Best Simultaneous Approximation in \(L^p(I, E)\)

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Communicated by William A. Light

Received November 26, 2000; accepted in revised form December 31, 2001

DEDICATED TO THE MEMORY OF PROFESSOR DEEB HUSSEIN

Let \(G\) be a reflexive subspace of the Banach space \(E\) and let \(L^p(I, E)\) denote the space of all \(p\)-Bochner integrable functions on the interval \(I = [0, 1]\) with values in \(E\), \(1 \leq p < \infty\). Given any norm \(N(\cdot, \cdot)\) on \(\mathbb{R}^2\), \(N\) nondecreasing in each coordinate on the set \(\mathbb{R}^2_+\), we prove that \(L^p(I, G)\) is \(N\)-simultaneously proximinal in \(L^p(I, E)\). Other results are also obtained. © 2002 Elsevier Science (USA)

Key Words: simultaneous approximation.

1. INTRODUCTION

Throughout this paper, \(E\) is a Banach space, \(G\) is a closed subspace of \(E\), \(p\) is a real number in \([1, \infty)\), and \(n\) is any integer, \(n \geq 1\). The norm of \(v \in E\) is denoted by \(\|v\|\) and the norm of \(u := (u_k)_{k=1}^n \in E^n\) is defined by

\[
\|u\|_{p, n} := \left[ \sum_{k=1}^n \|u_k\|^p \right]^{1/p}.
\]

Professor Deeb Hussein passed away on July 28, 2001.
Also, we let \( L^p(I, E) \) be the Banach space of \( p \)-Bochner integrable functions defined on \( I \) with values in \( E \), where \( I = [0, 1] \) is the unit interval in \( R \). Here \( R \) is the set of real numbers. The norm of \( f \in L^p(I, E) \) is given by

\[
\|f\|_p := \left[ \int_I \|f(s)\|^p \, d\mu \right]^{1/p},
\]

where \( \mu \) is the Lebesgue measure on \( I \).

Finally, we let \( N \) be any norm on \( R^2 \) satisfying, for every \( (x_1, x_2), (y_1, y_2) \in R^2 \),

\[
N(x_1, x_2) \leq N(y_1, y_2), \quad \text{if} \quad |x_i| \leq |y_i|, \quad i = 1, 2.
\]

Note that Eq. (1.1) is equivalent to \( N \) is nondecreasing in each coordinate on the set \( R^2_+ := \{(x_1, x_2) : x_1, x_2 \geq 0\} \). Also, note that Eq. (1.1) is satisfied by all the \( l^p \)-norms on \( R^2 \), \( 1 \leq p \leq \infty \).

The norm of \( (u^1, u^2) \in (E^*)^2 \) is defined by

\[
|(u^1, u^2)|_{p,n} := N \left( \left[ \sum_{k=1}^n \|u^1_k\|^p \right]^{1/p}, \left[ \sum_{k=1}^n \|u^2_k\|^p \right]^{1/p} \right),
\]

where \( u^1 = (u^1_k)_{k=1}^n \), \( u^2 = (u^2_k)_{k=1}^n \). Note that, by Eq. (1.1), \( |\cdot|_{p,n} \) is a norm on \( (E^*)^2 \) making it a Banach space. The diagonal of \( G^n \) is given by

\[
D^n := \{(g_k)_{k=1}^n : (g_k)_{k=1}^n \in G^n \}.
\]

**Definition 1.** We say that \( g \in G \) is a best \( N \)-simultaneous approximation from \( G \) of the pair of elements \( u^1, u^2 \in E \) if, for every \( h \in G \),

\[
N(\|u^1 - g\|, \|u^2 - g\|) \leq N(\|u^1 - h\|, \|u^2 - h\|),
\]

or, in other words, if, for every \( h \in G \),

\[
|(u^1 - g, u^2 - g)|_{1,1} \leq |(u^1 - h, u^2 - h)|_{1,1}.
\]

