Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 209 (2007) 311–323
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

The group of endotrivial modules in the normal case

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Received 5 January 2006; accepted 30 May 2006
Available online 1 August 2006
Communicated by S. Donkin

Abstract

The group of endotrivial modules has recently been determined for a finite group having a normal Sylow p-subgroup. In this paper, we give and compare three different presentations of a torsion-free subgroup of maximal rank of the group of endotrivial modules. Finally, we illustrate the constructions in an example.

MSC: 20C20

1. Introduction

Endotrivial modules play an important role in the modular representation theory of finite groups, and this may explain why many group theorists have been studying them intensively, since the late seventies. The classification of endotrivial modules of a finite p-group was recently achieved in [11]. Thereafter, in a joint work, Jon Carlson, Daniel Nakano and the author tackled the question of the classification of endotrivial modules for an arbitrary finite group, and they were able to give an almost complete classification in the case of a finite group of Lie type in its defining characteristic (cf. [8]). The general case is still an open question, in the sense that no presentation by generators and relations of the group of endotrivial modules is yet known. However, the obstacles have been overcome in the case of a finite group having a normal Sylow p-subgroup. The results are presented in [8, Theorem 3.4], where the authors show that in this case, the group of endotrivial modules is generated by the classes of the indecomposable endotrivial modules that are extended from the Sylow p-subgroup. Then, by means of cohomological tools, they construct a minimal set of generators for the group of endotrivial modules.

The primary aim of these notes is to give an alternative construction of a torsion-free subgroup of maximal rank of the group of endotrivial modules of a finite group having a normal Sylow p-subgroup, that does not appeal to any cohomological knowledge. The method refers to a theorem proven by Dade, and it is presented in Section 3; after that basic facts about endotrivial modules are recapitulated in Section 2. In Section 4, we review the modules and the techniques used in [7,8], and we compare with the approach presented in the previous section. Finally, in Section 5, we work out thoroughly an "odd extraspecial" example.

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0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2006.05.031

2. Preliminaries

Throughout these notes, we let k be an algebraically closed field of prime characteristic p. If G is a finite group, we write $\mathbf{mod}(kG)$ for the category of finitely generated kG-modules and $\mathbf{stmod}(kG)$ for the stable module category. That is, the objects of $\mathbf{stmod}(kG)$ are the same as those of $\mathbf{mod}(kG)$, and the morphisms are equivalence classes of morphisms. Namely, two morphisms are equivalent if their difference factors through a projective module (cf. [6, Section 5]). In addition, we write k for the one-dimensional trivial kG-module, and if M is a finitely generated kG-module, then $\mathrm{End}_k M = \mathrm{Hom}_k(M, M)$ denotes the kG-module that is the k-algebra of k-linear endomorphisms of M. Recall that, for two kG-modules M and N, there is an isomorphism of kG-modules $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$, where $M^* = \mathrm{Hom}_k(M, k)$ is the k-linear dual of M, and the tensor product "$\otimes$" is the tensor product over the field k.

Definition 2.1. Let G be a finite group. A finitely generated kG-module M is *endotrivial* provided that $\mathrm{End}_k M \cong k$ in $\mathbf{stmod}(kG)$, or equivalently, $\mathrm{End}_k M \cong k \oplus (\mathrm{proj})$ in $\mathbf{mod}(kG)$, for some projective kG-module (proj).

We say that two endotrivial kG-modules are *equivalent* if they are isomorphic in $\mathbf{stmod}(kG)$. The set $T(G)$ of isomorphism classes in $\mathbf{stmod}(kG)$ of endotrivial kG-modules is an abelian group, called the *group of endotrivial modules*. The composition law is defined by $[M] + [N] = [M \otimes N]$.

In particular, in $T(G)$, we have $0 = [k]$ and $-[M] = [M^*]$.

Endotrivial kG-modules were defined by Dade (cf. [12]), in 1978, for finite p-groups, as a particular case of the capped endo-permutation kG-modules. A *capped endo-permutation kG-module*, for a finite p-group G, is a finitely generated kG-module whose endomorphism algebra is a permutation module having a trivial direct summand. Modulo a suitable equivalence relation, they form a finitely generated abelian group $D(G)$, and the group $T(G)$ identifies with a subgroup of $D(G)$.

Also, for a subgroup H of G, the restriction map $\mathrm{Res}^G_H : \mathbf{mod}(kG) \to \mathbf{mod}(kH)$ (also denoted by "$\cdot\downarrow^G_H$") induces a group homomorphism $\mathrm{Res}^G_H : T(G) \to T(H)$.

Non-trivial examples of endotrivial modules are the syzygies of the trivial module, whereas, in the case of a finite p-group, most of the relative syzygies are capped endo-permutation modules (not endotrivial in general; cf. [1]). Let us recall their definitions.

Definition 2.2. If X is a finite G-set, then $\Omega^1_X(k)$ is the *relative* (*to* X) *syzygy of* k, that is, the kernel of the augmentation map $kX \to k$.

If $X = G$, then we define the *syzygy* $\Omega^n_G(k)$ *of* k, for each $n \in \mathbb{Z}$, as follows. If $n \geq 1$, we let $\Omega^n_G(k)$ be the kernel of the $(n-1)$-st differential in a minimal projective resolution of k.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+1}"] & X_n \arrow[rr, "\partial_n"] \arrow[dr, two heads] & & X_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_2"] & X_1 \arrow[rr, "\partial_1"] \arrow[dr, two heads] & & X_0 \arrow[r, two heads, "\partial_0"] & k \\
& & \Omega^n_G(k) \arrow[ur, hook] & & & & \Omega^1_G(k) \arrow[ur, hook] & &
\end{tikzcd}$$

If $n \leq -1$, we let $\Omega^n_G(k) = \mathrm{Hom}_k(\Omega^{-n}_G(k), k)$, and we set $\Omega^0_G(k) = k$.

Let G be a p-group and suppose that $\Omega^1_X(k)$ is a capped endo-permutation kG-module. Then we let Ω_X denote its class in $D(G)$, or $T(G)$ in the case where it is endotrivial. In particular, for any finite group G, and for any integer n, the syzygies $\Omega^n_G(k)$ are indecomposable endotrivial modules and we have $[\Omega^n_G(k)] = n\Omega_G$ in $T(G)$. We refer the reader to Section 4 in [6] for more properties of the syzygies, and to Sections 2–5 in [4] for those of the relative syzygies.

Elementary abelian p-subgroups play an important role in the analysis of $T(G)$. In particular, we will need the following group theoretical notions for our purposes.

Definition 2.3. Let G be a finite group and p be a prime.

(1) The *p-rank of* G is the largest integer r such that G has an elementary abelian p-subgroup of rank r.
(2) We write $\mathcal{E}_{\geq 2}(G)$ for the poset of G-conjugacy classes of elementary abelian p-subgroups of G of p-rank at least 2.

Note that the p-rank of a finite group G is the p-rank of a Sylow p-subgroup of G. Moreover, if a Sylow p-subgroup P is normal in G, then $\mathcal{E}_{\geq 2}(G)$ identifies with the quotient of $\mathcal{E}_{\geq 2}(P)$ by the action by conjugation of G.

The number of connected components of $\mathcal{E}_{\geq 2}(G)$ and the torsion-free rank of $T(G)$ are closely related, as shown by the next two theorems. We state them here as they appear in [11] and in [8] respectively. Let us point out the fact that the first result was first obtained by J. Alperin (cf. [1, Theorem 4]).

