) and \( \beta'_i = \beta_i - k, \ i = 1, \ldots, n \) are also solutions of (5.8).

The following proposition (proof omitted, because it is analogous to the proofs of Propositions 10 and 11) shows that the solutions of (5.8) have stronger properties than the solutions of (5.6) and (5.7).

**Proposition 12.** Let \( \alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_n \) be indices that are solutions of (5.8). If we choose new indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) equal to the indices with prime except \( \beta_i = \beta'_i + 1 \), then the \( i \)th row of the new matrix \( PL \) is zero. If we choose \( \alpha_i = \alpha'_i + 1 \), then the \( i \)th column of the new matrix \( PL \) is zero. If we choose \( \beta_i = \beta'_i + 2 \), then the \( i \)th row of the new matrix \( PL \) is zero and the realization of the new \( \Phi = -D_{\beta}^{-\sim} P D_{\alpha}^{-1} \) is not controllable. If we choose \( \alpha_i = \alpha'_i + 2 \), then the \( i \)th column of the new matrix \( PL \) is zero and the realization of the new \( \Phi = -D_{\beta}^{-\sim} P D_{\alpha}^{-1} \) is not observable.

If \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) are strictly positive integers that are a solution of the ILP problem (5.8), then the matrix \( PL = PL(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \) is finite and no column and row of that matrix is zero.

Solutions of (5.8) cannot be applied to the spectral factorization (1.2), because \( \sum_{i=1}^{m} \alpha_i \) have to be equal to \( n_r + k_\alpha \). However, they can be useful for the problem of obtaining zeros of polynomial matrices, and in particular, the number \( n_r + n_\alpha \), that requires in the spectral factorization.5

Before we summarize the obtained results in an algorithm, note that the matrix pencil (4.1) is strictly equivalent to the matrix pencil

---

5 It have to be noticed that there is a large class of polynomial matrices \( P \) for which the ILP problem (5.6) yields the same value of \( n_\alpha + n_\beta \) as (5.8) does.
\[ \begin{bmatrix}
L_{\beta_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & L_{\beta_n}
\end{bmatrix} \begin{bmatrix}
L_{\alpha_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & L_{\alpha_m}
\end{bmatrix}, \quad (5.9)
\]

where

\[ L_i = L_i(\lambda) = \begin{bmatrix}
\lambda & -1 & 0 & \cdots & 0 & 0 \\
0 & \lambda & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & \cdots & \lambda & -1
\end{bmatrix} \text{ for } i \text{ rows.}
\]

The pencil form (5.9) is more suitable for numerical calculations. (It is block-upper-triangular.)

A closed form of the algorithm is as follows.

**Algorithm of Theorem 5 for spectral factorization of a polynomial matrix**

\[ P(\lambda) \].

**Input:** Matrices \(P_0, P_1, \ldots, P_d\).

**Output:** Indices \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\), and matrices \(\Sigma_{22}, F_{\alpha}\) and \(\overline{F}_{\beta}\).

(i) Find indices \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\) by solving ILP problem (5.8).

(ii) Assign the coefficients in the matrix pencil (5.9) by the coefficients \(P_0, \ldots, P_d\).

(iii) Find the integer \(n_r + k_\alpha\) by performing the deflation algorithm on the matrix pencil (5.9). Re-compute new indices \(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\), such that \(n_\alpha = \sum_{i=1}^{m} \alpha_i = n_r + k_\alpha\) by solving the ILP problem (5.6).

(iv) Re-form the matrix pencil (5.9) with the new indices.

(v) Extract a maximal dimension proper \(C_{\alpha}\)-deflating subspace of the matrix pencil \(\lambda M_\alpha - N_\alpha\), and find the matrices \(X, F_{\alpha}, \overline{F}_{\beta}\).

For conciseness, we omit the analogous algorithm of Theorem 6.

**Remark 5.5.** In (3.17) and (3.18), the spectral factor \(P_{\alpha}\) is generalized column reduced with column indices \(\alpha_1, \ldots, \alpha_m\), and \(P_{\beta}\) is generalized row reduced with row indices \(\beta_1, \ldots, \beta_n\).