Note that \( g \in G \) is a best \( N \)-simultaneous approximation from \( G \) of \( u^1, u^2 \in E \) if and only if \( (g, g) \) is a best approximation from \( D^1 \) of the pair \( (u^1, u^2) \in E^2 \), where the norm on \( E^2 \) is \( |\cdot|_{1,1} \). If every pair of elements \( u^1, u^2 \in E \) admits a best \( N \)-simultaneous approximation from \( G \) (equivalently, \( D^1 \) is proximinal in \( E^2 \)), then \( G \) is said to be \( N \)-simultaneously proximinal in \( E \).
The problem of best simultaneous approximations has been studied by many authors, e.g., [1, 7, 13–15]. Most of these works have dealt with the characterizations of best simultaneous approximations in spaces of continuous functions with values in a Banach space $E$. Some existence and uniqueness results were also obtained. Results on best simultaneous approximation in general Banach spaces may be found in [9] and [11]. Little or no work has been done in the spaces $L^p(I, E)$. It is the aim of this paper to establish some existence results in this area. Among other things, we prove that, if $G$ is a reflexive subspace of the Banach space $E$ and $1 < p < \infty$, then $L^p(I, G)$ is $N$-simultaneously proximinal in $L^p(I, E)$.

Before we continue we note, as pointed out in Definition 1, that problems of best simultaneous approximation can also be viewed as special cases of vector-valued approximation. Some recent work in this area is due to Pinkus [10].

2. BEST SIMULTANEOUS APPROXIMATION IN $L^p(I, E)$

Recall that the norm of $u \in E^*$, hence also of $u \in G^*$ and of $u \in D^*$, is $\|u\|_{p,n}$ where $p$ is a fixed real number in $[1, \infty)$.

We start this section with the following observations:

**Remark 1.** Since all norms on a finite dimensional vector space are equivalent and since $N([\sum_{k=1}^{n} (\cdot)^p])^{1/p}, [\sum_{k=1}^{n} (\cdot)^p]^{1/p})$ is a norm on $R^n \times R^n$, we have (where the norm on $(E^n)^2$ is $\|\cdot\|_{p,n}$):

1. A sequence $((u^m_k)_{k=1}^{n}, (v^m_k)_{k=1}^{n})_{m=1}^{\infty}$ in $(E^n)^2$ is bounded if and only if the sets $\{u^m_k : 1 \leq k \leq n, 1 \leq m < \infty\}$ and $\{v^m_k : 1 \leq k \leq n, 1 \leq m < \infty\}$ are both bounded in $E$.

2. A sequence $((u^m_k)_{k=1}^{n}, (v^m_k)_{k=1}^{n})_{m=1}^{\infty}$ in $(E^n)^2$ is convergent (weakly convergent) if and only if all the sequences $((u^m_k)_{m=1}^{\infty}, (u^m_k)_{m=1}^{\infty}) : 1 \leq k \leq n$, are convergent (weakly convergent) in $E$.

3. If $G$ is reflexive then $G^n$ and $D^n$ are reflexive.

Note that $(g_k)_{k=1}^{n} \in G^n$ is a best $N$-simultaneous approximation from $G^n$ of the pair of elements $(u^1_k)_{k=1}^{n}, (u^2_k)_{k=1}^{n} \in E^n$ if and only if, for every $(h_k)_{k=1}^{n} \in G^n$,

$$|(u^1_k-g_k)_{k=1}^{n}, (u^2_k-g_k)_{k=1}^{n})|_{p,n} \leq |((u^1_k-h_k)_{k=1}^{n}, (u^2_k-h_k)_{k=1}^{n})|_{p,n}.$$ 

It follows immediately that

**Remark 2.** Let $(g_k)_{k=1}^{n} \in G^n$ be a best $N$-simultaneous approximation from $G^n$ of the pair of elements $(u^1_k)_{k=1}^{n}, (u^2_k)_{k=1}^{n} \in E^n$. Then, for each $k \in \{1, \ldots, n\}$, $g_k = 0$ whenever $u^1_k = u^2_k = 0$. 
From Remark 1 we obtain that, if $G$ is reflexive, then $D^*$ is reflexive and, consequently, proximinal in $(E^*)^2$. Hence, we have:

**LEMMA 1.** If $G$ is reflexive then, for every $n \geq 1$, $G^n$ is N-simultaneously proximinal in $E^n$.

Now, note that $g \in L^n(I, G)$ is a best $N$-simultaneous approximation from $L^n(I, G)$ of $f_1, f_2 \in L^n(I, E)$ if and only if, for all $h \in L^n(I, G)$,

$$N(\|f_1-g\|_p, \|f_2-g\|_p) \leq N(\|f_1-h\|_p, \|f_2-h\|_p).$$

The main result of this section is:

**THEOREM 1.** If, for every $n \geq 1$, $G^n$ is N-simultaneously proximinal in $E^n$, then every pair of simple functions $f_1, f_2 \in L^n(I, E)$ admits a best $N$-simultaneous approximation $g$ from $L^n(I, G)$.

**Proof.** Let $f_1, f_2$ be two simple functions in $L^n(I, E)$. Then $f_j(s) := \sum_{k=1}^{p} u_j^k \chi_{h_k}(s)$, $j = 1, 2$, where the $I_k$'s are disjoint measurable subsets of $I$ satisfying $\bigcup_{k=1}^{p} I_k = I$ and $\chi_{h_k}$ is the characteristic function of $I_k$. Since $f_1$ and $f_2$ represent classes of functions, we may assume that $\mu(I_k) > 0, 1 \leq k \leq n$. By assumption, there exists an N-simultaneous best approximation $(w_k)_{k=1}^{p}$ from $G^*$ of the pair of elements $(\mu^{1/p}(I_k) u_k^1)^{k-1}_{k-1}, (\mu^{1/p}(I_k) u_k^2)^{k-1}_{k-1} \in E^*$. This implies, if $g_k := \frac{1}{\mu^{1/p}(I_k)} w_k$, that $g := \sum_{k=1}^{p} g_k \chi_{h_k} \in L^n(I, G)$ and, since $w_k = \mu^{1/p}(I_k) g_k$, that

$$N(\|f_1-g\|_p, \|f_2-g\|_p) \leq N(\|f_1-h\|_p, \|f_2-h\|_p),$$

for all $h := \sum_{k=1}^{p} h_k \chi_{h_k} \in L^n(I, G)$. In other words, we have

$$N(\|f_1-g\|_p, \|f_2-g\|_p) \leq N(\|f_1-h\|_p, \|f_2-h\|_p),$$

for all $h := \sum_{k=1}^{p} h_k \chi_{h_k} \in L^n(I, G)$. We need to show that Eq. (2.2) holds for all simple functions (hence, by density, for all functions) $h \in L^n(I, G)$. So let $h$ be any simple function in $L^n(I, G)$. Then $h := \sum_{k=1}^{p} h_k \chi_{h_k}(s)$, where the $J_k$'s are disjoint, $\bigcup_{k=1}^{p} J_k = I$. Then

$$f_j = \sum_{1 \leq k \leq n} u_{j_k}^1 \chi_{J_k \cap J_j}, \quad j = 1, 2, \quad \text{and} \quad h = \sum_{1 \leq k \leq n} h_{k}^{1} \chi_{J_k \cap J_j}.$$
and, for each \( i, 1 \leq i \leq m, \)

\[ h_{ki} = h_i, \quad 1 \leq k \leq n. \]

Again, we obtain from the assumption that there exists a best N-simultaneous approximation \((w_{ki}^*)_{1 \leq i \leq m}^{1 \leq k \leq n}\) from \(G_{nm}\) of the pair of elements \((\mu^{1/p}(I_k \cap J_i), u_{ki}^1)_{1 \leq i \leq m}^{1 \leq k \leq n} \in E_{nm}\). Note that, by Remark 2, \(w_{ki}^* = 0\) whenever \(\mu^{1/p}(I_k \cap J_i) u_{ki}^1 = \mu^{1/p}(I_k \cap J_i) u_{ki}^2 = 0\). Let \(w_{ki}^*: = \mu^{1/p}(I_k \cap J_i) g_{ki}^*,\)

\[ g_{ki}^*: = \sum_{1 \leq i \leq m} g_{ki}^* \chi_{I_k \cap J_i} \in L^p(I, G) \]

and

\[ ((\mu^{1/p}(I_k \cap J_i)(u_{ki}^2 - g_{ki}^*))_{2}^{m, k, i=1}^{p, m, i=1, k=1, n, m})_{p, nm} \leq ((\mu^{1/p}(I_k \cap J_i)(u_{ki}^2 - g_{ki}^*))_{2}^{m, k, i=1}^{p, m, i=1, k=1, n, m}). \]