Let P be a finite p-group. Assume that P has p-rank at least 2 and is not semi-dihedral. Let $F_1, \ldots, F_m$ denote a set of representatives of the components $\mathcal{D}_1, \ldots, \mathcal{D}_m$ of the poset $\mathcal{E}_{\geq 2}(P)$. By [11], we know that we can make such a choice satisfying that all F_i's have p-rank 2, all but possibly F_m are maximal, and F_m is normal in P. For each $1 \leq i \leq m$ let S_i be a non-central subgroup of P of order p that is contained in F_i. By [11, Lemma 2.2], if F_i is maximal then $C_P(S_i) = S_i \times L_i$, where L_i has normal p-rank one. That is, L_i is cyclic if p is odd, and can be either cyclic or generalized quaternion in the case $p = 2$.

Theorem 2.4 (*[11], Theorem 3.1*). *The group $T(P)$ of endotrivial kP-modules is torsion-free of rank m and it is generated by Ω_P and the classes of the modules N_i for $i = 1, \ldots, (m-1)$, where N_i is the unique indecomposable summand with vertex P of*

$$\begin{array}{ll} M_i^{\otimes 2} & \text{if } C_P(S_i)/S_i \text{ is cyclic of order } \geq 3, \\ M_i & \text{if } p = 2 \text{ and } |C_P(S_i)/S_i| = 2, \\ M_i^{\otimes 4} & \text{if } p = 2 \text{ and } C_P(S_i)/S_i \text{ is generalized quaternion}, \end{array}$$

and where $M_i = \Omega_P^{-1}(k) \otimes \Omega^1_{P/S_i}(k)$. Moreover,

$$\mathrm{Res}^P_{F_j} N_i \cong \begin{cases} \Omega^{a_i}_{F_i}(k) \oplus (\mathrm{proj}) & \text{if } i = j \\ k \oplus (\mathrm{proj}) & \text{otherwise} \end{cases}$$

where

$$\begin{array}{ll} a_i = -2p & \text{if } C_P(S_i)/S_i \text{ is cyclic of order } \geq 3, \\ a_i = -2 & \text{if } p = 2 \text{ and } |C_P(S_i)/S_i| = 2, \\ a_i = -8 & \text{if } p = 2 \text{ and } C_P(S_i)/S_i \text{ is generalized quaternion}. \end{array}$$

In particular, if $T(P)$ is cyclic, then $T(P) = \langle \Omega_P \rangle$.

We refer the reader to [9,11,12] for a detailed description of the group of endotrivial modules in the case that $m = 1$, since we will not consider that situation later.

Let G be a finite group having a normal Sylow p-subgroup P, and let $F_1, \ldots, F_m$ be as above. Choose representatives $E_1, \ldots, E_n$ of the components $\mathcal{C}_1, \ldots, \mathcal{C}_n$ of the poset $\mathcal{E}_{\geq 2}(G)$.

Remark 2.5. If P has p-rank at least 3 and $m \geq 2$, then, for any index $1 \leq i \leq m-1$, the subgroups F_i and F_m are not G-conjugate. Indeed, since P is normal in G, then P contains all p-subgroups of G, and since the p-rank is at least 3, then F_m is contained in an elementary abelian p-subgroup F of rank 3. Thus, if ${}^g F_i = F_m < F$, for some $g \in G$, then $F_i < F^g < P$. But this means that F_i is not maximal in P, which implies $i = m$. In other words, there is only one connected component of elementary abelian p-subgroups of rank at least 3, which is thus invariant by any automorphism of P, in particular here by G-conjugation.

This observation allows us, without loss of generality and only for the convenience of the notation, to choose the indexes of the F_i's so that $E_i = F_i$, for $1 \leq i \leq n-1$ and $E_n = F_m$. The structure of $T(G)$ is as follows (cf. [8]).

Theorem 2.6. *The group $T(G)$ is finitely generated, and hence, it splits as a direct sum $TT(G) \oplus TF(G)$, where $TT(G)$ denotes the torsion subgroup and $TF(G)$ is a torsion-free subgroup of maximal rank (as a $\mathbb{Z}$-module).*

Moreover, $T(G)$ is generated by the classes of the indecomposable endotrivial kG-modules whose restriction to P is an indecomposable module, and $TF(G) \cong \mathbb{Z}^n$.

In particular, if $TT(P)$ is trivial, then $TT(G)$ is generated by the classes of the one-dimensional kG-modules. Also, if $n = 1$, then $TF(G) = \langle \Omega_G \rangle$.

Besides the purpose of the classification of endotrivial kG-modules, the fact that the syzygies of k are indecomposable endotrivial modules has the following consequence.

Corollary 2.7. *Let P be a normal Sylow p-subgroup of a finite group G, and let $(X., \partial.)$ be a kG-projective resolution of the trivial module k. Then, $(X., \partial.)$ is minimal $\Longleftrightarrow$ $(X.\downarrow^G_P, \partial.\downarrow^G_P)$ is a minimal kP-projective resolution of k.*

Proof. "$\Longleftarrow$" is obvious.

"$\Longrightarrow$" A projective kG-resolution of k is minimal if and only if it is the splice of short exact sequences, one for each integer $n \geq 0$,

$$0 \longrightarrow \Omega^{n+1}_G(k) \xrightarrow{i_n} X_n \xrightarrow{\partial_n} \Omega^n_G(k) \longrightarrow 0.$$

By restriction to P, we get then

$$0 \longrightarrow \Omega^{n+1}_P(k) \oplus (\text{proj}) \xrightarrow{i_n\downarrow^G_P} X_n\downarrow^G_P \xrightarrow{\partial_n\downarrow^G_P} \Omega^n_P(k) \oplus (\text{proj}) \longrightarrow 0.$$

But the kG-modules $\Omega^n_G(k)$ are indecomposable endotrivial and $P \lhd G$, and so, by Theorem 2.6, $\Omega^n_G(k)\downarrow^G_P$ is indecomposable. Therefore the projective factors "(proj)" are all zero. In other words, for each integer $n \geq 0$, the above short exact sequence is in fact

$$0 \longrightarrow \Omega^{n+1}_P(k) \xrightarrow{i_n\downarrow^G_P} X_n\downarrow^G_P \xrightarrow{\partial_n\downarrow^G_P} \Omega^n_P(k) \longrightarrow 0.$$

Hence, $(X.\downarrow^G_P, \partial.\downarrow^G_P)$ is a minimal kP-projective resolution of k. $\square$

3. Dade's result and its consequences

Throughout this section, we assume that G is a finite group and that P is a normal Sylow p-subgroup of G. We continue with the notation used in the previous section.

The only cases in which generators for the group $TF(G)$ are not given by Theorem 2.6 are for $TF(G)$ not cyclic, i.e. for $n \geq 2$, according to our notation. Hence, we will assume that $n \geq 2$. In this case, we also have $m \geq 2$, and so $TT(P) = 0$, by Theorem 2.4. Moreover, this also implies that the center of P is cyclic non-trivial. Thus, it has a unique cyclic subgroup Z of order p, which is contained in any maximal elementary abelian p-subgroup. Furthermore, $Z \lhd G$, since Z is characteristic in P.

A key tool for determining which endotrivial kP-modules extend to G is provided by the following theorem, proven by Dade and never published (as far as we are aware of).

