Groups Associated with Some Types of Infinite Dimensional Lie Algebras

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1. INTRODUCTION

In this note, we shall discuss a Hopf algebraic approach to the construction of ind-affine groups (affine groups of Shafarevich types) associated with some types of infinite dimensional Lie algebras.

It is well known that a complex Lie algebra determines a Lie group up to local isomorphisms, and therefore determines a unique connected simply connected Lie group. As for the algebraic groups, G. Hochschild has shown that for a finite dimensional Lie algebra \( L \) over an algebraically closed field \( F \) of characteristic 0 whose radical is nilpotent, there exists a connected simply connected algebraic group \( G \) such that the Lie algebra of \( G \) is isomorphic to \( L \). In particular, he has shown that if \( L = [L, L] \), the dual Hopf algebra \( U(L)^0 \) of the universal enveloping algebra \( U(L) \) of \( L \) is the coordinate ring of the algebraic group \( G \) (cf. G. Hochschild [3]). For a semi-simple Lie algebra, there are some other methods to construct such an algebraic group attached to it. One method is to use linear representation of the given Lie algebra (C. Chevalley); the other is to determine the group by generators and relations (R. Steinberg). These methods have been generalized to construct groups attached to Kac-Moody Lie algebras and have been investigated by many authors (G. V. Kac [7, 8], G. V. Kac and D. H. Peterson [9], J. Tits [12, 13]). G. V. Kac [9] has shown that the group has a structure of an infinite dimensional algebraic group in the sense of Shafarevich (cf. 3.3 and I. R. Shafarevich [10]). In this paper, generalizing Hochschild's theory, we shall give a method to construct ind-affine groups (a generalization of groups introduced by Shafarevich) directly from the given Lie algebra of some types called integrable. Here the definition of integrable Lie algebra is somewhat different from that given by

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V. G. Kac [8] and it may be regarded as a generalization of some types of (finite dimensional) algebraic Lie algebras.

Let $L$ be a Lie algebra over a field $F$ of characteristic 0 generated by a system of generators $\Gamma$. A representation $(\pi, V)$ of $L$, where $V$ is not necessarily of finite dimension is called $\Gamma$-integrable if $\pi(x)$ is locally nilpotent for all $x \in \Gamma$. A Lie algebra $L$ generated by $\Gamma$ is called $\Gamma$-integrable if $\Gamma$ is finite, $(\text{ad}, L)$ is $\Gamma$-integrable, and there exists a faithful $\Gamma$-integrable representation. Note that finite dimensional Lie algebras whose radicals are nilpotent and also Kac-Moody Lie algebras are integrable (cf. Example 4.1). Now, let $L$ be a $\Gamma$-integrable Lie algebra, and let $U$ be the universal enveloping algebra of $L$. For any positive integer $n$, let $M_n$ be the left ideal of $U$ generated by $x^n$ for all $x \in \Gamma$. Then, $U$ has a structure of a topological coalgebra $U_\Gamma$ with a base $\{M_n\}_{n \in N}$ of neighbourhoods of 0 (cf. Definition 2.7). The topological dual $U^\Gamma_F$ of $U_\Gamma$ is a commutative topological algebra. We can construct a topological Hopf algebra containing $U^\Gamma_F$ as a dense subalgebra which plays the role of a dual Hopf algebra in the finite dimensional semi-simple case and it defines an ind-affine group. We do not know under what condition the constructed group is ind-affine algebraic or under what condition the Lie algebra of the group is isomorphic to the given Lie algebra.

As was shown by G. Hochschild for pro-affine algebraic groups (cf. [5]), we can show some universality property for the constructed groups but contrary to the finite dimensional case, we do not know how they depend on the choice of a finite integrable system of generators $\Gamma$ in the given Lie algebra.

For simplicity, throughout the paper, $F$ is an algebraically closed field of characteristic 0.

2. Topological Hopf Algebras and Their Duals

2.1. Topological Vector Spaces

A vector space $V$ over $F$ is called a topological vector space if it is a topological group with respect to addition with a base $\mathcal{B} = \{V_x\}_{x \in I}$ of neighbourhoods of 0 consisting of subspaces of $V$. We give $F$ the discrete topology. $V$ is called separated if $\{0\}$ is closed or equivalently $\bigcap_x V_x = 0$. For a given family $\mathcal{B} = \{V_x\}_{x \in I}$ of subspaces of $V$ such that

(N) For any $\alpha, \beta \in I$, there exists an element $\gamma \in I$ such that $V_\gamma \subseteq V_\alpha \cap V_\beta$,

$V$ has a unique structure of topological vector space with $\mathcal{B}$ a base of neighbourhoods of 0. Hereafter, we call $\mathcal{B}$ simply a base. We can define naturally topologies on subspaces, quotient spaces, dual spaces, direct
products, direct sums, projective limits, and inductive limits of topological vector spaces. For the basic concepts and properties on topological vector spaces and topological coalgebras discussed in this section, we refer the reader to M. Takeuchi [11].

2.2 Tensor Product and Completed Tensor Product

Let $V, W$ be topological vector spaces with bases $\mathcal{B}_V = \{ V_\alpha \}_{\alpha \in I}$, $\mathcal{B}_W = \{ W_\beta \}_{\beta \in J}$. Then the tensor product $V \otimes W$ is a topological vector space with a base

$$\mathcal{B}_{V \otimes W} = \{ V_\alpha \otimes W_\beta + V \otimes W_\beta \; ; \; \alpha \in I, \; \beta \in J \}$$

which is called the tensor product topology. The topological vector space $\hat{V} = \lim V/V_\alpha$ with projective limit topology, where $V/V_\alpha$ are discrete, is called the completion of $V$. There exists a canonical continuous linear map $V \rightarrow \hat{V}$ such that the image of $V$ is dense in $\hat{V}$. The map is an injection if and only if $V$ is separated. If $\hat{V} = V$, then $V$ is called complete. The completion of $V \otimes W$ is called the completed tensor product and denoted by $\hat{V} \hat{\otimes} \hat{W}$. Any continuous linear map $\rho : V \rightarrow W$ of topological vector spaces induces the continuous linear map $\hat{\rho} : \hat{V} \rightarrow \hat{W}$.

2.3. Dual of Topological Vector Spaces

Let $V$ be a topological vector space with a base $\mathcal{B}_V = \{ V_\alpha \}_{\alpha \in I}$ and let $\hat{V} = \lim V/V_\alpha$ be the completion of $V$. The vector space $V^0$ of all continuous linear maps from $V$ into $F$ can be identified with $\lim (V/V_\alpha)^*$ which is a topological vector space by inductive limit topology, where $(V/V_\alpha)^*$ is considered as a topological dual to the discrete space $V/V_\alpha$. For topological vector spaces $V$ and $W$ and a continuous linear map $\rho : V \rightarrow W$, there exists a linear map $\rho^0 : W^0 \rightarrow V^0$ defined by

$$\rho^0(f)(x) = f(\rho(x)) \quad \text{for all} \quad x \in V \quad \text{and} \quad f \in W^0,$$

which is continuous. Let $V$ be a topological vector space with a base $\mathcal{B} = \{ V_\alpha \}_{\alpha \in I}$. The canonical continuous map $(V/V_\alpha)^* \rightarrow V^0$ induces a continuous map $V^{00} \rightarrow (V/V_\alpha)^0$. Since $(V/V_\alpha)^0$ is canonically isomorphic as a vector space to $V/V_\alpha$, the map induces a canonical continuous bijection $\omega_V : V^{00} \rightarrow \hat{V}$, but it is not necessarily a homeomorphism (cf. [11, 4.6]).