Note that, if \(\lambda_{ki} := \mu(I_k \cap J_i)/\mu(I_k)\), then \(\sum_{k=1}^{n} \lambda_{ki} = 1\) and

\[ \sum_{1 \leq i \leq m} \lambda_{ki} u_{ki}^2 - g_{ki}^* ||v||^p = \sum_{i=1}^{n} \mu(I_k) \sum_{k=1}^{m} \lambda_{ki} u_{ki}^2 - g_{ki}^* ||v||^p, \quad j = 1, 2. \]

Therefore, since \(||v||^p\) is a convex function of \(v \in E\) for \(p \geq 1,\)

\[ \sum_{i=1}^{m} \lambda_{ki} ||u_{ki}^2 - g_{ki}^*||^p \geq \sum_{i=1}^{m} \lambda_{ki} u_{ki}^2 - \sum_{i=1}^{m} \lambda_{ki} g_{ki}^* ||v||^p = ||u_k^2 - \beta_k||^p, \quad j = 1, 2, \]

where \(\beta_k := \sum_{i=1}^{m} \lambda_{ki} g_{ki}^*\) and where the equality follows from Eq. (2.3) and the fact that \(\sum_{i=1}^{m} \lambda_{ki} = 1\). Therefore we get

\[ (2.5) \quad \sum_{1 \leq i \leq m} \mu(I_k \cap J_i) ||u_{ki}^2 - g_{ki}^*||^p \geq \sum_{k=1}^{n} \mu(I_k) ||u_k^2 - \beta_k||^p, \quad j = 1, 2. \]

Hence, using Eqs. (2.1) then (1.1) and (2.5) then (2.4), we get

\[ ((\mu^{1/p}(I_k)(u_{ki}^2 - g_{ki}^*))_{2}^{m, k, i=1}^{p, m, i=1, k=1, n, m})_{p, nm} \leq ((\mu^{1/p}(I_k)(u_{ki}^2 - \beta_k))_{2}^{m, k, i=1}^{p, m, i=1, k=1, n, m})_{p, nm} \]

\[ \leq ((\mu^{1/p}(I_k \cap J_i)(u_{ki}^2 - g_{ki}^*))_{2}^{m, k, i=1}^{p, m, i=1, k=1, n, m})_{p, nm} \]

\[ \leq ((\mu^{1/p}(I_k \cap J_i)(u_{ki}^2 - h_{ki})))_{2}^{m, k, i=1}^{p, m, i=1, k=1, n, m})_{p, nm} \]

\[ \leq ((\mu^{1/p}(I_k \cap J_i)(u_{ki}^2 - h_{ki}))_{2}^{m, k, i=1}^{p, m, i=1, k=1, n, m})_{p, nm}. \]
In other words, 

\[
N(||f_1 - g||_p, ||f_2 - g||_p) \leq N(||f_1 - h||_p, ||f_2 - h||_p)
\]

for all simple functions \( h \in L^p(I, G) \) and, consequently, for all functions \( h \in L^p(I, G) \), since the set of simple functions is dense in \( L^p(I, G) \). The proof is complete.

**Corollary 1.** If \( G \) is reflexive, then every pair of simple functions \( f_1, f_2 \in L^p(I, E) \) admits a best \( N \)-simultaneous approximation \( g \) from \( L^p(I, G) \).

From the proof of the theorem we obtain:

**Remark 3.** If, for every \( n \geq 1 \), \( G^n \) is \( N \)-simultaneously proximinal in \( E^n \), then every pair of simple functions in \( L^p(I, E) \) admits a simple function in \( L^p(I, G) \) as an \( N \)-simultaneous approximation.