Theorem 3.1 (*Theorem 7.1, [13]*). *Let G be a finite group having a normal Sylow p-subgroup P, let k be an algebraically closed field of characteristic p, and let M be an endo-permutation kP-module. Then M extends to a kG-module if and only if M is G-stable.*

By definition, if H is a normal subgroup of G and M is a kH-module, then we define the conjugate kH-module ${}^gM$ of M by setting $\underbrace{h \cdot {}^gm}_{\text{in } {}^gM} = {}^g(\underbrace{h^g \cdot m}_{\text{in } M}), \forall\, h \in H, \forall\, m \in M, \forall\, g \in G$. We say then that M is *G-stable* if we have an isomorphism of kH-modules $M \cong {}^gM, \forall\, g \in G$. Equivalently, a kH-module M is G-stable if and only if ${}^gM \cong M, \forall\, g \in G/H$.

Since an endotrivial module is an endo-permutation module, and since a module is endotrivial if and only if its restriction to a Sylow p-subgroup is, we have the following.

Corollary 3.2. *The map $\mathrm{Res}^G_P : T(G) \to T(P)$ induces an isomorphism of abelian groups $TF(G) \cong TF(P)^{G/P}$, where $TF(P)^{G/P}$ is the subset of the G/P-fixed points. That is, $TF(P)^{G/P}$ is the subgroup of $TF(P)$ generated by the classes of the G-stable endotrivial kP-modules.*

Proof. By Theorem 2.6, we have $\ker(\mathrm{Res}^G_P) = TT(G)$, and the image of Res^G_P is necessarily a subgroup of $T(P)^{G/P}$. Hence, Res^G_P induces an injection $TF(G) \to TF(P)^{G/P}$. Now, by Theorem 3.1, an endotrivial kP-module M extends to a kG-module (necessarily endotrivial) if and only if M is G-stable. That is, the above map is also surjective. $\square$

At this stage, determining a set of generators for $TF(G)$ comes down to a question of linear algebra. Indeed, if $x \in T(P)$ then x is uniquely expressible as a $\mathbb{Z}$-linear combination $x = \sum_{i=1}^{m} x_i e_i$, where $e_i = [N_i]$, for the module N_i defined in Theorem 2.4, for $1 \le i \le m-1$, and $e_m = \Omega_P$.

Lemma 3.3. *For all $g \in G$, and for all $1 \le i \le m$, we have ${}^g N_i = N_{\sigma_g(i)}$, for the permutation σ_g, acting on the set $\{1, \ldots, m\}$, defined by ${}^g S_j =_P S_{\sigma_g(j)}, \forall\, 1 \le j \le m$.*

Here, the notation "$=_P$" means "is P-conjugate to".

Proof. Fix an index $1 \le i \le m-1$, and write $S = S_i$ and $N = N_i$. Let $g \in G$ and consider the map $\varphi : {}^g k[P/S] \longrightarrow k[P/{}^g S]$, defined by $\varphi({}^g(uS)) = {}^g u\, {}^g S$ on a k-basis $\{{}^g(uS) \mid uS \in P/S\}$ of the permutation kP-module ${}^g k[P/S]$. Let us verify that φ is an isomorphism of kP-modules. It is immediate that φ is an isomorphism of k-vector spaces, and so we only need to check that φ commutes with the action of P:

$$v \cdot \varphi({}^g(uS)) = v \cdot {}^g u\, {}^g S = {}^g(v^g\, u)\, {}^g S = \varphi\left({}^g(v^g\, uS)\right) = \varphi(v \cdot {}^g(uS)), \forall\, u \in P, \forall\, uS \in P/S, \forall\, g \in G.$$

It follows that we have exact sequences forming the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & {}^g\,\Omega^1_{P/S}(k) \arrow[r] \arrow[d, "\varphi\downarrow_{\ker({}^g\varepsilon)}"] & {}^g k[P/S] \arrow[r, "{}^g\varepsilon"] \arrow[d, "\varphi"] & {}^g k \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & \Omega^1_{P/{}^g S}(k) \arrow[r] & k[P/{}^g S] \arrow[r, "\varepsilon'"] & k \arrow[r] & 0
\end{tikzcd}$$

Since φ is an isomorphism, the left hand kP-modules ${}^g\,\Omega^1_{P/S}(k)$ and $\Omega^1_{P/{}^g S}(k)$ are isomorphic. Similarly, we have ${}^g\,\Omega^1_P(k) \cong \Omega^1_P(k)$ and the result follows. □

This lemma has an immediate consequence.

Proposition 3.4. $x = \sum_{i=1}^{m} x_i e_i \in T(P)^G \iff$ *$x_i = x_j$ whenever $S_i =_G S_j, \forall\, 1 \le i \le m-1$.*

We will need a group theoretical result, complementary to Lemma 2.2 in [11]. For this, let us fix an index $1 \le i \le n-1$ and write $S = S_i$, $E = E_i$ and $N = N_i$. Recall that our choice of the E_i's forces $E_i = F_j$ for some j.

Lemma 3.5. *The stabilizer of the P-conjugacy class of S is the subgroup $PN_G(E)$ of G.*

Proof. By [11, Lemma 2.2], we know that $C_P(S) = N_P(S) = S \times L$, where L has normal p-rank 1 and L contains the unique central subgroup Z of P of order p. Moreover, the index of $N_P(S)$ in $N_P(E)$ is p. In particular, we deduce that $E = S \times Z$, and that $N_P(E)/N_P(S)$ acts transitively (by conjugation) on the p non-central subgroups of order p of E.

Since $Z \lhd G$, we have $N_G(S) \le N_G(E)$, and so $PN_G(S) \le PN_G(E)$.

Conversely, for $g \in N_G(E)$, we have ${}^g S \le {}^g E = E$. Thus, there exists $v \in P$ (actually, we can take $v \in N_P(E) - N_P(S)$) such that ${}^g S = {}^v S$, and so, $v^{-1} g \in N_G(S)$. It follows then that $g \in PN_G(S)$, as was to be shown. □

Since $PN_G(E)$ is the stabilizer of the P-conjugacy class of S, Theorem 3.1 implies that N extends to a $k[PN_G(E)]$-module $\tilde{N}$, which is necessarily endotrivial.

Let C be a set of representatives of the left cosets $G/PN_G(E)$. Then the correspondence $c \mapsto {}^c N$ is a bijection from C to the G-conjugacy class of N.

Definition 3.6. Let H be a subgroup of G and let M be a kH-module. We define the *tensor induced module* $\mathrm{Ten}_H^G M$ as follows (cf. [14, Section 5.1]). It is the k-vector space

$$\bigotimes_{s \in [G/H]} (s \otimes M), \quad \text{where } [G/H] \text{ is a set of representatives of the left cosets } G/H$$

endowed with the structure of kG-module given by

$$g \cdot \otimes_s (s \otimes m_s) = \otimes_s (\tau_g(s) \otimes h_s m_s) = \otimes_s (s \otimes h_{\tau_g^{-1}(s)} m_{\tau_g^{-1}(s)}),$$

where $gs = \tau_g(s) h_s$, with $h_s \in H$ and a permutation τ_g of $[G/H]$.

