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# Infinitely many homoclinic solutions for nonautonomous $p(t)$-Laplacian Hamiltonian systems[☆]

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ABSTRACT

By using the Symmetric Mountain Pass Theorem, we establish some existence criteria which guarantee that the second-order ordinary $p(t)$-Laplacian systems of the form $\frac{d}{dt}\left(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)\right) - a(t)|u(t)|^{p(t)-2}u(t) + \nabla W(t, u(t)) = 0$ have infinitely many homoclinic solutions, where $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $p \in C(\mathbb{R}, \mathbb{R})$ and $p(t) > 1$, $a \in C(\mathbb{R}, \mathbb{R})$, and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ are non-periodic in $t$.



## 1. Introduction

In recent years there has been an increasing interest in the study of various mathematical problems with variable exponents. The main references in this field can be found in an overview paper [1]. For the applications of the $p(t)$-Laplacian equations, we refer to the works [2–4]. The existence of solutions of $p(t)$-Laplacian Dirichlet problems has been studied by several authors (see e.g. [5–7]). The purpose of the present paper is to study the homoclinic solution of the second-order ordinary $p(t)$-Laplacian system

$$\frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) - a(t)|u(t)|^{p(t)-2}u(t) + \nabla W(t, u(t)) = 0, \tag{1.1}$$

where $p \in C(\mathbb{R}, \mathbb{R})$ and $p(t) > 1$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $a : \mathbb{R} \to \mathbb{R}$, and $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. This is relatively a new topic for study. As usual, we say that a solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t) \to 0$ as $t \to \pm\infty$. In addition, if $u(t) \not\equiv 0$ then $u(t)$ is called a nontrivial homoclinic solution.

System (1.1) has been studied by Fan et al. in a series of papers [5–7]. Such $p(t)$-Laplacian systems have been applied to describe the physical phenomena with "pointwise different properties"; in particular, (1.1) first arose in the nonlinear

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elasticity theory (see [8]). As expected, the $p(t)$-Laplacian operator possesses more complicated nonlinearity than that of the $p$-Laplacian, for example, it is not homogeneous, this causes many troubles, and some classic theories and methods, such as the theory of Sobolev spaces, are not applicable.

It is well-known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic orbits of (1.1) emanating from 0.

If $p(t) \equiv p$ is a constant, system (1.1) reduces to the ordinary $p$-Laplacian system

$$\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0. \tag{1.2}$$

In recent years, the existence and multiplicity of homoclinic orbits for Hamiltonian systems have been investigated in many papers via variational methods, and several results have been obtained based on various hypotheses on the potential functions when $p = 2$, see, e.g., [9–16]. For the system (1.2), if $a(t)$ and $W(t, x)$ are $T$-periodic in $t$, Rabinowitz [16] showed the existence of homoclinic orbits as a limit of $2kT$-periodic solutions. The related results can be found in [17–21]. If $a(t)$ and $W(t, x)$ are non-periodic in $t$, the problem of existence of homoclinic orbits for the system (1.2) is quite different from the ones just described, because of the lack of compactness of the Sobolev embedding. In [22], Rabinowitz and Tanaka studied (1.2) with $p = 2$ without a periodicity assumption. Their main result can be stated as follows.

**Theorem A** (*[22]*). *Assume that a and W satisfy the following conditions:*

(A) $a \in C(\mathbb{R}, (0, \infty))$ *and* $a(t) \to +\infty$ *as* $|t| \to \infty$.

(W1) $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *and there is a constant* $\mu > 2$ *such that*

$$0 < \mu W(t, x) \le (\nabla W(t, x), x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}.$$

(W2) $|\nabla W(t, x)| = o(|x|)$ *as* $|x| \to 0$ *uniformly with respect to* $t \in \mathbb{R}$.

(W3) *There is a* $\overline{W} \in C(\mathbb{R}^N, \mathbb{R})$ *such that*

$$|W(t, x)| + |\nabla W(t, x)| \le |\overline{W}(x)|, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

(W4) $W(t, -x) = W(t, x)$, $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^N$.

*Then there exists an unbounded sequence of homoclinic solutions for the system (1.2).*

An immediate generalization of Theorem A to (1.1) by using the Symmetric Mountain Pass Theorem does not seem to be possible. The difficulty lies in the verification of the last condition of the Symmetric Mountain Pass Theorem, which is very different from the Mountain Pass Theorem. In this paper, motivated by the works of [10,17,22–25] we shall show how Symmetric Mountain Pass Theorem can be applied to establish the existence of infinitely many homoclinic solutions of the system (1.1). In what follows we shall also not assume the periodicity of the functions $a(t)$ and $W(t, x)$. In particular, when $p(t) = 2$, our results not only generalize Theorem A, but also relax conditions (W1) and (W2), and remove completely the condition (W3).

Our main results are the following theorems.

**Theorem 1.1.** *Assume that* $p$, $a$ *and* $W$ *satisfy* (A),(W4) *and the following assumptions:*

(P) $1 < p^- := \inf_{t \in \mathbb{R}} p(t) \le \sup_{t \in \mathbb{R}} p(t) := p^+ < \infty$.

(W5) $W(t, x) = W_1(t, x) - W_2(t, x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, *and there is* $R > 0$ *such that*

$$\frac{1}{a(t)}|\nabla W(t, x)| = o(|x|^{p^+-1}) \quad \text{as } x \to 0$$

*uniformly in* $t \in (-\infty, -R] \cup [R, +\infty)$.

(W6) *There is a constant* $\mu > p^+$ *such that*

$$0 < \mu W_1(t, x) \le (\nabla W_1(t, x), x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\};$$

(W7) $W_2(t, 0) \equiv 0$ *and there is a constant* $\varrho \in (p^+, \mu)$ *such that*

$$W_2(t, x) \ge 0, \qquad (\nabla W_2(t, x), x) \le \varrho W_2(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

*Then there exists an unbounded sequence of homoclinic solutions for system (1.1).*

**Theorem 1.2.** *Assume that* $p$, $a$ *and* $W$ *satisfy* (P), (A), (W4), (W6) *and the following assumptions:*

(W5′) $W(t, x) = W_1(t, x) - W_2(t, x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, *and*

$$\frac{1}{a(t)}|\nabla W(t, x)| = o(|x|^{p^+-1}) \quad \text{as } x \to 0$$

*uniformly in* $t \in \mathbb{R}$.

(W7′) $W_2(t, 0) \equiv 0$ *and there is a constant* $\varrho \in (p^+, \mu)$ *such that*

$$(\nabla W_2(t, x), x) \le \varrho W_2(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

*Then there exists an unbounded sequence of homoclinic solutions for system* (1.1).

**Theorem 1.3.** *Assume that* $p$, $a$ *and* $W$ *satisfy* (P), (A), (W4), (W5′) *and the following assumptions:*

(W8) *For any* $r > 0$, *there exist* $\alpha, \beta > 0$ *and* $\nu < p^-$ *such that*

$$0 \le \left( p(t) + \frac{1}{\alpha + \beta |x|^\nu} \right) W(t, x) \le (\nabla W(t, x), x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

(W9) *For any* $\gamma > 0$ *and* $\varepsilon > 0$

$$\lim_{s \to +\infty} s^{-p^+} \int_{t-\varepsilon}^{t+\varepsilon} \min_{|x| \ge 1} W(\tau, sx) d\tau = +\infty$$

*uniformly with respect to* $t \in [-\gamma, \gamma]$.