2.4. Dual of the Tensor Product

Let $V, W$ be topological vector spaces with bases $\mathcal{B}_V = \{ V_\alpha \}_{\alpha \in I}$, $\mathcal{B}_W = \{ W_\beta \}_{\beta \in J}$, and let $V^0, W^0$ be their topological duals. Then $V^0 \otimes W^0$ is isomorphic as a vector space to

$$\lim (V/V_\alpha)^* \otimes \lim (W/W_\beta)^* = \lim ((V/V_\alpha)^* \otimes (W/W_\beta)^*).$$
We define on the vector space $V^0 \otimes W^0$ the inductive limit topology of the space \( \lim \left( \left( \frac{V}{V_\alpha} \right)^* \otimes \left( \frac{W}{W_\beta} \right)^* \right) \) and call this topology the dual tensor product topology and denote the topological space by $V^0 \otimes_d W^0$. The continuous linear maps \( \left( \frac{V}{V_\alpha} \right)^* \otimes \left( \frac{W}{W_\beta} \right)^* \rightarrow V^0 \otimes W^0 \) induce a continuous bijection
\[
u: V^0 \otimes_d W^0 \rightarrow V^0 \otimes W^0.
\]
We denote by $V^0 \hat{\otimes}_d W^0$ the completion of $V^0 \otimes_d W^0$. Then the map $\nu$ induces a continuous linear map
\[\hat{\nu}: V^0 \hat{\otimes}_d W^0 \rightarrow V^0 \hat{\otimes} W^0.\]

Further, the canonical continuous maps \( \left( \frac{V}{V_\alpha} \right)^* \otimes \left( \frac{W}{W_\beta} \right)^* \rightarrow \left( \frac{V}{V_\alpha} \otimes \frac{W}{W_\beta} \right)^* \) induce a continuous linear map $V^0 \otimes_d W^0 \rightarrow \left( V \otimes W \right)^0 = \lim \left( \frac{V}{V_\alpha} \otimes \frac{W}{W_\beta} \right)^*$. Also, the canonical continuous linear maps \( \left( \frac{V}{V_\alpha} \right)^* \otimes \left( \frac{W}{W_\beta} \right)^* \rightarrow V^0 \otimes_d W^0 \) induce a continuous linear map
\[(V \hat{\otimes} W)^0 \rightarrow V^0 \hat{\otimes}_d W^0.\]

Thus, we have the following commutative diagram of continuous linear maps:

\[
\begin{array}{ccc}
V^0 \otimes_d W^0 & \longrightarrow & V^0 \hat{\otimes}_d W^0 \\
\downarrow \nu & & \downarrow \hat{\nu} \\
V^0 \otimes W^0 & \longrightarrow & V^0 \hat{\otimes} W^0
\end{array}
\]

2.5. Topological Coalgebras and Algebras

A topological coalgebra is a triple $(C, \Delta, \varepsilon)$ consisting of a topological vector space $C$ and continuous linear maps $\Delta: C \rightarrow C \hat{\otimes} C$ and $\varepsilon: C \rightarrow F$ satisfying
\[
(\Delta \hat{\otimes} 1) \cdot \Delta = (1 \hat{\otimes} \Delta) \cdot \Delta
\]

\[\Delta \cdot (1 \otimes \varepsilon) = (\varepsilon \otimes 1) \cdot \Delta \quad \text{is the canonical map } C \rightarrow \hat{C}.
\]

The completion $\hat{C}$ of $C$ is a topological coalgebra with the structure maps $\hat{\Delta}$ and $\hat{\varepsilon}$, and the canonical map $C \rightarrow \hat{C}$ is a continuous coalgebra homomorphism. A topological algebra is a triple $(A, \mu, \eta)$ consisting of a topological vector space $A$ and continuous linear maps $\mu: A \otimes A \rightarrow A$ and $\eta: F \rightarrow A$ satisfying
\[
\mu \cdot (\mu \otimes 1) = \mu \cdot (1 \otimes \mu)
\]

\[\mu \cdot (\eta \otimes 1) = \mu \cdot (1 \otimes \eta) = \text{id.}\]
Let $C^0$ be the topological dual of a topological coalgebra $C$; then there exist canonical continuous maps
\[
\mu: C^0 \otimes_a C^0 \xrightarrow{\text{cano}} (C \hat{\otimes} C)^0 \xrightarrow{} C^0
\]
\[
\eta: F \xrightarrow{\varepsilon^0} C^0
\]
which satisfy the axiom of the algebra structure maps. We see that $C^0$ is a topological algebra by dual tensor product topology and it is called the dual algebra of $C$. Let $A^0$ be the topological dual of a topological algebra $A$. The structure map $\mu: A \otimes A \to A$ induces a canonical continuous map $\hat{\mu}: A \hat{\otimes} A \to A$. Since $A^0 = A^0$, there exist canonical continuous linear maps
\[
A: A^0 \hat{\otimes} A^0 \xrightarrow{\text{cano}} (A \hat{\otimes} A)^0 \xrightarrow{} A^0 \otimes_a A^0
\]
\[
\varepsilon: A^0 \to F
\]
which satisfy the axiom of coalgebra structure maps. We see that $A^0$ is a topological coalgebra by dual tensor product topology and it is called the dual coalgebra of $A$.

2.6. Duals of Dual Algebras and Dual Coalgebras

Let $V$, $W$ be topological vector spaces with bases $\mathcal{B}_V = \{V_\alpha\}_{\alpha \in I}$, $\mathcal{B}_W = \{W_\beta\}_{\beta \in J}$. By definition
\[
V^0 \otimes_a W^0 = \varinjlim \left((V/V_\alpha)^* \otimes (W/W_\beta)^*\right)
\]
is a topological vector space with inductive limit topology. The topological dual $(V^0 \otimes_a W^0)^0$ of $V^0 \otimes_a W^0$ is denoted by $V^{oo} \otimes_{dd} W^{oo}$. Then there exists a canonical continuous bijection
\[
V^{oo} \otimes_{dd} W^{oo} \to V \hat{\otimes} W.
\]
If $C$ (resp. $A$) is a topological coalgebra (resp. topological algebra), then $C^{oo}$ (resp. $A^{oo}$) is a topological coalgebra (resp. topological algebra) by the dual of the dual tensor product topology. Further, there exists a continuous bijection of the coalgebra homomorphism $C^{oo} \to C$ (resp. algebra homomorphism $A^{oo} \to A$).