Proposition 3.7. *The kG-module $\mathrm{Ten}_{PN_G(E)}^G \tilde{N}$ is endotrivial.*

Proof. It is enough to check that $\mathrm{Res}_P^G \mathrm{Ten}_{PN_G(E)}^G \tilde{N}$ is endotrivial. By the tensor version of Mackey formula (cf. [14, Prop. 5.2.1]), and since $P \lhd G$, we have isomorphisms of kP-modules

$$\begin{aligned} \mathrm{Res}_P^G \mathrm{Ten}_{PN_G(E)}^G \tilde{N} &\cong \bigotimes_{c \in C} \mathrm{Ten}_{P \cap {}^c(PN_G(E))}^P \mathrm{Res}_{P \cap {}^c(PN_G(E))}^{{}^c(PN_G(E))} {}^c\tilde{N} \\ &= \bigotimes_{c \in C} \mathrm{Res}_P^{{}^c(PN_G(E))} {}^c\tilde{N} \cong \bigotimes_{c \in C} {}^c\left(\mathrm{Res}_P^{PN_G(E)} \tilde{N}\right) \cong \bigotimes_{c \in C} {}^c N \end{aligned}$$

which is a tensor product of endotrivial kP-modules and hence is endotrivial. □

For any elementary abelian p-group E of rank at least 2, we may identify the group $T(E) = \langle \Omega_E \rangle$ with $\mathbb{Z}$, via $n\Omega_E \mapsto n$.

Definition 3.8. Let H be a finite group and let $E_1, \ldots, E_l$ be representatives of the connected components of $\mathcal{E}_{\geq 2}(H)$. The product of all restriction maps $\mathrm{Res}_{E_i}^H : T(H) \longrightarrow T(E_i)$, composed with the isomorphism $T(E_i) \cong \mathbb{Z}$, $1 \leq i \leq l$, yields a well-defined homomorphism

$$\mathbf{res}_{\mathcal{E}(H)} : T(H) \longrightarrow \prod_{i=1}^l T(E_i) \longrightarrow \mathbb{Z}^l.$$

For any endotrivial kH-module M, the element $\mathbf{res}_{\mathcal{E}(H)}([M]) \in \mathbb{Z}^l$ is called the *type* of M.

Throughout these notes, $\delta_{i,j}$ denotes the Kronecker symbol.

Proposition 3.9. *We have* $\ker(\mathbf{res}_{\mathcal{E}(G)}) = TT(G)$. *Moreover,*

$$\mathbf{res}_{\mathcal{E}(G)}([\mathrm{Ten}_{PN_G(E_j)}^G \tilde{N}_j]) = (\delta_{i,j} a_i)_{i=1}^n,$$

where a_i is the integer defined in Theorem 2.4.

Proof. The first statement has been proved in [8, Proposition 2.3], so we only need to show the second one.

By transitivity of the restriction and Proposition 3.7, we have

$$\mathrm{Res}_{E_i}^G \mathrm{Ten}_{PN_G(E_j)}^G \tilde{N}_j \cong \bigotimes_{c \in C_j} \mathrm{Res}_{E_i}^P {}^c N_j$$

where C_j is a set of representatives of the left cosets $G/PN_G(E_j)$. We may assume that $1 \in C_j$ for all j. Now, for any $c \in C_j$, the kP-module ${}^c N_j$ is endotrivial and defined in Theorem 2.4 as the "unique" indecomposable direct summand with vertex P of a r-fold tensor product of the module

$${}^c\left(\Omega_P^{-1}(k) \otimes \Omega_{P/S_j}^1(k)\right) \cong \Omega_P^{-1}(k) \otimes \Omega_{P/{}^cS_j}^1(k)$$

where the integer r is either 1, 2 or 4. By "unique", we mean unique up to isomorphism.

Even though it is not an endotrivial module, $\mathrm{Res}_{E_i}^P \Omega_{P/{}^cS_j}^1(k)$ has a "unique" indecomposable summand V with vertex E_i, and V is isomorphic to $\Omega_{(P/{}^cS_j)\downarrow_{E_i}^P}^1(k)$. By the Mackey formula (for E_i-sets) we have that

$$(P/{}^cS_j)\downarrow_{E_i}^P \cong \coprod_{x \in [E_i \backslash P/{}^cS_j]} E_i/\left(E_i \cap {}^{xc}S_j\right).$$

If $i \neq j$ then $E_j \neq_G E_i$ and no G-conjugate of S_j is contained in E_i, since $E_i = S_i \times Z$ and $Z \lhd G$. It follows that

$$(P/{}^cS_j)\downarrow^P_{E_i} \cong \coprod_{x\in[E_i\backslash P/{}^cS_j]} E_i.$$

Thus, $V \cong \Omega^1_{E_i}(k)$ if $i \neq j$, by [4, Lemma 3.2.7], and therefore ${}^cN_j$ is trivial.

Otherwise, we have $i = j$. If ${}^{xc}S_i < E_i$, for $x \in P$, then there is $y \in N_P(E_i) - C_P(S_i)$ such that ${}^{xc}S_i = {}^yS_i$, since $N_P(E_i)/C_P(S_i)$ acts transitively on the p non-central subgroups of order p of E_i. It follows that $y^{-1}xc \in N_G(S_i)$, and so $c \in PN_G(S_i) = PN_G(E_i)$. By the choice of representatives in C_i, this implies $c = 1$. Then, Theorem 2.4 shows that $\mathrm{Res}^P_{E_i} N_i \cong \Omega^{a_i}_{E_i} \oplus (\mathrm{proj})$.

Finally, if $i = j$ and $c \notin PN_G(E_i)$, then the previous argument shows that ${}^{xc}S_i$ is not contained in E_i, for any $x \in P$. Therefore, $\mathrm{Res}^P_{E_i}{}^cN_i \cong k \oplus (\mathrm{proj})$, as in the case $i \neq j$.

In conclusion, by tensoring all the pieces together, we deduce that

$$\mathrm{Res}^G_{E_i}\mathrm{Ten}^G_{PN_G(E_j)}\tilde{N}_j \cong \bigotimes_{c\in C_j}\mathrm{Res}^P_{E_i}{}^cN_j \cong \begin{cases}\Omega^{a_i}_{E_i}(k)\oplus(\mathrm{proj}) & \text{if } i = j\\ k\oplus(\mathrm{proj}) & \text{if } i \neq j.\end{cases} \quad \square$$

Theorem 3.10. *Let G be a finite group having a normal Sylow p-subgroup P. Consider the same notation as above, and write the group $T(G)$ of endotrivial kG-modules as a direct sum $T(G) = TT(G) \oplus TF(G)$, where $TT(G)$ is the torsion subgroup and $TF(G)$ is torsion-free. Then, we may choose the set*

$$\left\{\Omega_G, x_i = [\mathrm{Ten}^G_{PN_G(E_i)}\tilde{N}_i] \mid 1 \le i \le n-1\right\} \quad \text{as a basis for the free } \mathbb{Z}\text{-module } TF(G).$$

Moreover, we have $\mathbf{res}_{\mathcal{E}(G)}(\Omega_G) = (1)^n_{i=1}$, and $\mathbf{res}_{\mathcal{E}(G)}(x_j) = (\delta_{i,j}a_i)^n_{i=1}$.

Proof. The kG-modules $\mathrm{Ten}^G_{PN_G(E_i)}\tilde{N}_i$ are endotrivial and their classes are linearly independent, by Theorem 2.4 and the proof of Proposition 3.7. Hence they generate a torsion-free submodule of $T(G)$ of rank equal to the rank of $TF(G)$.