*Then there exists an unbounded sequence of homoclinic solutions for system* (1.1).

**Remark 1.1.** Obviously, when $p(t) = 2$, both conditions (W5) and (W5′) are weaker than (W2). Therefore, both Theorems 1.1 and 1.2 with $p(t) = 2$ generalize Theorem A, respectively, by relaxing conditions (W1) and (W2) and removing condition (W3).

The rest of this paper is organized as follows: in Section 2, we introduce some notations, collect some preliminary results for the space $W_a^{1,p(t)}$, and establish the corresponding variational structure. In Section 3, we complete the proofs of Theorems 1.1–1.3. In Section 4, we give some examples to illustrate our results.

## 2. Preliminaries

In this section, we recall some results from the critical point theory, and list necessary properties of the space $W_a^{1,p(t)}$. Let $\Omega \subset \mathbb{R}$ be a measurable subset with meas $\Omega > 0$. Let $E = \{u | u$ is a measurable function in $\Omega\}$. Elements in $E$ that are equal to each other almost everywhere are considered as the same.

Define

$$L_a^{p(t)}(\Omega, \mathbb{R}^N) = \left\{ u \in S(\Omega, \mathbb{R}^N) \,\middle|\, \int_\Omega a(t) |u(t)|^{p(t)} dt < \infty \right\}$$

with the norm

$$|u|_{p(t),a} = \inf \left\{ \lambda > 0 \,\middle|\, \int_\Omega a(t) \left| \frac{u}{\lambda} \right|^{p(t)} dt \le 1 \right\}.$$

Define

$$E = W_a^{1,p(t)}(\Omega, \mathbb{R}^N) = \{ u \in L_a^{p(t)}(\Omega, \mathbb{R}^N) | \dot{u} \in L^{p(t)}(\Omega, \mathbb{R}^N) \}$$

with the norm

$$\|u\| = \inf \left\{ \lambda > 0 | \int_\Omega \left( \left| \frac{\dot{u}}{\lambda} \right|^{p(t)} + a(t) \left| \frac{u}{\lambda} \right|^{p(t)} \right) dt \le 1 \right\}.$$

We call the space $L_a^{p(t)}$ a generalized Lebesgue space, which is a special kind of generalized Orlicz space. The space $W_a^{1,p(t)}$ is called a generalized Sobolev space, which is a special kind of generalized Orlicz–Sobolev space. For the basic theory of generalized Orlicz space and generalized Orlicz–Sobolev space, see [26,27]. One can find the general theory of spaces $L_a^{p(t)}$ and $W_a^{1,p(t)}$ in [24].

**Lemma 2.1** (*[24]*). *Let*

$$\rho(u) = \int_\Omega a(t) |u|^{p(t)} dt, \quad \forall u \in L_a^{p(t)},$$

*then*

(i) $|u|_{p(t),a} < 1 \ (= 1; > 1) \Longleftrightarrow \rho(u) < 1 \ (= 1; > 1)$;

(ii) $|u|_{p(t),a} > 1 \Longrightarrow |u|_{p(t),a}^{p^-} \le \rho(u) \le |u|_{p(t),a}^{p^+}$,
$|u|_{p(t),a} < 1 \Longrightarrow |u|_{p(t),a}^{p^+} \le \rho(u) \le |u|_{p(t),a}^{p^-}$;

(iii) $|u|_{p(t),a} \to 0 \Longleftrightarrow \rho(u) \to 0$; $|u|_{p(t),a} \to \infty \Longleftrightarrow \rho(u) \to \infty$.

**Lemma 2.2** ([24]). *Let*

$$\varphi(u)=\int_{\Omega}(|\dot{u}|^{p(t)}+a(t)|u|^{p(t)})dt,\quad \forall u\in W_a^{1,p(t)},$$

*then*

(i) $\|u\|<1\ (=1;>1)\Longleftrightarrow \varphi(u)<1\ (=1;>1)$;
(ii) $\|u\|>1\Longrightarrow \|u\|^{p^-}\le\varphi(u)\le\|u\|^{p^+}$,
$\|u\|<1\Longrightarrow \|u\|^{p^+}\le\varphi(u)\le\|u\|^{p^-}$;
(iii) $\|u\|\to 0\Longleftrightarrow \varphi(u)\to 0$; $\|u\|\to\infty\Longleftrightarrow\varphi(u)\to\infty$.

**Lemma 2.3** ([24]). *Let* $\rho(u)=\int_{\Omega}a(t)|u|^{p(t)}dt,\ \forall u,u_n\in L_a^{p(t)}$ $(n=1,2,\ldots)$, *then the following statements are equivalent to each other*

(i) $\lim_{n\to\infty}|u_n-u|_{p(t),a}=0$;
(ii) $\lim_{n\to\infty}\rho(u_n-u)=0$;
(iii) $u_n\to u$ *a.e.* $t\in\Omega$ *and* $\lim_{n\to\infty}\rho(u_n)=\rho(u)$.

**Lemma 2.4** ([24]). *If*

$$\frac{1}{p(t)}+\frac{1}{q(t)}=1,$$

*then*

(i) $(L^{p(t)})^*=L^{q(t)}$, *where* $(L^{p(t)})^*$ *is the conjugate space of* $L^{p(t)}$;
(ii) $\forall u\in L^{p(t)},\ v\in L^{q(t)}$, *we have*

$$\left|\int_{\Omega}u(t)v(t)dt\right|\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(t)}|vh|_{q(t)}\le 2|u|_{p(t)}|v|_{q(t)},$$

*where* $\frac{1}{p(t)}+\frac{1}{q(t)}=1$, $q^-=\inf_{t\in\mathbb{R}}q(t)$.

*Now, we provide the variational structure of the system* (1.1).

*Let* $I:E\to\mathbb{R}$ *be defined by*

$$I(u)=\int_{\mathbb{R}}\frac{1}{p(t)}(|\dot{u}|^{p(t)}+a(t)|u|^{p(t)})dt-\int_{\mathbb{R}}W(t,u(t))dt. \tag{2.1}$$

*For convenience, we denote*

$$J(u)=\int_{\mathbb{R}}\frac{1}{p(t)}(|\dot{u}|^{p(t)}+a(t)|u|^{p(t)})dt,\qquad F(u)=\int_{\mathbb{R}}W(t,u(t))dt. \tag{2.2}$$

**Lemma 2.5** ([24]).

(i) $J\in C^1(E,\mathbb{R})$, *and*

$$\langle J'(u),v\rangle=\int_{\mathbb{R}}(|\dot{u}(t)|^{p(t)-2}(\dot{u}(t),\dot{v}(t))+a(t)|u(t)|^{p(t)-2}(u(t),v(t)))dt,\quad\forall u,v\in E;$$

(ii) $J':E\to E^*$ *is a mapping of type* $(S_+)$, *i.e., if* $u_n\rightharpoonup u$ *and* $\overline{\lim}_{n\to\infty}(J'(u_n),u_n-u)\le 0$, *then* $u_n$ *has a convergent subsequence in E.*

If (A), (W5) or (W5′) hold, then $I\in C^1(E,\mathbb{R})$ and one can easily check that

$$\langle I'(u),v\rangle=\int_{\mathbb{R}}[|\dot{u}(t)|^{p(t)-2}(\dot{u}(t),\dot{v}(t))+a(t)|u(t)|^{p(t)-2}(u(t),v(t))-(\nabla W(t,u(t)),v(t))]dt. \tag{2.3}$$

Furthermore, the critical points of $I$ in $E$ are classical solutions of (1.1) with $u(\pm\infty)=0$.