In the special case where \( N \) is the \( p \)-norm on \( R^2 \), we obtain a stronger result than that of Theorem 1:

**Theorem 2.** If \( N(x_1, x_2) := (|x_1|^p + |x_2|^p)^{1/p} \) and if \( L^p(I, D^1) \) is proximinal in \( L^p(I, E^2) \), then \( L^p(I, G) \) is \( N \)-simultaneously proximinal in \( L^p(I, E) \).

**Proof.** First we note that

\[
N(||f_1||_p, ||f_2||_p) = \left( \int_I ||f_1(s)||^p \, d\mu + \int_I ||f_2(s)||^p \, d\mu \right)^{1/p} = \left( \int_I (||f_1(s)||^p + ||f_2(s)||^p) \, d\mu \right)^{1/p} = \left( \int_I [N(||f_1(s)||, ||f_2(s)||)]^p \, d\mu \right)^{1/p}.
\]

This implies that \([L^p(I, E)]^2\) is isometric to \( L^p(I, E^2) \) and that \( D^1_{L^p(I, G)} := \{(g, g) : g \in L^p(I, G)\} \) is isometric to \( L^p(I, D^1) \). Hence we obtain, from the assumption, that \( D^1_{L^p(I, G)} \) is proximinal in \([L^p(I, E)]^2\). The theorem now follows from the fact that \( L^p(I, G) \) is \( N \)-simultaneously proximinal in \( L^p(I, E) \) if and only if \( D^1_{L^p(I, G)} \) is proximinal in \([L^p(I, E)]^2\). End of the proof.

On the question of proximinality of \( L^p(I, H) \) in \( L^p(I, X) \), where \( X \) is a Banach space and \( H \) is a closed subspace satisfying some conditions (in our case \( X = E^2 \) and \( H = D^1 \), many results have been established by various authors, e.g., [2, 4–6, 8, 12] to mention a few. For some of the strongest
results on this question, we refer the reader to [12] and [8]. Therefore, one can obtain several corollaries from Theorem 2. In particular, if $G$ is reflexive then, by Remark 1, $D^1$ is reflexive and, consequently, $L^p(I, D^1)$ is proximinal in $L^p(I, E)$, [12].

Note that if $G$ is reflexive and $1 < p < \infty$, then it follows, by Remark 1 and by [3, IV.1. Corollary 2], that $L^p(I, G)$ and $L^p(I, D^1)$ are reflexive. Therefore, for $p > 1$, we obtain directly, by Lemma 1, the following more general result than those of Corollary 1 and Theorem 2:

**Theorem 3.** If $G$ is reflexive and $1 < p < \infty$, then $L^p(I, G)$ is $N$-simultaneously proximinal in $L^p(I, E)$.

The case where $p = 1$ and $N$ is arbitrary is more difficult and will be studied in Section 3.

3. BEST SIMULTANEOUS APPROXIMATION IN $L^1(I, E)$

First, we establish some preliminary results needed for the proof of our main theorem:

**Lemma 2.** If $|x_j| < |y_j|$ in $R$, $j = 1, 2$, then $N(x_1, x_2) < N(y_1, y_2)$.

**Proof.** From the assumption we get that there exist $\alpha_1, \alpha_2 \in [0, 1)$ such that $|x_j| = \alpha_j |y_j|$, $j = 1, 2$. Let $\lambda := \max \{\alpha_1, \alpha_2\}$. Then $\lambda < 1$ and $|x_j| < \lambda |y_j|$, $j = 1, 2$.

Therefore by Eq. (1.1) we get

$$N(x_1, x_2) \leq \lambda N(y_1, y_2).$$

But $\lambda < 1$ and from the assumption $N(y_1, y_2) > 0$. Therefore $N(x_1, x_2) < N(y_1, y_2)$.  