Moreover, the proof of Proposition 3.7 shows that in $T(P)$ we have

$$\mathrm{Res}^G_P[\mathrm{Ten}^G_{PN_G(E_i)}\tilde{N}_i] = \sum_{c\in C_i}[{}^cN_i]$$

where C_i denotes a set of the representatives of the left cosets $G/PN_G(E_i)$. Since $\{{}^cN_i \mid c \in C_i\}$ is a sequence without repetition that contains all the G-conjugates of N_i, it also makes the element $\sum_{c\in C_i}[{}^cN_i]$ of $T(P)$ satisfy the minimal necessary condition for being G-stable, by Proposition 3.4. This proves the first statement. The second claim has been proved in Proposition 3.9. $\square$

4. Alternative constructions

Throughout this section, we let G be a finite group having a normal Sylow p-subgroup P. We assume $n \ge 2$, and we choose the representatives $E_1, \ldots, E_n$ and $F_1, \ldots, F_m$ of the connected components of $\mathcal{E}_{\ge 2}(G)$ and $\mathcal{E}_{\ge 2}(P)$, respectively, such that $E_i = F_i, \forall\, 1 \le i \le n-1$ and $E_n = F_m$, as before. Write also $Z = \langle z\rangle$ for the unique central subgroup of P of order p. Finally, let a_i be the integer defined in Theorem 2.4.

The purpose of this section is to review the two constructions of endotrivial modules providing free sets of generators for $TF(G)$, as given in [8]. Then, we compare the modules obtained for each construction. We assume that the reader is familiar with the notions of cohomology of finite groups and (cohomological) varieties of modules given in [6] or [14]. The notation we use in this section is as follows. For an inclusion of finite groups $H \to K$, we denote the induced restriction map in cohomology $\mathrm{H}^*(K,k) \to \mathrm{H}^*(H,k)$ as either Res^K_H or $\downarrow^K_H$ indiscriminately. Meanwhile, we denote by $\mathrm{Res}^*_{K,H} : V_H(k) \to V_K(k)$ the induced map on the varieties.

Let us now recall the presentation of $TF(G)$ given in [8].

Theorem 4.1 *([8, Theorem 3.4]). For each $1 \leq i \leq n-1$ there exists an endotrivial kG-module U_i such that*

$$U_i{\downarrow}^G_{E_i} \cong \Omega^{a_i}_{E_i} \oplus (\text{proj}) \quad \textit{and} \quad U_i{\downarrow}^G_{E_j} \cong k \oplus (\text{proj}), \quad \forall\, j \neq i,$$

where the a_i's are the integers defined in Theorem 2.4.

The classes Ω_G, $[U_i]$, $1 \leq i \leq n-1$ form a free set of generators for $TF(G)$.

The structure of the modules U_i is described in the proof, which uses the construction originating from [10] and quoted by the authors as the "deconstruction method". Briefly, it consists in analysing the variety $V = V_{G/Z}(\overline{\Omega^{a_i}_G(k)})$, where $\overline{\Omega^{a_i}_G(k)}$ is the kG/Z-module that is the quotient of $\Omega^{a_i}_G(k)$ by the submodule $(z-1)^{p-1}\Omega^{a_i}_G(k)$. The arguments of [7, Theorem 7.2] show that for each index j, the variety V decomposes as a union of varieties $V_j \cup V'_j$ such that

$$V_j \cap V'_j = \{0\}, \quad V_j = \text{Res}^*_{G/Z,E_j/Z}(V_{E_j/Z}(k)) \quad \text{and} \quad \text{Res}^*_{G/Z,E_l/Z}(V_{E_l/Z}(k)) \subseteq V'_j, \quad \forall\, l \neq j.$$

Then, by [7, Corollary 4.3], there exists an endotrivial module of the desired type, whose construction appeals to previous results (and ultimately from a theorem of [2], applied to the Lyndon–Hochschild–Serre spectral sequence of the group extension $1 \to Z \to G \to G/Z \to 1$). Indeed, it turns out that each V_j is the variety of the kG/Z-module that is the quotient $U_j/(z-1)^{p-1}U_j$ of an indecomposable endotrivial kG-module U_j.

We now turn to the third construction. Note that it does not require $P \lhd G$, and that, instead, it necessitates an additional assumption on the cohomology group $\text{H}^*(G,k)$.

Proposition 4.2 ([7, Corollary 4.6]). *Suppose that Z is a subgroup of order p in the center of a Sylow p-subgroup P of G. Suppose that for $d > 0$ the group $\text{H}^{2d}(G,k)$ has an element ζ such that $\text{Res}^G_Z(\zeta) \neq 0$. Then G has an endotrivial module of type $(2d\delta_{i,j})^n_{i=1}$ for each $1 \leq j \leq n$.*

Let us recall how these endotrivial modules are obtained, starting from a generalization of the definition of Carlson's L_ζ's modules (cf. [6, Section 6]).

Definition 4.3. Let $\zeta \in \text{H}^s(G,S)$ be non-zero, where S is a one-dimensional kG-module and $s \geq 1$. Let $\tilde{\zeta} \in \text{Hom}_G(\Omega^s_G(k), S)$ represent ζ. We define the kG-module $L_{\zeta,S} = \ker \tilde{\zeta}$.

Among many properties of these modules, all that we need to know for our concerns is that the modules $L_{\zeta,S}$ are defined up to isomorphism by the class $\zeta \in \text{H}^s(G,S)$. In particular, for $s = 2d$, as in Proposition 4.2, we have a short exact sequence of kG-modules

$$0 \longrightarrow L_{\zeta,S} \longrightarrow \Omega^{2d}_G(k) \xrightarrow{\tilde{\zeta}} S \longrightarrow 0$$

which does not depend on the choice of $\tilde{\zeta}$, and which does not involve any projective summand. Since $\Omega^{2d}_Z(k) = k$, we have that $\Omega^{2d}_G(k){\downarrow}^G_Z \cong k \oplus (\text{proj})$. Thus, the assumption that $\zeta{\downarrow}^G_Z \neq 0$ implies that $L_{\zeta,S}{\downarrow}^G_Z$ is projective. By the Quillen Dimension Theorem (cf. [14, Corollary 8.3.3]), it follows that the variety $V_G(L_{\zeta,S})$ of $L_{\zeta,S}$ is disconnected, since the number n of connected components of $\mathcal{E}_{\geq 2}(G)$ is at least 2. More precisely, $V_G(L_{\zeta,S})$ is a union of n subspaces $V_1, \ldots, V_n$, each V_i being the variety of a kG-submodule L_i of $L_{\zeta,S}$, such that $L_{\zeta,S}$ decomposes as the direct sum $L_1 \oplus \cdots \oplus L_n$. Note that V_i is a line for each $1 \leq i \leq n-1$. Similar reasoning applies to the group P instead of G, and since $\Omega_G(k){\downarrow}^G_P \cong \Omega_P(k)$, we conclude that $L_{\zeta,S}{\downarrow}^G_P \cong L_{\zeta{\downarrow}^G_P}$ splits as a direct sum $X_1 \oplus \cdots \oplus X_m$, where $m \geq 2$ is the number of connected components of $\mathcal{E}_{\geq 2}(P)$. Moreover, if $V_P(L_{\zeta,S}{\downarrow}^G_P) = W_1 \cup \cdots \cup W_m$, with $W_i = V_P(X_i)$, then G acts by permutation on the W_i's and hence on the X_i's. It follows that $L_i{\downarrow}^G_P \cong \oplus_{g \in [G/G_i]}\, g{\cdot}X_i$, for the stabilizer $G_i = PN_G(F_i)$ of the P-conjugacy class of F_i, as in the previous section.