**Remark 5.6.** The problem with zeros in \(C_{\alpha} \cap C_{\beta}\) (typically-imaginary axis, or unit circle) of \(P(\lambda)\) is also solvable if the sufficient conditions of Theorems 1–6 hold (see Examples 5.2 and 5.3).
**Remark 5.7.** The algorithm can work with para-hermitian matrices $P$, for which $\alpha_1 = \beta_1, \ldots, \alpha_m = \beta_m$. In that case the matrix $\mathcal{F}$ in (3.5) is symmetric, $P_\beta = P_\alpha^\sim$, and the $J$-spectral factorization (3.17) or (3.18) is completed by congruency transformation of the symmetric matrix $\Sigma_{22}$ as $\Sigma_{22} = \hat{\Phi}^T J \hat{\Phi} = \hat{\Phi}^T \text{diag}(I_p, -I_q, 0) \hat{\Phi}$, for some nonsingular matrix $\hat{\Phi}$. Actually, the algorithm is a generalization of the $J$-spectral factorization algorithm of [19], where besides para-hermitian, $P$ is assumed nonsingular. The singularity of $P$ is a further progress of this paper, in respect to [19], as we illustrate by the following example.

**Example 5.2.** This example coincides with Example 3 in [5], although the solution is different, due to the different problem formulation. Given the para-hermitian matrix

$$P(\lambda) = \begin{bmatrix}
\lambda^4 + \lambda^2 + 1 & -\lambda & \lambda^4 + \lambda^2 - 2\lambda + 1 \\
\lambda & 1 & \lambda + 2 \\
\lambda^4 + \lambda^2 + 2\lambda + 1 & -\lambda + 2 & \lambda^4 + \lambda^2 + 5
\end{bmatrix}.$$ 

It is easy to check that this matrix is of generic rank $\rho = 2$, and its zeros are $\pm i$, of double multiplicity. To find spectral factorization of $P$ by our algorithm, find at first $\alpha_1 = \alpha_3 = \beta_1 = \beta_3 = 2, \alpha_2 = \beta_2 = 0$. Then, define

$$A_1 = A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b_1 = b_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_3 \end{bmatrix},$$

$$D = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \lambda \end{bmatrix},$$

where the indices $\alpha$ and $\beta$ in these matrices are omitted. We have

$$\mathcal{F}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{F}_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathcal{F}_{13} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{F}_{22} = 1, \quad \mathcal{F}_{23} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathcal{F}_{33} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

$$\Sigma_{11} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 1 & 1 & 5 & 0 \\ -1 & -1 & 0 & -1 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Sigma_{22} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$ 

We have rank$(\Sigma_{22}) = \rho = 2$, so the polynomial matrix $P$ is nonderogatory. The finite generalized eigenvalues of the associated pencil $\lambda M - N$ of $P$ are $\pm i$, of double
multiplicity. Also, \( \epsilon_1 = \eta_1 = 2, \) \( n_r = n_l = 2, \) \( k_\alpha + k_\beta = 4. \) Further we check the condition (2) of Theorem 5. We have \( J = \{1, 2\}, \) \( I = \{3\}, \) \( \beta_{J_{\text{min}}} = \beta_3 = 2, \) \( \eta_{\text{max}} = \eta_1 = 2. \) Therefore, the condition 2) of Theorem 5 is satisfied.

Choose arbitrary zeros of the spectral factor that will be adjusted as \(-1\) and \(-2.\) Compute 4-dimensional deflating subspace of \( \lambda M - N \) corresponding to \( \pm i, -1 \) and \(-2\) and compute matrices \( X = V_2V_1^{-1} \) and \( F = -V_3V_1^{-1} : \)

\[
X = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 2 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}, \quad
F = \begin{bmatrix}
8.4285 & -2.1440 & 6.4285 & -5.1440 \\
0 & 1 & 2 & 1 \\
-7.4285 & 2.1440 & -5.4285 & 5.1440
\end{bmatrix}.
\]