2.7. Topological Hopf Algebras and Their Duals

A topological vector space $H$ over $F$ is called a topological Hopf algebra if there exist continuous linear maps
\[
\mu: H \otimes H \to H, \quad \eta: F \to H, \quad \Delta: H \to H \hat{\otimes} H
\]
\[
\varepsilon: H \to F, \quad S: H \to H
\]
such that \((H, \mu, \eta)\) is a topological algebra, \((H, A, \varepsilon)\) is a topological coalgebra, and the structure map satisfies

\[
\hat{\mu} \cdot (S \hat{\otimes} 1) \cdot \Delta = \hat{\mu} \cdot (S \hat{\otimes} 1) \cdot \Delta = \hat{\eta} \cdot \varepsilon.
\]

If \(H\) is a topological Hopf algebra, then the completion \(\hat{H}\) of \(H\) has also a structure of a topological Hopf algebra induced by the structure of \(H\). We call \(\hat{H}\) the completion of \(H\). For a topological Hopf algebra \(H\), the topological dual \(H^0\) has a structure of a topological Hopf algebra with respect to the dual tensor product topology. We call \(H^0\) the dual Hopf algebra of \(H\). Also \(H^{00}\) has a structure of a topological Hopf algebra with respect to the dual of the dual tensor product topology and there exists a bijective continuous homomorphism of Hopf algebras from \(H^{00}\) onto \(\hat{H}\).

2.8. Examples

(1) Let \(H\) be a Hopf algebra with discrete topology; then the dual space \(H^*\) of \(H\) has a structure of a topological Hopf algebra with respect to the dual space topology. This is the dual Hopf algebra of \(H\).

(2) Let \(A\) be an algebra and let \(\{M_\alpha\}_{\alpha \in I}\) be a family of subspaces of \(A\) with a directed index set \(I\) such that

(A_1) \quad M_\alpha \subseteq M_\beta \text{ if } \alpha \geq \beta,
(A_2) \quad \eta(F) \cap M_\alpha \text{ for all } \alpha \in I,
(A_3) \text{ for any } \alpha \in I, \text{ there exists } \beta \in I \text{ such that }
\[\mu(M_\beta \otimes A + A \otimes M_\beta) \subseteq M_\alpha.\]

Then \(A\) is a topological algebra with a base \(\{M_\alpha\}_{\alpha \in I}\). Also, let \(C\) be a coalgebra and let \(\{N_\beta\}_{\beta \in I}\) be a family of subspaces of \(C\) with a directed index set \(I\) such that

(C_1) \quad N_\alpha \subseteq N_\beta \text{ if } \alpha \geq \beta,
(C_2) \quad \varepsilon N_\alpha = 0 \text{ for all } \alpha \in I,
(C_3) \text{ for any } \alpha \in I, \text{ there exists } \beta \in I \text{ such that }
\[\Delta N_\beta \subseteq C \otimes N_\alpha + N_\alpha \otimes C.\]

Then \(C\) is a topological coalgebra with a base \(\{N_\alpha\}_{\alpha \in I}\).

(3) Let \(H\) be a Hopf algebra and let \(\{M_\alpha\}_{\alpha \in I}\) be a family of proper two sided ideals with a directed index set \(I\) satisfying (C_1), (C_2), (C_3), and further

(S) \quad \text{For any } \alpha \in I, \text{ there exists } \beta \in I \text{ such that }
\[SM_\beta \subseteq M_\alpha.\]
Then $H$ has a structure of a topological Hopf algebra with a base \( \{ M_x \}_{x \in I} \). Also, let $H$ be a Hopf algebra and let \( \{ N_x \}_{x \in I} \) be a family of coideals with a directed index set $I$ satisfying (A1), (A2), (A3), and (S). Then $H$ has a structure topological Hopf algebra with a base \( \{ N_x \}_{x \in I} \).

In particular, let $H$ be a Hopf algebra and let \( \{ M_x \}_{x \in I} \) be the set of all proper finite codimensional two sided ideals of $H$ and the order of $I$ is defined by $x \geq \beta$ if and only if $M_x \subset M_\beta$. Then the family \( \{ M_x \}_{x \in I} \) satisfies (C1), (C2), (C3), and (S). Thus $H$ has a structure of a topological Hopf algebra with a base \( \{ M_x \}_{x \in I} \). The dual Hopf algebra $H^\text{op}$ is a discrete Hopf algebra. This is the usual dual Hopf algebra of $H$.

### 3. Ind-Affine Groups

#### 3.1. Ind-Affine Varieties

Let $A$ be a complete commutative algebra over $F$. $A$ is called an algebra of Shafarevich type if $A$ is a projective limit \( \lim A_x \) of discrete commutative algebras $A_x$ such that the index set $I$ is a countable directed set and the homomorphisms $\mu_{\alpha \beta}: A_\beta \to A_\alpha$ are surjective for all $\alpha \leq \beta, \alpha, \beta \in I$. From the condition, the canonical homomorphisms $u_\alpha: A \to A_x$ are surjective and $A$ can be identified with \( \lim A/N_x \), where $N_x = \ker u_\alpha$ and \( \{ N_x \}_{x \in I} \) is a base of $A$ such that $\bigcap_{\alpha} N_x = 0$.

The set $X_A = \text{TA}_{\text{Alg}}(A, F)$ has a structure of topological space with the family of closed sets consisting of

$$V(M) = \{ x \in X_A ; x(f) = 0 \quad \text{for all} \quad f \in M \}$$

for all ideals $M$ of $A$. The topology is called Zariski topology and $X_A$ is called an ind-affine variety associated with $A$. In particular, if $A$ is discrete, then the associated variety $X_A$ is called an affine variety and further, if $A$ is a finitely generated algebra, then $X_A$ is called an affine algebraic variety.

Ind-affine variety is an inductive limit of affine varieties. For the algebra $A = \lim A_x$ of Shafarevich type, if $A_x$ are finitely generated, then the associated variety $X_A$ is an inductive limit of affine algebraic varieties and it is called an ind-affine algebraic variety.

Let $A, B$ be algebras of Shafarevich type and let $X_A, X_B$ be the associated ind-affine varieties. For a continuous homomorphism $u: B \to A$ of topological algebras, the continuous map

$$u: X_A \to X_B, \quad x \to x \cdot u$$

is called a morphism of $X_A$ into $X_B$. Denote by $\text{Var}_F(X_A, X_B)$ the set of all morphisms from $X_A$ into $X_B$. $\text{Var}_F(X_A, F)$ has the structure of a
commutative algebra over $F$ and is called the coordinate ring of $X_A$ which can be identified with the topological algebra $A_{\text{red}} = A/R$, where $R$ is the intersection of all closed maximal ideals $\ker x$ for all $x \in X_A$, namely

$$R = \{ f \in A ; x(f) = 0 \quad \text{for all} \quad x \in X_A \},$$

which we call the radical of $A$.

3.1.1. PROPOSITION. Let $A = \varprojlim A_\alpha$ be an algebra of Shafarevich type and let $R, R_\alpha$ be the radicals of $A$ and $A_\alpha$, respectively. Then

$$A_{\text{red}} = A/R = \varprojlim A_\alpha/R_\alpha = \varprojlim (A_\alpha)_{\text{red}}$$

and it is also an algebra of Shafarevich type.