For each index $1 \leq i \leq n$, set $L'_i = \oplus_{j \in J_i} L_j$, where $J_i = \{j \mid 1 \leq j \leq n, i \neq j\}$.

Proposition 4.4. *Each L_i gives rise to an indecomposable endotrivial kG-module M_i, of type $(2d\delta_{i,j})_{j=1}^{n}$, built as the push-out in the following commutative diagram.*

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& L_i' \arrow[r, equal] \arrow[d] & L_i' \arrow[d] & & \\
0 \arrow[r] & L_{\zeta,S} \arrow[r] \arrow[d] & \Omega_G^{2d}(k) \arrow[r, "\tilde{\zeta}"] \arrow[d] & S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_i \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d] & S \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

Proof. The proof is a paraphrase of an argument used in the demonstration of [8, Theorem 3.1], and goes as follows.

Let $1 \le i, j \le n$. We need to show that

$$M_i{\downarrow}_{E_j}^{G} \;\cong \begin{cases} \Omega_{E_j}^{2d}(k) \oplus (\text{proj}) & \text{if } i = j \\ k \oplus (\text{proj}) & \text{otherwise.} \end{cases}$$

We have $L_i{\downarrow}_P^G = \oplus_{g \in [G/G_i]}\, g \cdot X_i$, with $X_i{\downarrow}_{E_j}^P$ projective (and hence injective) if and only if $j \neq i$, i.e. $E_j \neq F_i$, by our choice of the representatives of the connected components of $\mathcal{E}_{\ge 2}(P)$ and $\mathcal{E}_{\ge 2}(G)$. This implies that exactly one of the restrictions to E_j of the exact sequences consisting in the second column or the bottom row is split, depending on whether X_j is a summand in $L_i{\downarrow}_P^G$ or not. Namely, the restriction to E_j of the second column splits if and only if $X_j \mid L_i{\downarrow}_P^G$, that is, if and only if $X_j = g \cdot X_i$, for some $g \in G$. This happens if and only if $i = j$, and in this case we have

$$M_j{\downarrow}_{E_j}^{G} \oplus (\text{proj}) \cong \Omega_P^{2d}(k){\downarrow}_{E_j}^{P} \cong \Omega_{E_j}^{2d}(k) \oplus (\text{proj}).$$

Hence $M_j{\downarrow}_{E_j}^{G} \cong \Omega_{E_j}^{2d}(k) \oplus (\text{proj})$.

Otherwise, $i \neq j$ and the restriction to E_j of the bottom row splits, since the left term is injective. In this case, we have $M_i{\downarrow}_{E_j}^{G} \cong k \oplus (\text{proj})$.

Thus $M_i{\downarrow}_{E_j}^{G}$ is endotrivial for all $1 \le i, j \le n$, and the type of M_i is $(2d\delta_{i,j})_{i=1}^{n}$, as asserted. □

We discuss the possible additional assumptions needed for this construction to be carried out. That is, we want to give a sufficient and necessary condition on the cohomology group $\mathrm{H}^{2d}(P, k)$, that would detect when there exists a one-dimensional kG-module S such that $\mathrm{H}^{2d}(G, S)$ contains an element that restrict non-trivially to Z.

Let H be a normal subgroup of G. Then $\mathrm{Hom}_H(M{\downarrow}_H^G, k)$ is a kG-module, for any kG-module M, and since the restriction map commutes with the differentials in complexes of kG-and kH-modules, it induces a map of kG-modules $\mathrm{Res}_H^G\colon \mathrm{H}^s(G, k) \longrightarrow \mathrm{H}^s(H, k)$ in cohomology (cf. [14, Section 4.1]). In fact, H acts trivially on $\mathrm{Hom}_H(M{\downarrow}_H^G, k)$, and so, we can consider Res_H^G as a map of kG/H-modules. Similarly, for G, P, Z and $2d$ as in Proposition 4.2, the map $\mathrm{Res}_Z^P\colon \mathrm{H}^{2d}(P, k) \longrightarrow \mathrm{H}^{2d}(Z, k)$ is a map of kG/P-modules. Since p does not divide the order of G/P, the map Res_Z^P splits. Now, the fact that $\mathrm{H}^{2d}(Z, k)$ is one dimensional implies that, if Res_Z^P is non-zero, then any non-zero element $\beta \in \mathrm{H}^{2d}(Z, k)$ lifts to an element $\zeta \in \mathrm{H}^{2d}(P, k)$ that restricts non-trivially to Z and such that $(g \cdot \zeta){\downarrow}_Z^P = g \cdot \zeta{\downarrow}_Z^P = \mu_g \beta$, for some $|G : P|$-root of unity $\mu_g \in k$, for any $g \in G$.

On the other hand, we also have that the action of G on the one-dimensional k-vector space $\mathrm{H}^s(Z, k)$, for any integer $s \ge 0$, defines a representation $\rho : G \to \mathrm{Aut}(\mathrm{H}^s(Z, k))$ of G. Set H for the kernel of ρ, that is, $H = \{g \in G \mid g \cdot \zeta = \zeta, \forall\, \zeta \in \mathrm{H}^s(Z, k)\}$ is the stabilizer of $\mathrm{H}^s(Z, k)$ in G. In particular, H is a normal subgroup of

G containing P, and the quotient group G/H is cyclic of order prime to p, since it is isomorphic to a finite subgroup of the group of units in k. Hence, $\mathrm{H}^s(Z,k)$ is a kG/H-module.

Lemma 4.5. *Let $s \geq 0$ be an integer and assume there exists $\zeta \in \mathrm{H}^s(P,k)$, such that $\zeta\downarrow^P_Z \neq 0$. Then $\mathrm{Tr}^H_P\zeta \in \mathrm{H}^s(H,k)$ and we have $(\mathrm{Tr}^H_P\zeta)\downarrow^H_Z \neq 0$, where Tr^H_P denotes the transfer in cohomology (cf. [14, Section 4.2]).*

Proof. Let $\zeta \in \mathrm{H}^s(P,k)$, for some $s \geq 0$, and assume that $\zeta\downarrow^P_Z \neq 0$. By definition of H, we have

$$\mathrm{Res}^H_Z \mathrm{Tr}^H_P \zeta = \sum_{g\in[H/P]} g\cdot\zeta\downarrow^P_Z = |H:P| \cdot \zeta\downarrow^P_Z .$$

Since $|H:P|$ is invertible in k, we have that $\mathrm{Res}^H_Z\mathrm{Tr}^H_P\zeta \neq 0$ if $\zeta\downarrow^P_Z \neq 0$. □

This observation leads us to a sufficient and necessary condition for determining when the last construction of endotrivial modules can be applied, provided that it applies for $T(P)$.

Proposition 4.6. *There exist a one-dimensional kG-module S and $\eta \in \mathrm{H}^s(G,S)$ such that $\eta\downarrow^G_Z \neq 0$ if and only if there exists $\zeta \in \mathrm{H}^s(P,k)$ such that $\zeta\downarrow^P_Z \neq 0$.*

In particular, for $s = 2d$, there exists then an indecomposable endotrivial module of type $(2d\delta_{i,j})^n_{i=1}$, for each index $1 \leq j \leq n$.