**Lemma 3.** If $g \in L^p(I, G)$ is a best $N$-simultaneous approximation from $L^p(I, G)$ of the pair of elements $f_1, f_2 \in L^p(I, E)$ then, for every measurable subset $A$ of $I$ and every $h \in L^p(I, G)$,

$$\int_A \|f_{j_0}(s) - g(s)\|^p d\mu \leq \int_A \|f_{j_0}(s) - h(s)\|^p d\mu,$$

for some $j_0 \in \{1, 2\}$.
Proof. If \( \mu(A) = 0 \) then there is nothing to prove. Suppose that, for some \( A \) satisfying \( \mu(A) > 0 \) and for some \( h_0 \in L^p(I, G) \), the inequality does not hold for \( j = 1 \) and for \( j = 2 \). Now, define \( g_n \in L^p(I, G) \) by

\[
g_n(s) := \begin{cases} 
g(s) & \text{if } s \in I - A \\
h_n(s) & \text{if } s \in A. \end{cases}
\]

Then we have, for \( j = 1, 2 \),

\[
\left[ \int_A \|f_j(s) - g_n(s)\|^p \, d\mu \right]^{1/p} = \left[ \int_A \|f_j(s) - h_n(s)\|^p \, d\mu + \int_{I - A} \|f_j(s) - g(s)\|^p \, d\mu \right]^{1/p} < \left[ \int_A \|f_j(s) - g(s)\|^p \, d\mu + \int_{I - A} \|f_j(s) - g(s)\|^p \, d\mu \right]^{1/p} = \left[ \int_I \|f_j(s) - g(s)\|^p \, d\mu \right]^{1/p}.
\]

This together with Lemma 2 imply that

\[
N(\|f_1 - g\|_p, \|f_2 - g\|_p) < N(\|f_1 - g\|_p, \|f_2 - g\|_p)
\]

which contradicts the fact that \( g \) is a best \( N \)-simultaneous approximation from \( L^p(I, G) \) of the pair of elements \( f_1, f_2 \).

As a corollary we get:

**Corollary 2.** If \( g \) is a best \( N \)-simultaneous approximation from \( L^p(I, G) \) of the pair of elements \( f_1, f_2 \in L^p(I, E) \) then, for every measurable subset \( A \) of \( I \),

\[
\int_A \|g(s)\|^p \, d\mu \leq 2 \max \left\{ \int_A \|f_1(s)\|^p \, d\mu, \int_A \|f_2(s)\|^p \, d\mu \right\}.
\]

**Proof.** Since, for \( j = 1, 2 \),

\[
\left[ \int_A \|g(s)\|^p \, d\mu \right]^{1/p} \leq \left[ \int_A \|f_j(s) - g(s)\|^p \, d\mu \right]^{1/p} + \left[ \int_A \|f_j(s)\|^p \, d\mu \right]^{1/p},
\]

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we obtain, by using Lemma 3 with \( h = 0 \), that for some \( j_0 \in \{1, 2\} \)

\[
\left[ \int_A \|g(s)\|^p \, d\mu \right]^{1/p} \leq 2 \left[ \int_A \|f_{j_0}(s)\|^p \, d\mu \right]^{1/p}
\]

\[
\leq 2 \max \left\{ \left[ \int_A \|f_1(s)\|^p \, d\mu \right]^{1/p}, \left[ \int_A \|f_2(s)\|^p \, d\mu \right]^{1/p} \right\},
\]

which completes the proof. \( \square \)

We note that, as a corollary of Lemma 3, we get that, if \( g \in G^* \) is a best \( N \)-simultaneous approximation from \( G^* \) of the pair of elements \( u^1, u^2 \in E^* \) then, for each \( h \in G^* \) and each \( k, 1 \leq k \leq n \),

either \( \|u^1_k - g_k\| \leq \|u^1_k - h_k\| \) or \( \|u^2_k - g_k\| \leq \|u^2_k - h_k\| \).