\textbf{Proof.} Since the homomorphisms $u_{\beta \alpha} : A_\beta \rightarrow A_\alpha$ are surjective for all $\alpha \leq \beta$, the canonical maps $u_\alpha : A \rightarrow A_\alpha$ are surjective and $u_{\alpha \beta}$ maps $R_\beta$ onto $R_\alpha$. Thus, we can see that the canonical homomorphism $A \rightarrow \varprojlim A_\alpha/R_\alpha$ is surjective and its kernel $\bigcap_\alpha u^{-1}_\alpha (R_\alpha)$ is equal to $R$. In fact, since $u_\alpha (R) \subset R_\alpha$ for all $\alpha \in \mathcal{A}$, $R \subset \bigcap_\alpha u^{-1}_\alpha (R_\alpha)$. Conversely, if $f \notin R$, then there exists an element $x \in X_A$ such that $x(f) \neq 0$. On the other hand, $x \in X_{A_\alpha}$ for some $\alpha \in \mathcal{A}$, namely, $x(N_\alpha) = 0$ and $x(f + N_\alpha) \neq 0$. Thus, $u_\alpha (f) \notin R_\alpha$ and $f \notin \bigcap_\alpha u^{-1}_\alpha (R_\alpha)$.

Note that if $A$ is a (discrete) finitely generated commutative algebra, then $R$ is the nilradical nil $A$ of $A$ and $A_{\text{red}} = A/(\text{nil } A)$. In general, if $A$ is a (discrete) commutative algebra, then $A$ is an inductive limit $\varinjlim A_\alpha$ of finitely generated subalgebras $A_\alpha$ of $A$. The map $\varphi_\alpha$ from the set $\{ \ker x \}_{x \in X_A}$ of maximal ideals of $A$ into the set $\{ \ker x_\alpha \}_{x_\alpha \in X_{A_\alpha}}$ of maximal ideals of $A_\alpha$ defined by $\varphi_\alpha (M) = M \cap A_\alpha$ is surjective. Thus, we have $R_\alpha = R \cap A_\alpha$ and

$$R = \varinjlim (R \cap A_\alpha) = \varinjlim (\text{nil } A_\alpha) = \text{nil } A.$$

Therefore, we have

3.1.2. PROPOSITION. Let $A = \varprojlim A_\alpha$ be an algebra of Shafarevich type. Then $A_{\text{red}} = \varprojlim A_\alpha/(\text{nil } A_\alpha)$.

For $x \in X_A$, let $M_x = \{ f \in A_{\text{red}} ; x(f) = 0 \}$ be a closed ideal of $A_{\text{red}}$ and let $M_x^{(2)}$ be the closure of $M_x^2$ in $A_{\text{red}}$. The topological vector space $\Omega_x = M_x^{(2)}/M_x^{(2)}$ is called the cotangent space to $X_A$ at $x$ and the topological dual $T_x(X_A) = \text{Mod}_F(\Omega_x, F)$ is called the tangent space to $X_A$ at $x$. $T_x(X_A)$ is the vector space over $F$ consisting of all continuous linear maps $\delta$ from $A_{\text{red}}$ into $F$ such that $\delta(fg) = \delta(f) g(x) + f(x) \delta(g)$ for all $f, g \in A_{\text{red}}$.\n
The direct product $X_A \times X_B$ has a structure of Ind-affine variety associated with the algebra $A \hat{\otimes} B$ of Shafarevich type which we call the direct product of $X_A$ and $X_B$.

3.2. Ind-Affine Groups

Let $A$ be a topological Hopf algebra whose underlying algebra is of Shafarevich type. We call $A$ a Hopf algebra of Shafarevich type. Then $G = X_A$ has a structure of a group such that the map

$$G \times G \rightarrow G, \quad (x, y) \rightarrow xy^{-1}$$

is a morphism of ind-affine varieties and $G$ is called an ind-affine group. In particular, if the underlying variety of $G$ is ind-affine algebraic, affine, or affine algebraic, we call $G$ an ind-affine algebraic group, affine group, or affine algebraic group, respectively. Ind-affine algebraic groups were introduced by I. R. Shafarevich and were called infinite dimensional algebraic groups (cf. [10]). Let $G$ be an affine group associated with a (discrete) commutative Hopf algebra $A$. Then the Hopf algebra $A$ is an inductive limit $\lim A_z$ of finitely generated sub-Hopf algebras $A_z$ of $A$. Therefore, $G$ is a projective limit of affine algebraic groups $G_z = \text{Alg}_F(A_z, F)$ and it is called also a pro-affine algebraic group (cf. [4]).

3.2.1. Proposition. Let $A$ be a Hopf algebra of Shafarevich type. Then the radical $R$ of $A$ is a closed Hopf ideal of $A$ and $A_{\text{red}}$ has also a structure of Hopf algebra of Shafarevich type.

Proof. It is obvious that $R$ is a closed ideal. If $f \in R$, then $\varepsilon(f) = 0$ for the identity of $G$. Thus, $\varepsilon(R) = 0$. Also, if $f \in R$, then $x(Sf) = x^{-1}(f) = 0$ for all $x \in G$. Therefore, $S(R) \subset R$. It remains to show that $A(R) \subset R \otimes A + A \otimes R$. Let $u_z: A \rightarrow A_z$, $u_\beta \otimes u_\beta: A \otimes A \rightarrow A_\beta \otimes A_\beta$ be the canonical homomorphism. For any $f \in A$ and $x \in I$, there exists $\beta \in I$ such that $\Delta(u_\beta(f)) \subset A_\beta \otimes A_\beta$ and $\Delta(u_\beta(f)) = u_\beta \otimes u_\beta(\Delta(f))$. It suffices to show that if $f \in R$, then $\Delta(u_\beta(f)) \subset A_\beta \otimes R_\beta + R_\beta \otimes A_\beta$. Let $f \in R$ and $\Delta(u_\beta(f)) = \sum_{i \in I} f_i \otimes g_i$, where $f_i, g_i \in A_\beta$ and $I$ is a finite set and we may assume that $\{g_i\}_{i \in I}$ are linearly independent over $F$. Let $I = J \cup K$, where $J \cap K = \emptyset$ and $g_i \notin R_\beta (i \in J)$ and $g_i \in R_\beta (i \in K)$. Then, there exists a set $\{x_i\}_{i \in J}$ of distinct elements of $X_\alpha$ such that $x_i(g_i) \neq 0 (i \in J)$. Further, there exists a set $\{g_i\}_{i \in J}$ of elements of $A_\beta$ which spans the vector subspace of $A_\beta$ spanned by $\{g_i\}_{i \in J}$ such that $x_i(g_i) = \delta_{ij} (i, j \in J)$ (cf. Lemma 1.1 of [6]). Thus, we may express $\sum_{i \in J} f_i \otimes g_i = \sum_{i \in J} f'_i \otimes g'_i$, where $f'_i$ are linear combinations of $\{f_i\}_{i \in J}$. Since $u_\beta(f) \in R_\beta$, $$(x x_k)(u_\beta(f)) = x(f'_k) = 0 \text{ for all } x \in X_\alpha.$$