Proof. The "*if*" part is clear.

Assume there exists $\zeta \in \mathrm{H}^s(P,k)$ such that $\zeta\downarrow^P_Z \neq 0$, and let H be the stabilizer of $\mathrm{H}^s(Z,k)$ in G, as above. By Lemma 4.5, we have that $\mathrm{Tr}^H_P\zeta \in \mathrm{H}^s(H,k)$ and that $(\mathrm{Tr}^H_P\zeta)\downarrow^H_Z \neq 0$. By the Eckmann–Shapiro Lemma (cf. [14, Proposition 4.1.3]), we have $\mathrm{H}^s(H,k) \cong \mathrm{H}^s(G,k\uparrow^G_H)$. Thus, the image of $\mathrm{Tr}^H_P\zeta$ under this isomorphism is an element of $\mathrm{H}^s(G,k\uparrow^G_H)$ that restricts non-trivially to the subgroup Z. Now, $k\uparrow^G_H$ is a direct sum of one-dimensional kG-modules, since G/H is a cyclic group of order prime to p, as noted above, and since k is algebraically closed. Therefore, there is a one-dimensional direct summand S of $k\uparrow^G_H$ and $\eta \in \mathrm{H}^s(G,S)$ such that $\eta\downarrow^G_Z \neq 0$, as was to be shown.

The last assertion is then the application of Proposition 4.2. □

We end this section with the comparison of the modules obtained in each construction, and, for convenience, we use the same notation. First, observe that a module $\mathrm{Ten}^G_{G_i}\tilde{N}_i$ of Section 3 is not indecomposable and it has a large dimension, in general. Moreover the corresponding kP-module N_i is the *cap* of a (larger, in general) endo-permutation module. That is, N_i is the unique direct summand with vertex P of a capped endo-permutation kP-module. Consequently, if one wishes to compute $T(G)$ with a computer support, the generators for $TF(G)$ provided by the $\mathrm{Ten}^G_{G_i}\tilde{N}_i$ are likely to involve time-consuming algorithms. Instead, the modules of Section 4 are indecomposable and their construction by an algebra software is likely to be more handy.

In order to compare the three presentations of $TF(G)$, we use the injectivity of the restriction map $\mathbf{res}_{\mathcal{E}(G)}\colon TF(G) \to \prod^n_{i=1} T(E_i)$. Then, there is the simple matter of comparing the computations in Theorems 3.10 and 4.1, and Proposition 4.2.

Remark 4.7. In $\mathbf{stmod}(kG)$, we have isomorphisms $\mathrm{Ten}^G_{G_i}\tilde{N}_i \cong U_i^* \cong M_i^*$, $\forall\, 1 \leq i \leq n$. In particular, since U_i and M_i are indecomposable, we have $U_i \cong M_i$ in $\mathbf{mod}(kG)$, for all $1 \leq i \leq n$.

It is remarkable, and remains unexplained, that the modules U_i and M_i are isomorphic, since their construction is apparently different. Moreover, the modules M_i cannot be built in general, whereas the modules U_i always exist.

5. Odd extraspecial example

Let p be an odd prime and let P be a Sylow p-subgroup of $\mathrm{PSL}_3(p)$. That is, P is an extraspecial p-group of order p^3 and exponent p. Let G be its normalizer in $\mathrm{PSL}_3(p)$.

There are $p+1$ conjugacy classes in P of maximal elementary abelian p-subgroups of rank 2, each consisting of a single maximal normal subgroup of P. Let us call these subgroups $F_1, \ldots, F_{p+1}$. Choose also non-central subgroups $S_i < F_i$, $1 \leq i \leq (p+1)$ of order p.

By Theorem 2.4, the group $T(P)$ of endotrivial modules is free abelian of rank $p+1$ and admits a presentation $T(P) = \langle \Omega_P, [N_i], 1 \le i \le p \rangle$, where $[N_i]$ denotes the class of the indecomposable endotrivial summand of the kP-module $\left(\Omega_P^{-1}(k) \otimes \Omega^1_{P/S_i}(k)\right)^{\otimes 2}$, for each index i. The elements $[N_i]$, for $1 \le i \le p+1$, verify a non-trivial relation in $T(P)$ (cf. also [5, Section 11])

$$\sum_{i=1}^{p+1} [N_i] = 2p\ \Omega_P, \quad \text{since both have the same type, namely } (2p, \ldots, 2p) \in \mathbb{Z}^{p+1}.$$

Let us also point out that, for each index i, the module $(\Omega_P^{-1}(k) \otimes \Omega^1_{P/S_i}(k))^{\otimes 2}$ is not endotrivial (and hence it is decomposable). Indeed, it has dimension $(p^3-1)^2(p^2-1)^2$, which is not congruent to $\pm 1 (\mathrm{mod}\ p^3)$, $\forall\ p > 2$.

Let us now consider $T(G)$. By Theorem 2.6, the torsion subgroup $TT(G)$ is generated by the one-dimensional kG-modules, since $TT(P)$ is trivial. That is, $TT(G)$ is isomorphic to the character group of the abelian p'-group $G/P \cong C_{p-1} \times C_{p-1}$. On the other hand, for $TF(G)$, we note that there are exactly two elementary abelian p-subgroups which are normal in G, and the $p-1$ others form a unique G-conjugacy class. Say $F_1, F_2 \lhd G$, and $F_3 =_G \cdots =_G F_{p+1}$. Then, we may choose $E_i = F_i$, $1 \le i \le 3$, and so we get $G_1 = G_2 = G$ and G_3 has index $p-1$ in G, where $G_i = PN_G(E_i) = N_G(E_i)$ for $1 \le i \le 3$. Indeed, by the Class Formula, the cardinality of the G-orbit of E_3 is $p-1$, which is hence the index in G of the stabilizer $PN_G(E_3)$ of the G-conjugacy class of E_3.

From Proposition 3.4 and Lemma 3.5, we get $T(P)^{G/P} = \langle \Omega_P, [N_1], [N_2] \rangle \cong \mathbb{Z}^3$. Note also that $\sum_{i=3}^{p+1}[N_i] = 2p\Omega_P - [N_1] - [N_2] \in T(P)^{G/P}$.

Theorem 3.1 implies that the modules N_1 and N_2 extend to G, into indecomposable endotrivial modules $\tilde{N}_1$ and $\tilde{N}_2$, respectively. Therefore, $TF(G) = \langle \Omega_P, [\tilde{N}_1], [\tilde{N}_2] \rangle$, by Theorem 3.10.

By Proposition 3.7, the module N_3 extends to an indecomposable endotrivial kG_3-module $\tilde{N}_3$, yielding then an endotrivial kG-module $\mathrm{Ten}_{G_3}^G \tilde{N}_3$. Moreover, $[\mathrm{Ten}_{G_3}^G \tilde{N}_3] + [\tilde{N}_1] + [\tilde{N}_2] \in TT(G)$.

Lastly, we remark that in the presentation of $TF(G)$ above, the underlying modules are indecomposable, which is not the case in general. But it need to be pointed out that the modules N_i are proper direct summands of (endo-permutation) modules of dimension $(p^3-1)^2(p^2-1)^2$, and that at this point we do not know what $\mathrm{Dim}(N_i)$ is. The second half of this section will give us the answer.