**Lemma 2.6** ([24,25]). *If* $u\in E$, *then* $u\in C(\mathbb{R},\mathbb{R}^N)$, *and* $u(t)\to 0,\ |t|\to\infty$. *Furthermore, the embedding* $E\hookrightarrow L^\infty(\mathbb{R},\mathbb{R}^N)$ *is continuous and compact.*

**Remark 2.1.** By Lemma 2.6, there exists a constant $C>0$ such that

$$\|u\|_\infty:=\|u\|_{L^\infty}\le C\|u\|_E. \tag{2.4}$$

**Lemma 2.7.** *Assume that* (W6) *and* (W7) *or* (W7′) *hold. Then for every* $(t,x)\in\mathbb{R}\times\mathbb{R}^N$,

(i) $s^{-\mu}W_1(t,sx)$ *is nondecreasing on* $(0,+\infty)$;
(ii) $s^{-\varrho}W_2(t,sx)$ *is nonincreasing on* $(0,+\infty)$.

*The proof of Lemma 2.7 is routine and so we omit it.*

**Lemma 2.8** ([28]). *Let $E$ be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy Palais–Smale (PS)-condition. Suppose that $I$ satisfies the following conditions:*

(i) $I(0) = 0$.
(ii) *There exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$.*
(iii) *For each finite dimensional subspace $E' \subset E$, there is $r = r(E') > 0$ such that $I(u) \leq 0$ for $u \in E' \setminus B_r(0)$, where $B_r(0)$ is an open ball in $E$ of radius $r$ centered at 0.*
*Then $I$ possesses an unbounded sequence of critical values.*

**Remark 2.2.** A deformation lemma can be proved with condition (C) replacing the usual (PS)-condition, and it turns out that Lemma 2.8 hold true under condition (C). (We say $I$ satisfies condition (C), i.e., for every sequence $\{u_k\} \subset E$, $\{u_k\}$ has a convergent subsequence if $I(u_k)$ is bounded and $(1 + \|u_k\|)\|I'(u_k)\| \to 0$ as $k \to \infty$.)

## 3. Proof of theorems

**Proof of Theorem 1.1.** It is clear that $I(0) = 0$. We first show that $I$ satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $c > 0$ such that

$$|I(u_k)| \leq c, \qquad \|I'(u_k)\|_{E^*} \leq c \quad \text{for } k \in \mathbb{N}. \tag{3.1}$$

We may assume that $\|u_k\| \geq 1$, otherwise, $\|u_k\|$ is bounded obviously. From (2.1), (2.3), (3.1), (W6), (W7) and Lemma 2.2, we obtain

$$\begin{aligned} c + \frac{c}{\mu}\|u_k\| &\geq I(u_k) - \frac{1}{\mu}\langle I'(u_k), u_k\rangle \\ &= \int_{\mathbb{R}} \left(\frac{1}{p(t)} - \frac{1}{\mu}\right)(|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})dt - \int_{\mathbb{R}} W(t, u(t))dt \\ &\quad + \frac{1}{\mu}\int_{\mathbb{R}} [(\nabla W_1(t, u_k(t)), u_k(t)) - (\nabla W_2(t, u_k(t)), u_k(t))]dt \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\mu}\right)\|u_k\|^{p^-} \\ &\quad + \frac{1}{\mu}\int_{\mathbb{R}} [(\nabla W_1(t, u_k(t)), u_k(t)) - \mu W_1(t, u_k(t)) + (\mu W_2(t, u_k(t)) - (\nabla W_2(t, u_k(t)), u_k(t)))]dt \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\mu}\right)\|u_k\|^{p^-}, \quad k \in \mathbb{N}. \end{aligned}$$

It follows from (P) that there exists a constant $A > 0$ such that

$$\|u_k\| \leq A \quad \text{for } k \in \mathbb{N}. \tag{3.2}$$

Thus passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u$ in $E$. For any given number $\varepsilon > 0$, by (W5), we can choose $\delta \in (0, 1)$ such that

$$|\nabla W(t, x)| \leq p^+\varepsilon a(t)|x|^{p^+-1}, \qquad |W(t, x)| \leq \varepsilon a(t)|x|^{p^+} \quad \text{for } |t| \geq R, \text{ and } |x| \leq \delta. \tag{3.3}$$

Since $u \in E$, by Lemma 2.6, we can also choose $R_1 > 0$ such that

$$|u(t)| < \frac{\delta}{2}, \quad \text{for } |t| \geq R_1. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$|\nabla W(t, u(t))| \leq p^+\varepsilon a(t)|u(t)|^{p^+-1}, \qquad |W(t, u(t))| \leq \varepsilon a(t)|u(t)|^{p^+} \quad \text{for } |t| \geq R' := \max\{R, R_1\}. \tag{3.5}$$

By Lemma 2.6, $u_k \to u$ in $L^\infty(\mathbb{R}, \mathbb{R}^N)$, and hence we can choose $K_1 \in \mathbb{N}$ such that

$$|u_k(t)| < \delta, \quad \forall k > K_1, \qquad |t| \geq R'. \tag{3.6}$$

Thus,

$$|\nabla W(t, u_k(t))| \leq p^+\varepsilon a(t)|u_k(t)|^{p^+-1}, \qquad |W(t, u_k(t))| \leq \varepsilon a(t)|u_k(t)|^{p^+} \quad \text{for } |t| \geq R'. \tag{3.7}$$

By Lemma 2.6 and (W5), it is easy to verify that there exists $K_2 \in \mathbb{N}$ such that

$$|W(t, u_k(t)) - W(t, u(t))| \leq \frac{\varepsilon}{2R'}, \qquad |\nabla W(t, u_k(t)) - \nabla W(t, u(t))| \leq \frac{\varepsilon}{2R'},, \quad \forall k > K_2, |t| \leq R'. \tag{3.8}$$

Then, for $\forall k > K := \max\{K_1, K_2\}$, by (3.4), (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned}
|F(u_k) - F(u)| &= \left| \int_{\mathbb{R}} (W(t, u_k(t)) - W(t, u(t)))dt \right| \\
&\le \int_{-R'}^{R'} |W(t, u_k(t)) - W(t, u(t))|\, dt + \int_{\mathbb{R}\setminus[-R',R']} |W(t, u_k(t)) - W(t, u(t))|dt \\
&\le \varepsilon + \varepsilon \int_{\mathbb{R}\setminus[-R',R']} a(t)(|u_k|^{p^+} + |u|^{p^+})dt \\
&\le \varepsilon + \varepsilon \int_{\mathbb{R}\setminus[-R',R']} a(t)(|u_k|^{p(t)} + |u|^{p(t)})dt. \\
&\le \varepsilon + \varepsilon \int_{\mathbb{R}} a(t)(|u_k|^{p(t)} + |u|^{p(t)})dt. \qquad (3.9)
\end{aligned}$$

It follows from (3.2) and Lemma 2.2 that $F(u_k) \to F(u)$.