Introduce the matrix \( \Psi = D + F^T. \) Spectral factorization is \( P = \Psi^{-1} \Sigma_22 \Psi, \) where

\[
\Psi = \begin{bmatrix}
\lambda^2 - 2.1440\lambda + 8.4285 & 0 & -5.1440\lambda + 6.4285 \\
\lambda & 1 & \lambda + 2 \\
2.1440\lambda - 7.4285 & 0 & \lambda^2 + 5.1440\lambda - 5.4285
\end{bmatrix}.
\]

It can be checked that the zeros of the right spectral factor \( \Psi, \) i.e. the eigenvalues of \( A - BF \) are \( \pm i, -1 \) and \(-2, \) as required.

To the end of the section we specify the obtained results on full-rank matrices \( P, \) and square and nonsingular matrices \( P. \) At first note that if \( P \) is wide then \( n_l = 0, \) and if it is tall then \( n_r = 0. \) There are four cases of factorizations:

(i) \( P \) is tall, \( P_\beta \) is tall and \( P_\alpha \) is square. Theorem 3 applies, no zero assignment necessary.

(ii) \( P \) is wide, \( P_\beta \) is square and \( P_\alpha \) is wide. Theorem 4 applies, no zero assignment necessary.

(iii) \( P \) is wide, \( P_\beta \) is wide and \( P_\alpha \) is square. Theorem 5 applies, zero assignment necessary.

(iv) \( P \) is tall, \( P_\beta \) is square and \( P_\alpha \) is tall. Theorem 6 applies, zero assignment necessary.

Cases (i) and (ii) can be solved by only orthogonal transformations. If the polynomial matrix \( P \) is square \( (n = m) \) and nonsingular and it is nonderogatory for the indices \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \) satisfying \( n_\alpha = \sum_{i=1}^m \alpha_i = k_\alpha, \) then by Proposition 8, there exists an inverse of \( \Sigma_22. \) It is easy to check that

\[
Q(\lambda M_\alpha - N_\alpha) = \begin{bmatrix}
\lambda I - H_\alpha & 0 \\
R & I
\end{bmatrix},
\]

where

\[
Q = \begin{bmatrix}
I & 0 & -B_\alpha \Sigma_22^{-1} \\
0 & I & \Sigma_21 \Sigma_22^{-1} \\
0 & 0 & -\Sigma_22^{-1}
\end{bmatrix}, \quad
R = \begin{bmatrix}
\Sigma_22^{-1} \Sigma_21, \quad \Sigma_22^{-1} B_\beta^T
\end{bmatrix}.
\]
\[ H_\alpha = \begin{bmatrix} A_\alpha - B_\alpha \Sigma_{22}^{-1} \Sigma_{21} & -B_\alpha \Sigma_{22}^{-1} B_\beta^T \\ -\Sigma_{11} + \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} & -\left( A_\beta - B_\beta \Sigma_{22}^{-1} \Sigma_{21} \right)^T \end{bmatrix}. \] (5.10)

The subspace \([V_1^T \ V_2^T \ V_3^T]^T\) is \(C_\alpha\)-deflating for the matrix pencil \(\lambda M_\alpha - N_\alpha\) if and only if the subspace \([V_1^T \ V_2^T]^T\) is \(C_\alpha\)-invariant for the matrix (5.10) and \(V_3 = -\Sigma_{22}^{-1} \Sigma_{21} V_1 - \Sigma_{22}^{-1} B_\beta^T V_2\). Therefore the problem of obtaining deflating subspaces (operating on matrix pencils) simplifies into a problem with invariant subspaces (operating on matrices).

The maximal \(C_\alpha\)-invariant subspace of the matrix \(H_\alpha\) can be found by orthogonal similarity transformation matrices (\(QR\) algorithm).

**Remark 5.8.** For square polynomial matrices with nonsingular \(\Sigma_{22}\), the matrix \(X\) in Theorem 5 satisfies the non-symmetric algebraic Riccati equation (ARE)

\[ A_\beta^T X + X A_\alpha - (X B_\alpha + \Sigma_{12}) \Sigma_{22}^{-1} \left( B_\beta^T X + \Sigma_{21} \right) + \Sigma_{11} = 0. \] (5.11)

See [18] for results on existence of a solution of (5.11).