We turn now to the presentation of $T(P)$ and $TF(G)$ described in Section 4, and for convenience, we adopt the same notation. In particular, since p is odd, we have $a_i = 2p$, for the integer a_i of Theorem 2.4. Hence, we can apply Proposition 4.2 to find generators for $T(P)$. Indeed, by [3, Theorem 10.1] (or [15, Theorems 6 and 7]), there exists $\zeta \in \mathrm{H}^{2p}(P,k)$ such that $\zeta\downarrow^P_Z \neq 0$. In fact, $\zeta \in \mathrm{H}^{2p}(P, \mathbb{F}_p)$ and ζ is obtained as the norm from any $F \in \mathcal{E}_{\ge 2}(P)$ to P of the inflation of a non-zero element $\eta \in \mathrm{H}^2(Z, \mathbb{F}_p)$, and where Z is identified with a quotient of F.

Let $\tilde{\zeta}\colon \Omega_P^{2p}(k) \to k$ represent ζ and set $L_\zeta = \ker \tilde{\zeta}$. Then L_ζ is the direct sum $L_1 \oplus \cdots \oplus L_{p+1}$ of indecomposable kP-modules. Moreover, all the modules L_i have the same dimension, since the automorphism group $\mathrm{Aut}(P)$ of P permutes transitively the subgroups $F_1, \ldots, F_{p+1}$ and hence $\mathrm{Aut}(P)$ also acts by permuting transitively the L_i's.

Now, the endotrivial modules M_i, for $1 \le i \le p+1$, are obtained as the push-out

$$\begin{tikzcd} L_\zeta \arrow[r] \arrow[d] & \Omega_P^{2p}(k) \arrow[d, dashed] \\ L_i \arrow[r, dashed] & M_i. \end{tikzcd}$$

In addition, by [10, Corollary 4.4], we have $\mathrm{Dim}(\Omega_P^{2p}(k)) = p^3(p+1)+1$. Therefore, we get $\mathrm{Dim}(L_i) = p^3$, and so $\mathrm{Dim}(M_i) = p^3+1$. Note that this is the minimal possible dimension that we could expect for M_i. Indeed, it has to be greater than 1 and congruent to 1 (mod $|P|$).

Last but not least, we handle $TF(G)$. Note that since $\mathrm{H}^{2p}(P,k)$ contains an element that restricts non-trivially to Z, we can apply any of the two constructions presented in Section 4. We choose the "deconstruction method" (cf. Theorem 4.1).

We consider the kG/Z-module

$$U = \Omega_G^{2p}(k) / (z-1)^{p-1}\Omega_G^{2p}(k).$$

Note that the action of $(z-1)^{p-1} = \sum_{i=0}^{p-1} z^i$ on a kZ-module M annihilates any non-free summand of M, and sends a free module onto its socle. By a dimension argument (cf. [10, Corollary 4.4]), we deduce that $\Omega_G^{2p}(k)\downarrow_Z^G \cong k \oplus (kZ)^{p^2(p+1)}$, and so $(z-1)^{p-1}\Omega_G^{2p}(k)$ has dimension $p^2(p+1)$. It follows that $\mathrm{Dim}(U) = p^2(p+1)(p-1)+1$.

In addition, in the notation of the discussion after Theorem 4.1, the variety $V = V_{G/Z}(U)$ decomposes as the union $V_1 \cup V_2 \cup V_3 \cup W$, where $W = V_1' \cap V_2' \cap V_3'$. This is because $\mathcal{E}_{\geq 2}(G)$ consists of three isolated vertices E_1, E_2, E_3. Then, each V_i is the variety of the kG/Z-module $\bar{U}_i = U_i/(z-1)^{p-1}U_i$, for an indecomposable endotrivial kG-module U_i satisfying

$$U_i\downarrow_{E_j}^G = \begin{cases} \Omega_{E_i}^{2p}(k) \oplus (\mathrm{proj}) & \text{if } i = j \\ k \oplus (\mathrm{proj}) & \text{if } i \neq j. \end{cases}$$

Since the group $\mathrm{Aut}(G)$ of automorphisms of G permutes transitively E_1, E_2 and E_3, the induced action on V permutes transitively V_1, V_2 and V_3. Consequently, the modules $\bar{U}_1, \bar{U}_2$ and $\bar{U}_3$ have the same dimension. Since the U_i's are endotrivial, we have $U_i\downarrow_Z^G = k \oplus (\mathrm{proj})$. It follows that $\mathrm{Dim}(U_i) = 1 + p\mathrm{Dim}(\bar{U}_i)$, for $1 \leq i \leq 3$, and thus the modules U_i have the same dimension.

Now, $E_i = F_i$, for $1 \leq i \leq 3$, and $E_3 =_G F_i, \forall\, 3 \leq i \leq p+1$. That is, the G-conjugacy classes of F_1 and F_2 coincide with their respective P-conjugacy classes. Therefore, $U_1\downarrow_P^G$ and $U_2\downarrow_P^G$ are isomorphic to the indecomposable endotrivial modules obtained using the techniques of Theorem 4.1 applied to P instead of G, and corresponding to F_1 and F_2 respectively. By Remark 4.7, we have $U_i\downarrow_P^G \cong M_i$ for $i = 1, 2$. Thus, $\mathrm{Dim}(U_i) = p^3+1$ and $\mathrm{Dim}(\bar{U}_i) = p^2$, for $i = 1, 2$ and 3. Note also that the isomorphism $U_3\downarrow_P^G \oplus (\mathrm{proj}) \cong \otimes_{i=3}^{p+1} M_i$ tells us that $\mathrm{Dim}((\mathrm{proj})) = (p^3+1)^{p-1} - (p^3+1)$. On the other hand, the variety W is the variety of a kG/Z-module of dimension $\mathrm{Dim}(U) - 3\mathrm{Dim}(\bar{U}_1) = p^2((p+1)(p-1)-3)+1$.

We can now answer the question concerning the dimension of the modules $\tilde{N}_1$, $\tilde{N}_2$ and $\mathrm{Ten}_{G_3}^G \tilde{N}_3$. Indeed, by duality and indecomposability, we have $\mathrm{Dim}(\tilde{N}_i) = p^3+1$ for $i = 1, 2$, whereas $\mathrm{Dim}(\mathrm{Ten}_{G_3}^G \tilde{N}_3) = (p^3+1)^{p-1}$.

In particular, $\mathrm{Ten}_{G_3}^G \tilde{N}_3$ is not indecomposable, since

$$\mathrm{Dim}(M_3) = (p-1)p^3 + 1 < (p^3+1)^{p-1} = \mathrm{Dim}(\mathrm{Ten}_{G_3}^G \tilde{N}_3), \quad \forall\, p > 2.$$

More precisely, we have $\mathrm{Ten}_{G_3}^G \tilde{N}_3 \cong M_3^* \oplus (\mathrm{proj})$, where

$$\mathrm{Dim}((\mathrm{proj})) = \mathrm{Dim}(\mathrm{Ten}_{G_3}^G \tilde{N}_3) - \mathrm{Dim}(M_3) = \sum_{i=2}^{p-1} \binom{p-1}{i} p^3 i.$$

Acknowledgment

The author would like to thank especially Jon Carlson for sharing his deep knowledge of the topic. In particular, she is grateful to him for the many enlightening discussions and suggestions that helped with generalizing several results.

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