On the other hand, by (2.4), (3.7), (3.8) and Young's inequality, for any $v \in E$, $k > K$, we have

$$\begin{aligned}
|(F'(u_k) - F'(u), v)| &\le \int_{\mathbb{R}} |\nabla W(t, u_k(t))v - \nabla W(t, u(t))v|dt \\
&\le \int_{-R'}^{R'} |\nabla W(t, u_k(t)) - \nabla W(t, u(t))\|v|dt \\
&\quad + \int_{\mathbb{R}\setminus[-R',R']} |\nabla W(t, u_k(t)) - \nabla W(t, u(t))\|v|dt \\
&\le \varepsilon\|v\|_\infty + \varepsilon \int_{\mathbb{R}\setminus[-R',R']} a(t)(|u_k|^{p^+} + |u|^{p^+})|v|dt \\
&\le C\varepsilon\|v\| + \varepsilon \int_{\mathbb{R}\setminus[-R',R']} a(t)\left(\frac{p(t)-1}{p(t)}|u_k|^{p(t)} + \frac{1}{p(t)}|v|^{p(t)}\right)dt \\
&\quad + \varepsilon \int_{\mathbb{R}\setminus[-R',R']} a(t)\left(\frac{p(t)-1}{p(t)}|u|^{p(t)} + \frac{1}{p(t)}|v|^{p(t)}\right)dt \\
&\le C\varepsilon\|v\| + \frac{p^+-1}{p^-}\varepsilon \int_{\mathbb{R}\setminus[-R',R']} a(t)(|u_k|^{p(t)} + |u|^{p(t)})dt \\
&\quad + \frac{2}{p^-}\varepsilon \int_{\mathbb{R}\setminus[-R',R']} a(t)|v|^{p(t)}dt \\
&\le C\varepsilon\|v\| + \frac{p^+-1}{p^-}\varepsilon \int_{\mathbb{R}} a(t)(|u_k|^{p(t)} + |u|^{p(t)})dt \\
&\quad + \frac{2}{p^-}\varepsilon \int_{\mathbb{R}} a(t)|v|^{p(t)}dt. \qquad (3.10)
\end{aligned}$$

Hence,

$$\begin{aligned}
\|F'(u_k) - F'(u)\| &= \sup_{\|v\|=1} |(F'(u_k) - F'(u), v)| \\
&\le C\varepsilon + \frac{p^+-1}{p^-}\varepsilon \int_{\mathbb{R}\setminus[-R',R']} a(t)(|u_k|^{p(t)} + |u|^{p(t)})dt + \frac{2}{p^-}\varepsilon. \qquad (3.11)
\end{aligned}$$

It follows from (3.2) and Lemma 2.2 that $F'(u_k) \to F'(u)$. Since $I'(u_k) = J'(u_k) - F'(u_k) \to 0$, we find

$$J'(u_k) \to F'(u),$$

which implies that $(J'(u_k), u_k - u) \to 0$. By Lemma 2.5, $J'$ is a mapping of type $(S_+)$, and hence, $u_k \to u$. Therefore, $I$ satisfies (PS) condition.

We shall now show that there exist constants $\rho, \alpha > 0$ such that $I$ satisfies assumption (ii) of Lemma 2.8. By (W5), there exists $\eta \in (0, 1)$ such that

$$|\nabla W(t, x)| \le \frac{1}{2}a(t)|x|^{p^+-1} \quad \text{for} |t| \ge R, \qquad |x| \le \eta. \qquad (3.12)$$

Since $W(t,0)=0$, it follows that

$$|W(t,x)| \le \frac{1}{2p^+}a(t)|x|^{p^+} \quad \text{for } |t|\ge R, \qquad |x|\le\eta. \tag{3.13}$$

Set

$$M=\sup\left\{\left.\frac{W_1(t,x)}{a(t)}\right|\, t\in[-R,R],\ x\in\mathbb{R}^N,\ |x|=1\right\}. \tag{3.14}$$

Also, set $\zeta=\min\{1/(2p^+M+1)^{1/(\mu-p^+)},\eta,C\}$. By (2.4), if $\|u\|=\frac{\zeta}{C}:=\rho<1$, then $|u(t)|\le\zeta\le\eta<1$ for $t\in\mathbb{R}$. By (3.14) and Lemma 2.7(i), we have

$$\begin{aligned}\int_{-R}^{R}W_1(t,u(t))dt &\le \int_{\{t\in[-R,R]:u(t)\neq 0\}}W_1\left(t,\frac{u(t)}{|u(t)|}\right)|u(t)|^{\mu}dt\\ &\le M\int_{-R}^{R}a(t)|u(t)|^{\mu}dt\\ &\le M\delta^{\mu-p^+}\int_{-R}^{R}a(t)|u(t)|^{p^+}dt\\ &\le \frac{1}{2p^+}\int_{-R}^{R}a(t)|u(t)|^{p^+}dt\\ &\le \frac{1}{2p^+}\int_{-R}^{R}a(t)|u(t)|^{p(t)}dt.\end{aligned} \tag{3.15}$$

Now let

$$\alpha=\frac{1}{2p^+}\left(\frac{\zeta}{C}\right)^{p^+}.$$

For $\|u\|=\rho\le 1$, from (2.1), (3.13), (3.15), (W6), (W7) and Lemma 2.2, we have

$$\begin{aligned}I(u) &= \int_{\mathbb{R}}\frac{1}{p(t)}(|\dot u|^{p(t)}+a(t)|u|^{p(t)})dt-\int_{\mathbb{R}}W(t,u(t))dt\\ &= \int_{\mathbb{R}}\frac{1}{p(t)}(|\dot u|^{p(t)}+a(t)|u|^{p(t)})dt-\int_{\mathbb{R}\setminus[-R,R]}W(t,u(t))dt-\int_{-R}^{R}W(t,u(t))dt\\ &\ge \int_{\mathbb{R}}\frac{1}{p(t)}(|\dot u|^{p(t)}+a(t)|u|^{p(t)})dt-\frac{1}{2p^+}\int_{\mathbb{R}\setminus[-R,R]}a(t)|u(t)|^{p^+}dt-\int_{-R}^{R}W_1(t,u(t))dt\\ &\ge \int_{\mathbb{R}}\frac{1}{p(t)}(|\dot u|^{p(t)}+a(t)|u|^{p(t)})dt-\frac{1}{2p^+}\int_{\mathbb{R}\setminus[-R,R]}a(t)|u(t)|^{p(t)}dt-\int_{-R}^{R}W_1(t,u(t))dt\\ &\ge \int_{\mathbb{R}}\frac{1}{p(t)}(|\dot u|^{p(t)}+a(t)|u|^{p(t)})dt-\frac{1}{2p^+}\int_{\mathbb{R}\setminus[-R,R]}a(t)|u(t)|^{p(t)}dt-\frac{1}{2p^+}\int_{-R}^{R}a(t)|u(t)|^{p(t)}dt\\ &\ge \frac{1}{2p^+}\int_{\mathbb{R}}(|\dot u|^{p(t)}+a(t)|u|^{p(t)})dt\\ &\ge \frac{1}{2p^+}\|u\|^{p^+}\\ &= \alpha.\end{aligned} \tag{3.16}$$

Clearly, (3.16) shows that $\|u\|=\rho$ implies that $I(u)\ge\alpha$, i.e., $I$ satisfies assumption (ii) of Lemma 2.8.