Hence, for every \( J \subset \{1, 2, \ldots, n\} \),

\[
\sum_{k \in J} \|g_k\|^p \leq 2 \max \left\{ \sum_{k \in J} \|u^1_k\|^p, \sum_{k \in J} \|u^2_k\|^p \right\}.
\]

We are now ready to establish the analogue of Theorem 3 for \( L^1(I, E) \):

**Theorem 4.** If \( G \) is reflexive then \( L^1(I, G) \) is \( N \)-simultaneously proximinal in \( L^1(I, E) \).

**Proof.** Let \( f_1, f_2 \in L^1(I, E) \) and let \( \{f_{jn}\}_{n=1}^\infty, j = 1, 2 \), be two sequences of simple functions in \( L^1(I, E) \) satisfying

\[
\lim_{n \to \infty} \|f_j - f_{jn}\|_1 = 0, \quad j = 1, 2.
\]

By Corollary 1 we obtain, for each \( n \geq 1 \), that the pair of simple functions \( f_{1n}, f_{2n} \) admits a best \( N \)-simultaneous approximation \( g_n \) from \( L^0(I, G) \). Hence we have, for each \( n \geq 1 \),

\[
N(\|f_{1n} - g_n\|_1, \|f_{2n} - g_n\|_1) \leq N(\|f_{1n} - h\|_1, \|f_{2n} - h\|_1),
\]

for every \( h \in L^1(I, G) \). By Corollary 3, we obtain that

\[
\int_A \|g_n\| \, d\mu \leq 2 \max \left\{ \int_A \|f_{1n}\| \, d\mu, \int_A \|f_{2n}\| \, d\mu \right\}
\]
for every \(n \geq 1\). It follows, since both \(\{f_{1n}\}_{n=1}^{\infty}\) and \(\{f_{2n}\}_{n=1}^{\infty}\) are uniformly integrable, i.e.,
\[
\sup_n \left\{ \sup_A \left\{ \int_A \|f_n(x)\| \, d\mu \right\} : A \subset I, \mu(A) \leq \varepsilon \right\} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0,
\]
and since \(f_n \rightarrow f_j\) in \(L^1(I, E)\), \(j = 1, 2\), that the sequence \(\{g_n\}_{n=1}^{\infty}\) in \(L^1(I, G)\) is bounded and uniformly integrable. Hence, since \(G\) is reflexive, we obtain, by Dunford’s theorem [3], that \(\{g_n\}_{n=1}^{\infty}\) is relatively weakly compact in \(L^1(I, E)\). Therefore, there exists a subsequence, say \(\{g_{n_k}\}_{k=1}^{\infty}\), which converges weakly to some element \(g \in L^1(I, E)\). It follows, since \(L^1(I, G)\) is closed and convex hence weakly closed, that \(g \in L^1(I, G)\). It follows from Eq. (1.1) that \(N(\|\cdot\|, \|\cdot\|)\) is convex and continuous, and hence weakly lower semicontinuous, on \(L^1(I, E)\). This together with Eq. (3.1) imply that, for every \(h \in L^1(I, G)\),
\[
N(\|f_1 - g\|, \|f_2 - g\|) \leq \liminf_n N(\|f_{1n} - g_n\|, \|f_{2n} - g_n\|)
\leq \liminf_n N(\|f_{1n} - h\|, \|f_{2n} - h\|)
= N(\|f_1 - h\|, \|f_2 - h\|).
\]
Therefore \(g\) is a best \(N\)-simultaneous approximation from \(L^1(I, G)\) of the pair \(f_1, f_2 \in L^1(I, E)\). 

Finally, we note the following:

**Remark 4.** It follows immediately that all the results and proofs in this paper are valid in the case where \(N\) is a norm on \(R^M, M \geq 2\), satisfying
\[
N(x) \leq N(y), \quad \text{if} \quad |x_i| \leq |y_i|, \quad 1 \leq i \leq M.
\]
In this case, \(g \in G\) is said to be a best \(N\)-simultaneous approximation from \(G\) of the elements \(u^1, u^2, \ldots, u^M \in E\) if, for every \(h \in G\),
\[
N(\|u^1 - g\|, \|u^2 - g\|, \ldots, \|u^M - g\|) \leq N(\|u^1 - h\|, \|u^2 - h\|, \ldots, \|u^M - h\|).
\]

**REFERENCES**