Therefore, $f'_k \in R_\beta$ and $\Delta(u_\beta(f)) \in R_\beta \otimes A_\beta + A_\beta \otimes R_\beta$. \[\square\]
The tangent space $T_e(G)$ to $G$ at the identity $e$ has a structure of a Lie algebra defined by
\[
[\delta, \delta'] = (\delta \hat{\otimes} \delta' - \delta' \hat{\otimes} \delta) \cdot A \quad \text{for all} \quad \delta, \delta' \in T_e(G).
\]
It is called the Lie algebra of $G$ and denoted by $\text{Lie } G$. Further, we see that
\[
G = \{ x \in (A_{\text{red}})^0; \Delta x = x \otimes x, \varepsilon x = 1 \}
\]
\[
\text{Lie } G = \{ x \in (A_{\text{red}})^0; \Delta x = x \otimes 1 + 1 \otimes x, \varepsilon x = 0 \},
\]
where $(A_{\text{red}})^0$ is the dual Hopf algebra of $A_{\text{red}}$. Also, $\text{Lie } G$ can be identified with the Lie algebra $\text{Der}^G(A_{\text{red}})$ of all continuous derivations $D$ satisfying $A \cdot D = (1 \otimes D) \cdot A$. In fact, the following linear maps are inverses of each other:
\[
T_e(G) \rightarrow \text{Der}^G(A_{\text{red}}), \quad \delta \rightarrow (1 \hat{\otimes} \delta) \cdot A
\]
\[
\text{Der}^G(A_{\text{red}}) \rightarrow T_e(G), \quad D \rightarrow \varepsilon \cdot D.
\]

Let $G = X_A$, $K = X_B$ be ind-affine groups. A continuous homomorphism $u: B \rightarrow A$ of topological Hopf algebras defines a homomorphism $\varphi = u^* u$ of $G$ into $K$ which is called a morphism of ind-affine groups. The homomorphism $u$ induces a homomorphism $u_{\text{red}}: B_{\text{red}} \rightarrow A_{\text{red}}$ and its dual $(u_{\text{red}})^0$ induces a homomorphism of Lie algebras
\[
d\varphi: \text{Lie } G \rightarrow \text{Lie } K
\]
which we call the differential of $\varphi$.

4. INTEGRABLE LIE ALGEBRAS

4.1. Definition and Examples

Let $L$ be a Lie algebra over $F$ with a system of generators $\Gamma$. A representation $(\pi, V)$ of $L$ is called $\Gamma$-integrable if $\pi(x)$ is locally nilpotent for all $x \in \Gamma$. If $L$ and $\Gamma$ satisfy the properties

(1) $\Gamma$ is finite,

(2) $(\text{ad}, L)$ is $\Gamma$-integrable,

(3) there exists a faithful $\Gamma$-integrable representation,

then $L$ is called $\Gamma$-integrable or simply integrable and $\Gamma$ is called an integrable system of generators.

Remark. V. G. Kac [8] has defined integrable Lie algebras and integrable representations using locally finite elements. Our definition is
somewhat different from that of [5], for we use locally nilpotent elements instead of locally finite elements and assume some additional conditions to discuss general infinite dimensional Lie algebras over an algebraically closed field of characteristic 0.

**Examples.** (1) Let \( L = Fx \) be a one dimensional Lie algebra and set \( \Gamma = \{x\} \). Take the representation \((\pi, V)\) of degree 2 for a fixed base defined by

\[
\pi: L \to \text{End}(V), \quad x \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Then \( \pi \) is a faithful \( \Gamma \)-integrable representation and \( L \) is integrable. Note that the representation \((\pi, V)\) of degree 2 defined by

\[
\pi: L \to \text{End}(V), \quad x \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

is faithful but not \( \Gamma \)-integrable.

(2) Let \( L = Fx + Fy \) be a two dimensional solvable Lie algebra defined by \([x, y] = y\) and take \( \Gamma = \{x, y\} \). The representation \((\pi, V)\) of \( L \) of degree 2 for a fixed base defined by

\[
\pi: L \to \text{End}(V), \quad x \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad y \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

is faithful but \( L \) is not \( \Gamma \)-integrable. Note that \( L \) is the Lie algebra of the linear algebraic group

\[
G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} ; a \in F^\times, b \in F \right\}.
\]

(3) Let \( L \) be a finite dimensional semi-simple Lie algebra over \( F \). Let \( L = H + \sum_{\alpha \in \Phi} Fx_{\alpha} \) be a Cartan decomposition of \( L \), where \( H \) is a Cartan subalgebra of \( L \) and \( \Phi \) is the root system of \( L \) with respect to \( H \). Take a base \( \Pi = \{ \alpha_1, \ldots, \alpha_l \} \) of \( \Phi \) and set \( \Gamma = \{ x_{\alpha_1}, \ldots, x_{\alpha_l}, x_{-\alpha_1}, \ldots, x_{-\alpha_l} \} \). Then \( \Gamma \) is a finite integrable system of generators for \( L \) and \( L \) is an integrable Lie algebra.

(4) Let \( L \) be a finite dimensional Lie algebra over \( F \). If the radical of \( L \) is nilpotent, then \( L \) is integrable. In fact, by Ado's Theorem, there exists a faithful representation \((\pi, V)\) of \( L \) such that \( \pi(x) \) is nilpotent for all \( x \in N \), where \( N \) is the maximal nilpotent ideal of \( L \) and in our case, \( N \) is the radical of \( L \). Since \( L \) has a Levi decomposition \( L = N + S \), where \( S \) is a maximal semi-simple subalgebra of \( L \). If we take a system \( \Gamma \) of generators
for $L$ consisting of a base of $N$ together with a generator system for $S$ as in Example (3), then it is a finite integrable system of generators for $L$ and $L$ is an integrable Lie algebra.

(5) Let $A$ be a generalized Cartan matrix, i.e., $l \times l$-matrix $A = (A_{ij})$ satisfying (C1) $A_{ii} = 2$, (C2) $A_{ij} \leq 0$ if $i \neq j$, and (C3) if $A_{ij} = 0$, then $A_{ji} = 0$. Let $L$ be the Kac–Moody Lie algebra over $F$ associated with $A$, i.e., the Lie algebra generated by $3l$ elements $\{x_1, ..., x_l, h_1, ..., h_l, y_1, ..., y_l\}$ with the following relations:

$$[h_i, h_j] = 0, \quad [h_i, x_j] = A_{ij} x_j, \quad [h_i, y_j] = -A_{ij} y_j,$$

$$[x_i, y_j] = 0 \quad \text{if} \quad i \neq j, \quad [x_i, x_j] = h_i,$$

$$(\text{ad} \ x_i)^{-A_{ij}+1}(x_j) = 0, \quad (\text{ad} \ y_i)^{-A_{ij}+1}(y_j) = 0 \quad \text{if} \quad i \neq j.$$

Set $\Gamma = \{x_1, ..., x_l, y_1, ..., y_l\}$; then it is a finite integrable system of generators for $L$ and $L$ is an integrable Lie algebra.