Now, we shall prove (iii). Let $E'$ be a finite dimensional subspace of $E$. Assume that $\dim E'=m$ and $u_1,u_2,\ldots,u_m$ is the base of $E'$ such that

$$\|u_i\|=c,\quad i=1,2,\ldots,m. \tag{3.17}$$

For any $u\in E'$, there exist $\lambda_i\in\mathbb{R},\ i=1,2,\ldots,m$ such that

$$u(t)=\sum_{i=1}^{m}\lambda_i u_i(t) \quad \text{for } t\in\mathbb{R}. \tag{3.18}$$

Let

$$\|u\|_* = \sum_{i=1}^{m} |\lambda_i|\, \|u_i\|. \tag{3.19}$$

It is easy to verify that $\|\cdot\|_*$ defined by (3.19) is a norm on $E'$. Since all the norms in a finite dimensional normed space are equivalent, there are constants $c > 0$ and $c' > 0$ such that

$$c'\|u\|_* \le \|u\| \le c\|u\|_\infty \quad \text{for } u \in E'. \tag{3.20}$$

Since $u_i \in E$, we can choose $R_1 > R$ such that

$$|u_i(t)| < \frac{c'\eta}{1+c'}, \qquad |t| > R_1, \quad i = 1, 2, \ldots, m, \tag{3.21}$$

where $\eta$ is given in (3.12). Set

$$\Theta = \left\{ \sum_{i=1}^{m} \lambda_i u_i(t) : \lambda_i \in \mathbb{R},\ i = 1, 2, \ldots, m;\ \sum_{i=1}^{m} |\lambda_i| = 1 \right\} = \{u \in E' : \|u\|_* = c\}. \tag{3.22}$$

Then, for $u \in \Theta$, let $t_0 = t_0(u) \in \mathbb{R}$ be such that

$$|u(t_0)| = \|u\|_\infty. \tag{3.23}$$

Now by (3.17)–(3.20), (3.22) and (3.23), we have

$$\begin{aligned} c'c &= c'c\sum_{i=1}^{m} |\lambda_i| = c'\sum_{i=1}^{m} |\lambda_i|\, \|u_i\| = c'\|u\|_* \\ &\le \|u\| \le c\|u\|_{L^\infty(\mathbb{R})} = c|u(t_0)| \\ &\le c\sum_{i=1}^{m} |\lambda_i||u_i(t_0)|, \quad u \in \Theta. \end{aligned} \tag{3.24}$$

This shows that $|u(t_0)| \ge c'$ and there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $|u_{i_0}(t_0)| \ge c'$, which, together with (3.21), implies that $|t_0| \le R_1$. Set $R_2 = R_1 + 1$ and

$$\gamma = \min\left\{ W_1(t, x) : -R_2 \le t \le R_2,\ \frac{c'}{2} \le |x| \le cC \right\}. \tag{3.25}$$

Since $W_1(t, x) > 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$, and $W_1 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, it follows that $\gamma > 0$. Now for any $u \in E$, it follows from (2.4) and Lemma 2.7 (ii) that

$$\begin{aligned} \int_{-R_2}^{R_2} W_2(t, u(t))dt &= \int_{\{t\in[-R_2,R_2]:|u(t)|>1\}} W_2(t, u(t))dt + \int_{\{t\in[-R_2,R_2]:|u(t)|\le 1\}} W_2(t, u(t))dt \\ &\le \int_{\{t\in[-R_2,R_2]:|u(t)|>1\}} W_2\left(t, \frac{u(t)}{|u(t)|}\right) |u(t)|^{\varrho} dt + \int_{-R_2}^{R_2} \max_{|x|\le 1} W_2(t, x)dt \\ &\le \|u\|_\infty^{\varrho} \int_{-R_2}^{R_2} \max_{|x|=1} W_2(t, x)dt + \int_{-R_2}^{R_2} \max_{|x|\le 1} W_2(t, x)dt \\ &\le C^{\varrho}\|u\|^{\varrho} \int_{-R_2}^{R_2} \max_{|x|=1} W_2(t, x)dt + \int_{-R_2}^{R_2} \max_{|x|\le 1} W_2(t, x)dt \\ &= M_1\|u\|^{\varrho} + M_2, \end{aligned} \tag{3.26}$$

where

$$M_1 = C^{\varrho} \int_{-R_2}^{R_2} \max_{|x|=1} W_2(t, x)dt, \qquad M_2 = \int_{-R_2}^{R_2} \max_{|x|\le 1} W_2(t, x)dt.$$

Since $\dot{u}_i \in L^{p(t)}(\mathbb{R}),\ i = 1, 2, \ldots, m$, we find that there exists $\epsilon \in (0, 1)$ such that

$$\begin{aligned} \int_{t-\epsilon}^{t+\epsilon} |\dot{u}_i(s)|ds &\le (2\epsilon)^{1/q^-} |\dot{u}_i(s)|_{p^-} \\ &\le \frac{c'}{2p^-} \quad \text{for } t \in \mathbb{R},\ i = 1, 2, \ldots, m. \end{aligned} \tag{3.27}$$

Then for $u \in \Theta$ with $|u(t_0)| = \|u\|_\infty$ and $t \in [t_0 - \epsilon, t_0 + \epsilon]$, it follows from (3.18), (3.22)–(3.24) and (3.27) that

$$\begin{aligned} |u(t)|^{p^-} &= |u(t_0)|^{p^-} + p^- \int_{t_0}^{t} |u(s)|^{p^- -2}(\dot{u}(s), u(s))ds \\ &\geq |u(t_0)|^{p^-} - p^- \int_{t_0-\epsilon}^{t_0+\epsilon} |\dot{u}(s)||u(s)|^{p^- -1}ds \\ &\geq |u(t_0)|^{p^-} - p^-|u(t_0)|^{p^- -1} \int_{t_0-\epsilon}^{t_0+\epsilon} |\dot{u}(s)|ds \\ &\geq |u(t_0)|^{p^-} - p^-|u(t_0)|^{p^- -1} \sum_{i=1}^{m} |\lambda_i| \int_{t_0-\epsilon}^{t_0+\epsilon} |\dot{u}_i(s)|ds \\ &\geq \frac{c'}{2}|u(t_0)|^{p^- -1} \\ &\geq \left(\frac{c'}{2}\right)^{p^-}. \end{aligned} \tag{3.28}$$