Analogously, consider the following matrix, associated to the matrix pencil (4.6)

\[ H_\beta = \begin{bmatrix} A_\beta - B_\beta \Sigma_{22}^{-1} \Sigma_{12}^T & -B_\beta \Sigma_{22}^{-1} B_\alpha^T \\ -\left( \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \right)^T & -\left( A_\alpha - B_\alpha \Sigma_{22}^{-1} \Sigma_{21} \right)^T \end{bmatrix}. \] (5.12)

It is easy to conclude that \(H_\beta = -H_\alpha^T\). In this case the problem of obtaining \(C_\beta\)-deflating subspace of the matrix pencil (4.6) in Theorem 6 reduces to obtaining \(C_\beta\)-invariant subspace of the matrix (5.12).

**Example 5.3.** Given polynomial matrix

\[ P(\lambda) = \begin{bmatrix} -1 & \lambda^3 + \lambda/2 + 1 \\ -\lambda^3 - \lambda + 1 & \lambda^4 - 3\lambda^2/4 - 1 \end{bmatrix}. \]

This polynomial matrix has singular matrices \(P_0\) and \(P_d\), hence the algorithm of [3] cannot be applied for obtaining the zeros of \(P\). The algorithm of [14] also cannot be applied because the matrix \(P\) is neither column nor row reduced. Choose for \(C_\alpha\) the right complex half-plane and for \(C_\beta\) the left half-plane. It is easy to see that the indices \(\alpha_1 = \beta_1 = 1, \alpha_2 = \beta_2 = 2\) yield matrix

\[ P_L = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}, \]

that is nonsingular. The nonderogacy condition is satisfied. Further we have

\[ \mathcal{F}_{11} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{F}_{12} = \begin{bmatrix} 1 & 0.25 & 0 \\ -0.25 & 0 & -1 \end{bmatrix}, \]

\[ \mathcal{F}_{22} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0.75 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]
By permutation of \( \mathcal{F} \) we obtain the matrix \( \Sigma \):

\[
\Sigma_{11} = \begin{bmatrix}
-1 & 1 & 0.25 \\
1 & -1 & 0 \\
0.5 & 0 & 0.75
\end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix}
0 & 0 \\
-0.5 & 0 \\
0 & 0
\end{bmatrix},
\]

\[
\Sigma_{22} = \begin{bmatrix}
0 & -1 \\
-1 & 1
\end{bmatrix} = P_L.
\]

The matrices \( A_{\alpha}, B_{\alpha}, A_{\beta} \) and \( B_{\beta} \) are

\[
A_{\alpha} = A_{\beta} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad B_{\alpha} = B_{\beta} = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}.
\]

The eigenvalues of the matrix \( H_{\alpha} \) (5.10) are

\[
0.7481 \pm 0.8211i, \quad -0.5614 \pm 0.8777i, \quad -0.3734 \quad \text{and} \quad 0.
\]

Choose \( 0 \in \mathbb{C}_{\alpha} \). The \( \mathbb{C}_{\alpha} \)-solution of the ARS (2.2), (2.3) and (2.4) is

\[
X = \begin{bmatrix}
-0.6684 & 0.6475 & -0.6715 \\
1.1684 & -0.8975 & 0.6715 \\
-0.1562 & 0.6992 & 2.6889
\end{bmatrix},
\]

\[
F_{\alpha} = \begin{bmatrix}
0.8246 & -1.0968 & -2.0174 \\
0.6684 & -0.3975 & 0.6715 \\
-0.6684 & 0.6684 & -0.1562 \\
-0.6715 & 0.6715 & 2.6889
\end{bmatrix},
\]

\[
\overline{F}_{\beta} = \begin{bmatrix}
-0.6684 & 0.6475 & -0.6715 \\
1.1684 & -0.8975 & 0.6715 \\
-0.1562 & 0.6992 & 2.6889
\end{bmatrix}.
\]