4.2. Representative Functions

Let $L$ be a Lie algebra over $F$ generated by a finite set $\Gamma$ with $l$ elements, and let $U = U(L)$ be the universal enveloping algebra of $L$. For any integer $n \geq 1$, let $M_n$ be the left ideal of $U$ generated by $x^n$ for all $x \in \Gamma$. Then the family $\{M_n\}_{n \in I}$, where $I$ is the ordered set of positive integers, satisfies conditions (C1), (C2), and (C3) of 2.8. Thus $U$ is a topological coalgebra with a base $\{M_n\}_{n \in I}$ which we denote by $U^\circ$. Let $U_R^\circ = \lim (U/M_n)^*$ be the topological dual of $U^\circ$, a commutative topological algebra with respect to dual tensor product topology.

Let $(\pi, V)$ be a $\Gamma$-integrable representation of $L$. Then $V$ can be regarded as a $U$-module. Take a base $\{v_x\}_{x \in J}$ of $V$ over $F$ and let

$$\pi(u) v_x = \sum_\beta \pi_{x\beta}(u) v_\beta, \quad u \in U.$$  

Then $\pi_{x\beta}$ is an $F$-valued function on $U$. We denote by $R(\pi)$ the subalgebra of the dual algebra $U^*$ of $U$ with respect to the discrete topology generated by $\pi_{x\beta}$ for all $x, \beta \in J$ and we call $R(\pi)$ the algebra of representative functions of $\pi$ which is independent of the choice of a base of $V$. Then, we have the following.

4.2.1. Proposition. Let $L$ be a Lie algebra over $F$ generated by a finite set $\Gamma$ and let $(\pi, V)$ be a $\Gamma$-integrable representation of $L$. Then $R(\pi) \subset U^\circ_R$, where $U^\circ_R$ is the dual algebra of the topological coalgebra $U_R$.

Proof. It suffices to show that the function $\pi_{x\beta}$ on $U_R$ is continuous. Since $\Gamma$ is a finite set and $\pi(x)$ is locally nilpotent for all $x \in \Gamma$, for a given
there exists an integer \( n \) such that \( \pi(x)^n v_x = \pi(x^n) v_x = 0 \) for all \( x \in \Gamma \). Therefore, \( \pi(ux^n) v_x = \pi(u) \pi(x^n) v_x = 0 \) for all \( x \in \Gamma \) and all \( u \in U \). Thus, we have \( \pi_{\alpha \beta}(M_n) = 0 \) for all \( \alpha, \beta \in J \).

4.3. Separability

We shall prove the following basic result which is essentially due to Harisch-Chandra’s Lemma (cf. [21]) on the representation of infinite-dimensional Lie algebras.

4.3.1. Proposition. Let \( L \) be a Lie algebra over \( F \) generated by a finite set \( \Gamma \) such that \( (\text{ad}, L) \) is \( \Gamma \)-integrable. Let \( U \) be the universal enveloping algebra of \( L \) and for an integer \( n > 0 \), let \( M_n \) be the left ideal of \( U \) generated by \( x^n \) for all \( x \in \Gamma \). Then \( L \) has a faithful \( \Gamma \)-integrable representation if and only if \( \bigcap_n M_n = 0 \), i.e., \( U_\Gamma \) is separated by the topology defined by \( \{ M_n \}_{n \in \mathbb{N}} \).

First, we shall prove a lemma.

4.3.2. Lemma. Let \( L \) be a Lie algebra over \( F \) and let \( U \) be the universal enveloping algebra of \( L \). Then for any \( x \in L \) and \( u \in U \), we have

\[
x^t u = \sum_{i=0}^{t} \binom{t}{i} ((\text{ad} x)^i u) x^{t-i}.
\]

Proof. We shall prove by induction on \( t \). If \( t = 1 \), then \( xu = ux + (\text{ad} x) u \). Now assume that the equation holds for any positive integer \( \leq t \). Then

\[
x^{t+1} u = x \left( \sum_{i=0}^{t} \binom{t}{i} ((\text{ad} x)^i u) x^{t-i} \right)
\]

\[
= \sum_{i=0}^{t} \binom{t}{i} ((\text{ad} x)^i u) x^{t-i+1} + (\text{ad} x) \left( \sum_{i=0}^{t} \binom{t}{i} ((\text{ad} x)^i u) x^{t-i} \right)
\]

\[
= ux^{t+1} + \sum_{i=1}^{t} \left( \binom{t}{i} \binom{t}{i-1} \right) ((\text{ad} x)^{t+1-i} u) x^{t-i+1} + (\text{ad} x)^{t+1} u
\]

\[
= \sum_{i=0}^{t+1} \binom{t+1}{i} ((\text{ad} x)^i u) x^{t+1-i}.
\]

Prove of 4.3.1. Since \( (\text{ad}, L) \) is \( \Gamma \)-integrable, from Lemma 4.3.2, the right multiplication \( v \mapsto vu \) by \( u \in U_\Gamma \) is a continuous map from \( U_\Gamma \) into \( U_\Gamma \). Therefore, its dual \( \pi(u) \) defined by \( \pi(u)f(v) = f(vu) \) is a continuous map from \( U_\Gamma^0 \) into \( U_\Gamma^0 \) and it defines a representation \( (\pi, U_\Gamma^0) \) of \( U_\Gamma \). Suppose \( \bigcap_n M_n = 0 \) and \( \pi(u) = 0 \). Then \( f(vu) = 0 \) for all \( v \in U \) and for all \( f \in U_\Gamma^0 \). Therefore, \( f(u) = 0 \) for all \( f \in U_\Gamma^0 \). This shows that \( u \in \bigcap_n M_n = 0 \) and \( \pi \) is a
faithful representation of $U_f$. Also the restriction of $\pi$ to $L$ is a faithful $I'$-integrable representation. Conversely, suppose $(\pi, V)$ is a faithful $I'$-integrable representation of $L$. For any positive integer $m$, define a representation $(\pi^{(m)}, V^{(m)})$ of $L$ as follows:

$$V^{(m)} = V \otimes V \otimes \cdots \otimes V \quad (m \text{ times})$$

$$\pi^{(m)}(x) = \sum_{i=1}^{m} 1 \otimes 1 \otimes \cdots \otimes \pi(x) \otimes 1 \otimes \cdots \otimes 1, \quad x \in L.$$

Then it is a $I'$-integrable representation of $L$ and also of $U_f$.

Let $N_m = \text{Ker}(\pi^{(m)})$. Then, $AN_m+n \subset N_m \otimes U + U \otimes N_n$ and $N = \bigcap_m N_m$ is a Hopf ideal of $U$. Therefore, $L \cap N = 0$ implies $N = 0$ (cf. [1, Corollary 2.4.14]). Since $\bigcap M_n \subset N$, we have $\bigcap M_n = 0$.

5. Groups Associated with Integrable Lie Algebras

5.1. One Parameter Subgroups

Let $L$ be an integrable Lie algebra over $F$ with a finite integrable system of generators $I$. Let $U = U(L)$ be the universal enveloping algebra of $L$; let $M_n$ be the left ideal of $U$ generated by $x^n$ for all $x \in I$; let $U_f$ be the topological coalgebra with a base $\{ M_n \}_{n \in I}$; let $\hat{U}_f = \lim U/M_n$ be the completion of $U_f$. Then $\hat{U}_f$ is a complete topological coalgebra and from Proposition 4.3.1, $U_f$ is a dense subcoalgebra of $\hat{U}_f$. The topological dual $U_f^\#$ of $U_f$ is a commutative topological algebra with respect to the dual tensor product topology. Note that there exists a canonical continuous bijection $U_f^\# \to \hat{U}_f$.