On the other hand, since $\|u\| \leq c$ for $u \in \Theta$, it follows from (2.4) that

$$|u(t)| \leq cC \quad \text{for } t \in \mathbb{R},\ u \in \Theta. \tag{3.29}$$

Hence, from (3.25), (3.28) and (3.29), we have

$$\int_{-R_2}^{R_2} W_1(t, u(t))dt \geq \int_{t_0-\epsilon}^{t_0+\epsilon} W_1(t, u(t))dt \geq 2\epsilon\gamma \quad \text{for } u \in \Theta. \tag{3.30}$$

Also, by (3.21) and (3.22), we have

$$|u(t)| \leq \sum_{i=1}^{m} |\lambda_i||u_i(t)| \leq \eta \quad \text{for } |t| \geq R_2,\ u \in \Theta. \tag{3.31}$$

From (3.13), (3.26), (3.30), (3.31) and Lemma 2.7, we find for $u \in \Theta$ and $\sigma > 1$,

$$\begin{aligned} I(\sigma u) &= \int_{\mathbb{R}} \frac{1}{p(t)}(|\sigma\dot{u}|^{p(t)} + a(t)|\sigma u|^{p(t)})dt - \int_{\mathbb{R}} W(t, \sigma u(t))dt \\ &= \int_{\mathbb{R}} \frac{1}{p(t)}(|\sigma\dot{u}|^{p(t)} + a(t)|\sigma u|^{p(t)})dt + \int_{\mathbb{R}} W_2(t, \sigma u(t))dt - \int_{\mathbb{R}} W_1(t, \sigma u(t))dt \\ &\leq \int_{\mathbb{R}} \frac{1}{p(t)}(|\sigma\dot{u}|^{p(t)} + a(t)|\sigma u|^{p(t)})dt + \sigma^{\varrho} \int_{\mathbb{R}} W_2(t, u(t))dt - \sigma^{\mu} \int_{\mathbb{R}} W_1(t, u(t))dt \\ &= \int_{\mathbb{R}} \frac{1}{p(t)}(|\sigma\dot{u}|^{p(t)} + a(t)|\sigma u|^{p(t)})dt + \sigma^{\varrho} \int_{\mathbb{R}\setminus(-R_2,R_2)} W_2(t, u(t))dt - \sigma^{\mu} \int_{\mathbb{R}\setminus(-R_2,R_2)} W_1(t, u(t))dt \\ &\quad + \sigma^{\varrho} \int_{-R_2}^{R_2} W_2(t, u(t))dt - \sigma^{\mu} \int_{-R_2}^{R_2} W_1(t, u(t))dt \\ &\leq \int_{\mathbb{R}} \frac{1}{p(t)}(|\sigma\dot{u}|^{p(t)} + a(t)|\sigma u|^{p(t)})dt - \sigma^{\varrho} \int_{\mathbb{R}\setminus(-R_2,R_2)} W(t, u(t))dt \\ &\quad + \sigma^{\varrho} \int_{-R_2}^{R_2} W_2(t, u(t))dt - \sigma^{\mu} \int_{-R_2}^{R_2} W_1(t, u(t))dt \\ &\leq \int_{\mathbb{R}} \frac{1}{p(t)}(|\sigma\dot{u}|^{p(t)} + a(t)|\sigma u|^{p(t)})dt + \frac{\sigma^{\varrho}}{2p^+} \int_{\mathbb{R}\setminus(-R_2,R_2)} a(t)|u(t)|^{p^+}dt + \sigma^{\varrho}(M_1\|u\|^{\varrho} + M_2) - 2\epsilon\gamma\sigma^{\mu} \\ &\leq \int_{\mathbb{R}} \frac{1}{p(t)}(|\sigma\dot{u}|^{p(t)} + a(t)|\sigma u|^{p(t)})dt + \frac{\sigma^{\varrho}\max\{\|u\|_\infty^{p^+-p^-}, 1\}}{2p^+} \int_{\mathbb{R}\setminus(-R_2,R_2)} a(t)|u(t)|^{p(t)}dt \\ &\quad + \sigma^{\varrho}(M_1\|u\|^{\varrho} + M_2) - 2\epsilon\gamma\sigma^{\mu} \\ &\leq \frac{\sigma^{p^+}\max\{c^{p^+}, c^{p^-}\}}{p^-} + \frac{\max\{c^{p^+}, c^{p^-}\}\max\{(cC)^{p^+-p^-}, 1\}\sigma^{\varrho}}{2p^+} \\ &\quad + M_1(c\sigma)^{\varrho} + M_2\sigma^{\varrho} - 2\epsilon\gamma\sigma^{\mu}. \end{aligned} \tag{3.32}$$

Since $\mu > \varrho \geq p^+$, we deduce that there is $\sigma_0 = \sigma_0(c, c', M_1, M_2, R_1, R_2, \epsilon, \gamma) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for } u \in \Theta \quad \text{and} \quad \sigma \geq \sigma_0.$$

Hence, it follows that

$$I(u) < 0 \quad \text{for } u \in E' \quad \text{and} \quad \|u\| \geq c\sigma_0.$$

This shows that (iii) of Lemma 2.8 also holds. Therefore, $I$ possesses an unbounded sequence $\{d_k\}_{k=1}^{\infty}$ of critical values with $d_k = I(u_k)$, where $u_k$ is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. If $\{\|u_k\|\}$ is bounded, then there exists $B > 0$ such that

$$\|u_k\| \leq B \quad \text{for } k \in \mathbb{N}. \tag{3.33}$$

In a similar manner as in (3.5) and (3.6), for the given $\eta$ in (3.13), there exists $R_3 > R$ such that

$$|u_k(t)| \leq \eta \quad \text{for } |t| \geq R_3,\ k \in \mathbb{N}. \tag{3.34}$$

Thus, from (2.1), (2.4), (3.13), (3.33) and (3.34), we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}_k|^{p(t)} + a(t)|u_k|^{p(t)})dt &= d_k + \int_{\mathbb{R}} W(t, u_k(t))dt \\
&= d_k + \int_{\mathbb{R}\setminus[-R_3,R_3]} W(t, u_k(t))dt + \int_{-R_3}^{R_3} W(t, u_k(t))dt \\
&\geq d_k - \frac{1}{2p^+}\int_{\mathbb{R}\setminus[-R_3,R_3]} a(t)|u_k(t)|^{p^+}dt - \int_{-R_3}^{R_3} |W(t, u_k(t))|dt \\
&\geq d_k - \frac{1}{2p^+}\int_{\mathbb{R}\setminus[-R_3,R_3]} a(t)|u_k(t)|^{p(t)}dt - \int_{-R_3}^{R_3} |W(t, u_k(t))|dt \\
&\geq d_k - \frac{\max\{(BC)^{p^+-p^-}, 1\}}{2p^+}\int_{\mathbb{R}}(|\dot{u}_k|^{p(t)} + a(t)|u_k|^{p(t)})dt \\
&\quad - \int_{-R_3}^{R_3} \max_{|x|\leq CB} |W(t, x)|dt.
\end{aligned} \tag{3.35}$$