The spectral factors \( P_{\alpha} \) and \( P_{\beta} \) are

\[
P_{\alpha} = \begin{bmatrix}
0.8246 + \lambda & -1.0968 - 2.0174\lambda \\
0.6684 & -0.3975 + 0.6715\lambda + \lambda^2
\end{bmatrix},
\]

\[
P_{\beta} = \begin{bmatrix}
-0.6684 & -0.6715 + \lambda \\
0.6684 + 0.1562\lambda - \lambda^2 & 0.6715 - 2.6889\lambda + \lambda^2
\end{bmatrix}.
\]

6. Applications

6.1. Inner–outer factorization

Given polynomial matrix \( Q(\lambda) \), a problem of inner–outer factorizations we define as finding inner rational matrix \( G_i \) and outer polynomial matrix \( Q_o \) such that

\[
Q(\lambda) = G_i(\lambda)Q_o(\lambda).
\]

(6.1)

A rational matrix \( G_i \) is inner if it is analytic in \( \mathbb{C}_{\alpha} \) and \( G_i^{-1}G_i = I \). A polynomial matrix \( Q_o \) is outer if it is of full row rank and have its zeros in \( \mathbb{C}_{\alpha} \).
The definition (6.1) holds for discrete-time case, namely, if $G_i$ is a discrete-time rational matrix, functions $G_i\sim$ are defined by $G_i\sim = G_i^T(\lambda^{-1})$.

Note that the definition (6.1) is a generalization of the usual definition of inner–outer factorization of rational matrices in [8, 11, 15, 16, 23, 24]. Indeed, if $G$ is a rational transfer matrix of the form $G = QH^{-1}$, in the usual inner–outer factorization $G = G_1G_0$ the inner factor remains the same, while the outer factor is $G_0 = Q_0H^{-1}$.

The inner–outer factorization (6.1) may be of minimal order or not. It is minimal if the number of poles of $G_i$ is minimal.

There are algorithms for inner–outer factorization [8, 11, 15, 16, 23, 24], mostly based on a state space. We develop a polynomial algorithm. The algorithm works with continuous-time, as well as with discrete-time systems.

If $Q$ is of full column rank, the following operation is avoid. Let $U$ be an unimodular polynomial matrix, with inverse denoted by $V = U^{-1}$, such that

\[ QU = [P, 0], \]  

where $P$ is of full column rank.

We can apply the algorithm of Theorem 3 and factorize the polynomial matrix $P$ on

\[ P = P_\beta P_\alpha, \]  

so that the zeros of $P_\alpha$ are in $C_\alpha$ and the zeros of $P_\beta$ are in $C_\beta$. Namely, by only orthogonal transformation matrices we extract the maximal proper $C_\alpha$-deflating subspace, that corresponds to finite generalized eigenvalues of the matrix pencil $\lambda M_\alpha - N_\alpha$.

The polynomial matrix $P_\alpha$ is square and nonsingular.\(^6\) Further we have

\[
Q\sim Q = V^\sim \begin{bmatrix} P^\sim & 0 \\ 0 & 0 \end{bmatrix} V = V^\sim \begin{bmatrix} P\sim P & 0 \\ 0 & 0 \end{bmatrix} V = V^\sim \begin{bmatrix} P_\alpha P_\sim^\beta P_\sim^\beta P_\beta P_\alpha & 0 \\ 0 & 0 \end{bmatrix} V.
\]

Compute the polynomial factorization of the para-hermitian matrix (continuous-time or discrete-time)

\[ P_\sim^\beta P_\sim = R^\sim R, \]  

where $R$ is square and nonsingular polynomial matrix with zeros in $C_\alpha$. Then

\[ Q_0 = [RP_\alpha, 0]V. \]  

The inner factor should be computed by solving the equation (6.1), i.e.

\[ [P_\beta P_\alpha, 0]V = G_i[RP_\alpha, 0]V. \]

Hence

\[ G_i = P_\beta R^{-1}. \]  