For an element $x \in I$, let $U_x$ be the sub-Hopf algebra of $U_f$ generated by $x$ which is the polynomial algebra $F(x)$ with one variable $x$ over $F$ and which has the comultiplication, counit, and antipode defined by

$$Ax = x \otimes 1 + 1 \otimes x, \quad \varepsilon x = 0, \quad Sx = -x.$$

Further, $U_x$ is a topological sub-Hopf algebra induced by the topology of $U_f$. The topological dual $H_x$ of $U_x$ is a discrete Hopf algebra and it is a polynomial algebra $F[\xi]$ over $F$, where $\xi$ is defined by $\xi(x^n) = \delta_{n1}$ for all non-negative integers $n$ and satisfies $\xi^m(x^n) = m! \delta_{mn}$. $F[\xi]$ has comultiplication, counit, and antipode defined by

$$\Delta \xi = \xi \otimes 1 + 1 \otimes \xi, \quad \varepsilon \xi = 0, \quad S\xi = -\xi.$$

The canonical continuous bijection $H_x^0 \to \hat{U}_x = F[[x]]$ is an iso-
morphism and we have the affine group \( G_x = \text{Alg}_F(H_x, F) = G(H_x^0) \) consisting of the elements
\[
\exp tx = \sum_{n=0}^{\infty} \frac{1}{n!} t^n x^n \quad \text{for all } t \in F,
\]
and \( G_x \) is isomorphic to the additive group \( G(F) \) of \( F \). Note that if \( (\pi, V) \) is a faithful representation of \( L \) for which \( \Gamma \) is integrable, \( \exp t\pi(x) \) is a well-defined automorphism of the vector space \( V \).

We shall construct in 5.4 an ind-affine group such that the group is contained in \( \hat{U}_\Gamma \) and contains \( G_x \) for all \( x \in \Gamma \) which we call the one parameter subgroups. To construct the group, we shall first introduce the \( \Gamma \)-filters on the topological coalgebra \( U_\Gamma \).

5.2. \( \Gamma \)-Filters of \( U_\Gamma \)

Let \( \Phi = \{D_p\}_{p \in I} \) be a family of subcoalgebras \( D_p \) of \( U_\Gamma \), where the index set \( I \) is the positive integers (or more generally, a countable ordered set) and \( D_p \) are topological subcoalgebras induced by the topology of \( U_\Gamma \). \( \Phi = \{D_p\}_{p \in I} \) is called a \( \Gamma \)-filter of \( U_\Gamma \) if it satisfies
\[
\begin{align*}
&(F_1) \quad D_p \subset D_q \text{ if } p \leq q, \quad \Gamma \subset D_1, \quad \lim D_p = U, \\
&(F_2) \quad 1 \in D_p \text{ for all } p \in I, \\
&(F_3) \quad \text{for any } p \in I, \text{ there exists } q \in I \text{ such that } D_p D_p \subset D_q, \\
&(F_4) \quad \text{for any } p \in I, \text{ there exists } q \in I \text{ such that } SD_p \subset D_q.
\end{align*}
\]

We denote by \( U_{\Gamma, \Phi} \) the topological coalgebra with inductive limit topology of the topological coalgebras \( D_p \). Then the identity map from \( U_{\Gamma, \Phi} \) onto \( U_\Gamma \) is a continuous bijective homomorphism of topological coalgebras. The dual algebra \( U_{\Gamma, \Phi}^0 \) of \( U_{\Gamma, \Phi} \) is the projective limit of the dual algebra
\[
D_p^0 = \lim (D_p/(D_p \cap M_n))^* \]
of \( D_p \). The dual of the identity map from \( U \) onto itself induces a continuous injective homomorphism
\[
\varphi: U_\Gamma^0 \to U_{\Gamma, \Phi}^0
\]
of topological algebras.
An element $u = x_i^{m_i} x_{i+1}^{m_{i+1}} \cdots x_{i+s}^{m_{i+s}}$ of $U$, where $x_{i+k} \in \Gamma$ and $x_{i+k} \neq x_{i+k+1} (1 \leq k < s)$, is called a $\Gamma$-monomial of $\Gamma$-degree $n = n_1 + n_2 + \cdots + n_s$ and of $\Gamma$-length $s$. We shall use the following two types of $\Gamma$-filters of $U_\Gamma$.

5.2.1. $\Gamma$-Filters Defined by $\Gamma$-Degree. Let $D_p$ be the subspace of $U_\Gamma$ spanned by $\Gamma$-monomials of degree $\leq p$. Then, $D_p$ is a topological subcoalgebra of $U_\Gamma$ and the family $\Phi = \{D_p\}_{p \in I}$ is a $\Gamma$-filter of $U_\Gamma$ which we call the $\Gamma$-filter defined by $\Gamma$-degree. Note that $D_p$ is stable under the antipode $S$ of $U$ and $\dim D_p < \infty$ for all $p \in I$.

5.2.2. $\Gamma$-Filters Defined by $\Gamma$-Length. Let $C_p$ be the subspace of $U_\Gamma$ spanned by $\Gamma$-monomials of $\Gamma$-length $\leq p$. Then, $C_p$ is a topological subcoalgebra of $U_\Gamma$ and the family $\Psi = \{C_p\}_{p \in I}$ is a $\Gamma$-filter of $U_\Gamma$ which we call the $\Gamma$-filter defined by $\Gamma$-length. Note that $C_p$ is stable under the antipode $S$ of $U$.

Let $\Phi, \Psi$ be the $\Gamma$-filters defined as above and let $U_{\Gamma,\Phi}, U_{\Gamma,\Psi}$ be the topological coalgebra with inductive limit topology of topological coalgebras $\{D_p\}_{p \in I}, \{C_p\}_{p \in I}$, respectively.

For any coalgebra $D_q$, there exists a coalgebra $C_p$ such that $D_q \subset C_p$. Therefore, the identity map from $U$ onto $U$ induces bijective continuous homomorphisms $\sigma, \tau, \varphi, \psi$ of coalgebras

$$U \xrightarrow{\sigma} U_{\Gamma,\Phi} \xrightarrow{\tau} U_{\Gamma,\Psi} \xrightarrow{\psi} U_\Gamma,$$

such that $\varphi = \psi \cdot \tau$. The dual maps of $\sigma, \tau, \varphi, \psi$ induce continuous injective homomorphisms of algebras

$$U^* \xrightarrow{\varphi^0} U_{\Gamma,\Phi} \xrightarrow{\tau^0} U_{\Gamma,\Psi} \xrightarrow{\psi^0} U^*,$$

such that $\varphi^0 = \tau^0 \cdot \psi^0$.