Hence, it follows that

$$d_k \leq \left(\frac{1}{p^-} + \frac{\max\{(BC)^{p^+-p^-}, 1\}}{2p^+}\right)\max\{\|u_k\|^{p^-}, \|u_k\|^{p^+}\} + \int_{-R_3}^{R_3} \max_{|x|\leq CB} |W(t, x)|dt < +\infty.$$

This contradicts to the fact that $\{d_k\}_{k=1}^{\infty}$ is unbounded, and so $\{\|u_k\|\}$ is unbounded. The proof is complete. □

**Proof of Theorem 1.2.** In the proof of Theorem 1.1, condition $W_2(t, x) \geq 0$ in (W7) is used only to prove (3.2) and assumption (ii) of Lemma 2.8. Therefore, it suffices to show that (3.2) and assumption (ii) of Lemma 2.8 still hold if we replace (W5) and (W7) by (W5′) and (W7′). For this, from (2.1) and (2.4), (3.1), (W5′) and (W7′), we obtain

$$\begin{aligned}
c + \frac{c}{\varrho}\|u_k\| &\geq I(u_k) - \frac{1}{\varrho}\langle I'(u_k), u_k\rangle \\
&= \int_{\mathbb{R}}\left(\frac{1}{p(t)} - \frac{1}{\varrho}\right)(|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})dt - \int_{\mathbb{R}} W(t, u(t))dt \\
&\quad + \frac{1}{\varrho}\int_{\mathbb{R}}[(\nabla W_1(t, u_k(t)), u_k(t)) - (\nabla W_2(t, u_k(t)), u_k(t))]dt \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\varrho}\right)\|u_k\|^{p^-} \\
&\quad + \frac{1}{\varrho}\int_{\mathbb{R}}[(\nabla W_1(t, u_k(t)), u_k(t)) - \varrho W_1(t, u_k(t)) + (\varrho W_2(t, u_k(t)) - (\nabla W_2(t, u_k(t)), u_k(t)))]dt \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\varrho}\right)\|u_k\|^{p^-}, \quad k \in \mathbb{N}.
\end{aligned}$$

Thus, it follows that there exists a constant $A > 0$ such that (3.2) holds. Next, by (W5′) there exists a $\eta \in (0, 1)$ such that

$$|\nabla W(t, x)| \leq \frac{1}{2}a(t)|x|^{p^+-1} \quad \text{for } t \in \mathbb{R}, \qquad |x| \leq \eta. \tag{3.36}$$

Since $W(t,0)=0$, it follows that

$$|W(t,x)| \le \frac{1}{2p^+}a(t)|x|^{p^+} \quad \text{for } t\in\mathbb{R}, \qquad |x|\le\eta. \tag{3.37}$$

If $\|u\| = \frac{\eta}{C} := \rho$, by (2.14), $|u(t)|\le\eta$ for $t\in\mathbb{R}$. Set

$$\alpha = \frac{1}{2p^+}\left(\frac{\eta}{C}\right)^{p^+}.$$

Hence, by (2.1), (3.37) and Lemma 2.2, we have

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})dt - \int_{\mathbb{R}} W(t,u(t))dt \\ &\ge \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})dt - \frac{1}{2p^+}\int_{\mathbb{R}} a(t)|u(t)|^{p^+}dt \\ &\ge \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})dt - \frac{1}{2p^+}\int_{\mathbb{R}} a(t)|u(t)|^{p(t)}dt \\ &\ge \frac{1}{2p^+}\int_{\mathbb{R}}[|\dot{u}(t)|^{p(t)} + a(t)|u(t)|^{p(t)}]dt \\ &\ge \frac{1}{2p^+}\|u\|^{p^+} \\ &= \alpha. \end{aligned} \tag{3.38}$$

Thus, $\|u\| = \rho$ implies that $I(u)\ge\alpha$, i.e., assumption (ii) of Lemma 2.8 holds. By following similar arguments as in Theorem 1.1, we can also easily verify that $I$ satisfies the assumption (iii). This completes the proof of Theorem 1.2. □

**Proof of Theorem 1.3.** We first show that $I$ satisfies condition (C). Assume that $\{u_k\}_{k\in\mathbb{N}}\subset E$ is a (C) sequence of $I$, that is, $\{I(u_k)\}$ is bounded and $(1+\|u_k\|)\|I'(u_k)\|\to 0$ as $k\to\infty$. Then, in view of (2.1) and (2.3), we have

$$\begin{aligned} C_1 &\ge p(t)I(u_k) - \langle I'(u_k), u_k\rangle \\ &= \int_{\mathbb{R}}[(\nabla W(t,u_k(t)), u_k(t)) - p(t)W(t,u_k(t))]dt. \end{aligned} \tag{3.39}$$

It follows from (W5′) that there exists $\eta\in(0,1)$ such that (3.37) holds. By (W8), we have

$$(\nabla W(t,x),x) \ge p(t)W(t,x)\ge 0 \quad \text{for } (t,x)\in\mathbb{R}\times\mathbb{R}^N, \tag{3.40}$$

and

$$W(t,x) \le (\alpha+\beta|x|^{\nu})[(\nabla W(t,x),x) - p(t)W(t,x)] \quad \text{for } (t,x)\in\mathbb{R}\times\mathbb{R}^N. \tag{3.41}$$

We may assume that $\|u_k\|\ge 1$, otherwise, $\|u_k\|$ is bounded obviously.

Now from (2.1), (2.4) and (3.39)–(3.41), we get

$$\begin{aligned} \frac{1}{p^+}\|u_k\|^{p^-} &\le \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})dt \\ &= I(u_k) + \int_{\mathbb{R}} W(t,u_k(t))dt \\ &\le I(u_k) + \int_{\mathbb{R}}(\alpha+\beta|u_k(t)|^{\nu})[(\nabla W(t,u_k(t)),u_k(t)) - p(t)W(t,u_k(t))]dt \\ &\le C_2 + \int_{\mathbb{R}}(\alpha+\beta|u_k(t)|^{\nu})[(\nabla W(t,u_k(t)),u_k(t)) - p(t)W(t,u_k(t))]dt \\ &\le C_2 + (\alpha+\beta\|u_k\|_\infty^{\nu})\int_{\mathbb{R}}[(\nabla W(t,u_k(t)),u_k(t)) - p(t)W(t,u_k(t))]dt \\ &\le C_2 + C_1(\alpha+\beta\|u_k\|_\infty^{\nu}) \\ &\le C_2 + C_1(\alpha + C^{\nu}\beta\|u_k\|^{\nu}). \end{aligned} \tag{3.42}$$

Since $\nu < p$-, we find that $\{\|u_k\|\}$ is bounded. Next, similar to the proof of Theorem 1.1, we can also prove that $\{u_k\}$ has a convergent subsequence in $E$. Hence, $I$ satisfies condition (C).