\(^6\) Theorem 1 can be applied also. Namely, for the inner–outer factorization, we do need the zeros of $P_\beta$ to be in $C_\beta$. However, in this case there is not a guaranty for minimality of the number of poles of the inner factor.
Since the extracted $C_\alpha$-deflating subspace is maximal, the number of zeros of $G_i$ is minimal. If $Q$ is of full row rank, the poles of $G_i$ equal the mirrors of the un-stable zeros of $Q$. Otherwise, new poles in $G_i$ are introduced. Namely, the zeros of $P_\beta^{-}\tilde{P}_\beta$ are equal to the zeros of $P_\beta$ union their images only if $P_\beta$ is square, i.e. $Q$ is of full row rank. If $Q$ is not of full row rank the zeros of $P_\beta$ union their images are only a subset of the zeros of $P_\beta^{-}\tilde{P}_\beta$.

The presented algorithm is based on three numerically reliable computations: (i) matrix triangularization in (6.2) (see [9]), (ii) extracting a deflating subspace in (6.3) by orthogonal transformations, and (iii) polynomial spectral factorization in (6.4), i.e. extracting an invariant subspace of a continuous- or discrete-time Hamiltonian matrix by orthogonal transformations.

A variant of the preceding inner–outer factorization is the following problem: Given polynomial matrix $Q$, find its factorization

$$Q = L_i \begin{bmatrix} Q_o & 0 \end{bmatrix}$$

(6.7)

where $L_i$ is square inner rational matrix, and $Q_o$ is an outer polynomial matrix. With reference to the above inner–outer factorization algorithm, partition $P_\beta$ in (6.3) as $P_\beta = \begin{bmatrix} P_{\beta_1}^T, P_{\beta_2}^T \end{bmatrix}^T$, and suppose that the polynomial matrix $P_{\beta_1}$ is nonsingular. (If it is not, we can pre-numerate the row indices of $Q$, so that the matrix $P_{\beta_1}$ is nonsingular.) Compute the following polynomial fractioning

$$P_{\beta_2} P_{\beta_1}^{-1} = S_2^{-1} S_1$$

for some coprime polynomial matrices $S_1$ and $S_2$ (see [6] for an algorithm), and compute the polynomial spectral factorization

$$S_1^{-}\tilde{S}_1 + S_2^{-}\tilde{S}_2 = \tilde{S}_3 S_3,$$

where the zeros of $S_3$ are in $C_\alpha$. Then the required factorization (6.7) consists of $Q_o$ (6.5) and

$$L_i = [G_i, H_i], \quad H_i = \begin{bmatrix} -\tilde{S}_1 & \tilde{S}_2 \end{bmatrix} S_3^{-1},$$

where $G_i$ is given by (6.6).

### 6.2. Greatest common divisor of polynomial matrices

To find the greatest common divisor to the right of given polynomial matrices $P_1, \ldots, P_k$, form the matrix $P = \begin{bmatrix} P_1^T, \ldots, P_k^T \end{bmatrix}^T$. By the algorithms of Theorems 1, 3 and 5, factorize the matrix $P$ on $P = P_\beta P_\alpha$, by extracting a maximal dimension $C$-deflating subspace. Then the greatest common divisor is $P_\alpha$. If the matrix $P$ is of full column rank, then the extracting of the deflating subspace can be done by orthogonal transformations only. This method differs from the standard methods for extracting greatest common divisor (see [9] for the triangularization method).
7. Conclusions

In this paper we have presented a numerical algorithm for polynomial spectral factorization, based on finding invariant or deflating subspace by application of orthogonal or near-orthogonal transformation matrices.

The question of disconjugacy, i.e. nonsingularity of matrix $V_1$, related to existence of a solution of ARS and to existence of spectral factorization, is not solved in this paper. Special cases (sufficient conditions) are solved:

(1) in case $P$ is para-hermitian and positive-definite on the imaginary axis [24], and on the unit circle [11],

(2) in case $\det \Sigma_2 \neq 0$ [18],

(3) in case $P$ is a scalar polynomial [4].

How can we generalize the conditions of these works?

References