It is obvious that $I$ is even and $I(0) = 0$ and so assumption (i) of Lemma 2.8 holds. The proof of assumption (ii) of Lemma 2.8 is the same as in the proof of Theorem 1.2. Now, we shall prove condition (iii) of Lemma 2.8. Let $E'$ be a finite dimensional subspace of $E$. Assume that $\dim E' = m$ and $u_1, u_2, \ldots, u_m$ is the base of $E'$ such that (3.17) and (3.18) hold. Since all the norms of a finite dimensional normed space are equivalent, there are two constants $c > 0$ and $c' > 0$ such that (3.20) holds. Let $R_1, R_2$ and $\Theta$ be the same as in the proof of Theorem 1.1. Then, (3.21), (3.23), (3.24), (3.27) and (3.28) hold. For the $R_2$ and $\epsilon \in (0, 1)$ given in the proof of Theorem 1.1, by (W9), there exists $\sigma_0 > 1$ such that

$$s^{-p^+}\int_{t-\epsilon}^{t+\epsilon}\min_{|x|\ge 1} W(\tau, sx)d\tau \ge \max\{c^{p^+}, c^{p^-}\}\left(\frac{2}{c'}\right)^{p^+} \quad \text{for } s \ge c'\sigma_0/2,\ t \in [-R_2, R_2]. \tag{3.43}$$

It follows from (2.1), (W9), (3.28), (3.40) and (3.43) that

$$\begin{aligned} I(\sigma u) &= \int_{\mathbb{R}} \frac{1}{p(t)}(|\sigma\dot{u}|^{p(t)} + a(t)|\sigma u|^{p(t)})dt - \int_{\mathbb{R}} W(t, \sigma u(t))dt \\ &\le \frac{\sigma^{p^+}\max\{c^{p^+}, c^{p^-}\}}{p^-} - \int_{t_0-\epsilon}^{t_0+\epsilon} W(t, \sigma u(t))dt \\ &\le \frac{\sigma^{p^+}\max\{c^{p^+}, c^{p^-}\}}{p^-} - \int_{t_0-\epsilon}^{t_0+\epsilon} \min_{|x|\ge 1} W(t, 2^{-1}c'\sigma x)dt \\ &\le \frac{\sigma^{p^+}\max\{c^{p^+}, c^{p^-}\}}{p^-} - \max\{c^{p^+}, c^{p^-}\}\sigma^{p^+} \\ &= -\frac{(p^- - 1)\max\{c^{p^+}, c^{p^-}\}\sigma^{p^+}}{p^-} \quad \text{for } u \in \Theta \text{ and } \sigma \ge \sigma_0. \end{aligned} \tag{3.44}$$

That is,

$$I(\sigma u) < 0 \quad \text{for} \quad u \in \Theta \text{ and } \sigma \ge \sigma_0,$$

where $\sigma_0 = \sigma_0(\epsilon, R_2) = \sigma_0(E') > 1$. Hence, we have

$$I(u) < 0 \quad \text{for } u \in E' \text{ and } \|u\| \ge c\sigma_0.$$

This shows that condition (iii) of Lemma 2.8 also holds. The rest of the proof is the same as that of Theorem 1.1. □

## 4. Examples

In this section, we give some examples to illustrate our results.

**Example 4.1.** Consider the second-order ordinary $p(t)$-Laplacian system

$$\frac{d}{dt}(|\dot{u}(t)|^{2+t+\frac{1}{t}}\dot{u}(t)) - a(t)|u(t)|^{2+t+\frac{1}{t}}u(t) + \nabla W(t, u(t)) = 0; \tag{4.1}$$

here, $p(t) = 4 + t + \frac{1}{t}$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $a \in C(\mathbb{R}, (0, \infty))$ such that $a(t) \to +\infty$ as $|t| \to \infty$. Let

$$W(t, x) = a(t)\left(\sum_{i=1}^{m} a_i|x|^{\mu_i} - \sum_{j=1}^{n} b_j|x|^{\varrho_j}\right),$$

where $\mu_1 > \mu_2 > \cdots > \mu_m > \varrho_1 > \varrho_2 > \cdots > \varrho_n > 6, a_i, b_j > 0,\ i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Let $\mu = \mu_m,\ \varrho = \varrho_1$, and

$$W_1(t, x) = a(t)\sum_{i=1}^{m} a_i|x|^{\mu_i}, \qquad W_2(t, x) = a(t)\sum_{j=1}^{n} b_j|x|^{\varrho_j}.$$

Then it is easy to verify that all conditions of Theorem 1.1 are satisfied. By Theorem 1.1, system (4.1) has an unbounded sequence of homoclinic solutions.

**Example 4.2.** Consider the second-order ordinary $p(t)$-Laplacian system

$$\frac{d}{dt}(|\dot{u}(t)|^{1+\frac{1}{1+t^2}}\dot{u}(t)) - a(t)|u(t)|^{1+\frac{1}{1+t^2}}u(t) + \nabla W(t, u(t)) = 0; \tag{4.2}$$

here, $p(t) = 3 + \frac{1}{1+t^2}$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $a \in C(\mathbb{R}, (0, \infty))$ such that $a(t) \to +\infty$ as $|t| \to \infty$. Let

$$W(t, x) = a(t)[a_1|x|^{\mu_1} + a_2|x|^{\mu_2} - b_1(\sin t)|x|^{\varrho_1} - b_2|x|^{\varrho_2}],$$

where $\mu_1 > \mu_2 > \varrho_1 > \varrho_2 > 4$, $a_1, a_2 > 0$, $b_1, b_2 > 0$. Let $\mu = \mu_2$, $\varrho = \varrho_1$, and

$$W_1(t, x) = a(t)(a_1|x|^{\mu_1} + a_2|x|^{\mu_2}), \qquad W_2(t, x) = a(t)[b_1(\sin t)|x|^{\varrho_1} + b_2|x|^{\varrho_2}].$$

Then it is easy to verify that all conditions of Theorem 1.2 are satisfied. By Theorem 1.2, system (4.2) has an unbounded sequence of homoclinic solutions.

**Example 4.3.** Consider the second-order ordinary $p(t)$-Laplacian system

$$\frac{d}{dt}(|\dot{u}(t)|^{-\frac{1}{2}+\frac{1}{t^2+1}}\dot{u}(t)) - a(t)|u(t)|^{-\frac{1}{2}+\frac{1}{t^2+1}}u(t) + \nabla W(t, u(t)) = 0; \tag{4.3}$$

here, $p(t) = \frac{3}{2} + \frac{1}{t^2+1}$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $a \in C(\mathbb{R}, (0, \infty))$ such that $a(t) \to +\infty$ as $|t| \to \infty$. Let

$$W(t, x) = a(t)(1 + \sin t)|x|^{5/2} \ln(1 + |x|).$$

Since

$$\begin{aligned}(\nabla W(t, x), x) &= a(t)(1 + \sin t)\left[\frac{5}{2}|x|^{5/2} \ln(1 + |x|) + \frac{|x|^{7/2}}{1 + |x|}\right] \\ &\geq \left(\frac{5}{2} + \frac{1}{1 + |x|}\right) W(t, x) \geq \left(\frac{3}{2} + \frac{1}{t^2 + 1} + \frac{1}{1 + |x|}\right) W(t, x) \geq 0\end{aligned}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. This shows that (W8) holds with $\alpha = \beta = \nu = 1$. In addition, it is easy to show that (W9) also holds. The verification of assumptions (W4) and (W5′) of Theorem 1.3 is also straightforward. Thus, Theorem 1.3 is applicable and the system (4.3) has an unbounded sequence of homoclinic solutions.

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