5.3. Construction of Hopf Algebras of Shafarevich Types.

Let $\Psi = \{C_p\}_{p \in I}$ be as in the previous section. We shall introduce another topology on $U_{\Gamma,\Psi}$. Let $p, q$ be positive integers and set

$$V_{q,n} = \sum_{\substack{m \geq n \\ x \in \Gamma}} C_q x^m, \quad W_{p,n} = C_p \cap V_{p+1,n}.$$
$W_{p,n}$ is a subset of $C_p \cap M_n$ and the underlying coalgebra of $C_p$ has a structure of a topological coalgebra with a base \{ $W_{p,n}$ \}$_{n \in I}$ which we denote by $C'_p$. Let $U'_{\Gamma, \varphi} = \lim C'_p$ be the topological coalgebra with inductive limit topology of topological coalgebras $C'_p$ and let $A_{\Gamma, \varphi}$ be the dual algebra of $U'_{\Gamma, \varphi}$. There exists a canonical bijective homomorphism of topological coalgebras

$$\theta: U'_{\Gamma, \varphi} \rightarrow U_{\Gamma, \varphi}.$$ 

The dual

$$\theta^0: U^0_{\Gamma, \varphi} \rightarrow A_{\Gamma, \varphi}$$

is an injective homomorphism of topological algebras.

Let $I_p$ be the set of sequences of indices $(i) = (i_1, \ldots, i_p)$, where $1 \leq i_k \leq l (1 \leq k \leq p)$ and let $R_p$ be the set of sequences of non-negative integers $r = (r_1, \ldots, r_p)$. We denote simply by $x'_{(i)}$ the $\Gamma$-monomial $x'_{i_1} \cdots x'_{i_p}$ of $C'_\Gamma$.

Since $\Gamma$ is a finite set and $(\text{ad}, L)$ is $\Gamma$-integrable, there exists an integer $s > 0$ such that

$$(\text{ad} \, x)^s \, y = 0 \quad \text{for all } x, y \in \Gamma.$$ 

We shall fix such an integer $s$ once and for all. Then, for a $\Gamma$-monomial $x'_{(i)}$, if $m > (r_1 + \cdots + r_p) \, s$ then

$$(\text{ad} \, x)^m \, x'_{(i)} = 0 \quad \text{for all } x \in \Gamma.$$ 

5.3.1. **Lemma.** Let $z = x'_{(i)}$ be a $\Gamma$-monomial in $C_p$, where $r = (r_1, \ldots, r_p) \in R_p$, $(i) - (i_1, \ldots, i_p) \in I_p$. Then, for any positive integers $n$ and $t$ such that

$$t > (r_1 + \cdots + r_p) \, s + n,$$

we have that $x'x'_{(i)} \in W_{p+1,n}$ for all $x \in \Gamma$.

**Proof.** First, we shall take up a simple case $z = x'_{i_1}$. If $x = x_1$, then the assertion is trivial. Assume $x \neq x_1$ and from Lemma 4.3.2, we have

$$x'x'_{(i)} = \sum_{k=0}^{t} \binom{t}{k} \,(\text{ad} \, x)^k \, x'_{(i)} \, x'^{-k}.$$ 

Therefore, if we take $t > (r_1 + \cdots + r_p) \, s + n$, then $x'x'_{(i)} \in W_{p+1,n}$.

Let $A_p = \lim (C_p/W_{p,n})^*$ be the dual algebra of $C_p$ and by definition, $A_{\Gamma, \varphi} = \lim A_p$. 

5.3.2. **Proposition.** \( \dim(C_p/W_{p,n}) < \infty. \)

**Proof.** Set \( t_p = n \) and define \( t_k (1 \leq k < p) \) inductively by

\[
t_k = (t_{k+1} + \cdots + t_p) s + n,
\]

and let \( R_{p,n} = \{ r = (r_k) \in R_p; r_k \leq t_k (1 \leq k \leq p) \}. \) We shall show that the set of monomials \( x_{(i)} \) for all \( r \in R_{p,n} \) and all \( (i) \in I_p \) span \( C_p \) modulo \( W_{p,n}. \) Since the set \( R_{p,n} \) and \( I_p \) are finite, this implies that \( \dim(C_p/W_{p,n}) < \infty. \) It suffices to show that for any monomial \( x'_{(i)} \), if \( r \notin R_{p,n} \) then \( x'_{(i)} \in W_{p,n}. \) Now, let \( r \notin R_{p,n} \) and let there exist an integer \( j (1 \leq j \leq p) \) such that \( r_j > t_j \) and \( r_k \leq t_k \) \( (j+1 \leq k \leq p). \) From the definition of \( t_k \) and Lemma 5.3.1, we have

\[
x'_{i_j} \cdot x'_{i_{j+1}} \cdots x'_{i_p} \in W_{p-j+1,n}
\]

and therefore, \( x'_{(i)} \in W_{p,n}. \) \]

5.3.3. **Theorem.** \( A_{\Gamma, \Psi} \) have a canonical structure of a Hopf algebra of Shafarevich type.

**Proof.** We first show that the multiplication \( U \otimes U \to U \) of \( U \) induces a continuous map from \( C'_p \otimes C'_p \) into \( C'_{2p}. \) This implies that the dual \( \Delta: U^* \to U^* \otimes U^* \) of the multiplication of \( U \) induces a well defined continuous coalgebra structure map \( \Delta: A_{\Gamma, \Psi} \to A_{\Gamma, \Psi} \hat{\otimes}_d A_{\Gamma, \Psi}. \) We will show that for any integer \( n > 0, \) there exists an integer \( m > 0 \) such that

\[
W_{p,m} C_p + C_p W_{p,m} \subset W_{2p+2,n}.
\]

It is clear that \( C_p W_{p,m} \subset W_{2p+2,n} \) for \( m > n. \) Thus, it remains to show that for any integer \( n > 0, \) there exists an integer \( m > 0 \) such that \( W_{p,m} C_p \subset W_{2p+2,n}. \) Take \( m > t_1 \) and \( x^m x'_{(i)} \in W_{p+1,n} \) for all \( (i) \in I_p \) and \( r \in R_{p,n}. \) Since the set of \( \Gamma \)-monomials \( \{ x_{(i)} ; (i) \in I_p, r \in R_{p,n} \} \) spans \( C_p \) modulo \( W_{p,n}, \) we have \( x^m C_p \subset W_{p+1,n} \) for all \( x \in \Gamma. \) Therefore, \( W_{p,m} \subset C_{p+1} W_{p+1,n} \subset W_{2p+2,n}. \)

Finally, we show that the antipode \( S: U \to U \) induces a continuous map from \( U'_{\Gamma, \Psi} \) into itself. This implies that the dual \( S^*: U^* \to U^* \) of \( S \) induces a continuous algebra homomorphism from \( A_{\Gamma, \Psi} \) into itself. Since \( S(C_p) \subset C_{p+2} \) for any integer \( p > 0, \) we shall show that the map \( S \) induces a continuous map from \( C'_p \) into \( C'_{p+2}. \) It suffices to show that for any integer \( n > 0, \) there exists an integer \( m > 0 \) such that \( S(W_{p,m}) \subset W_{p+2,n}. \) As we have shown above, if we take an integer \( m \) sufficiently large, then \( x^m C_p \subset W_{p+2,n} \) for all \( x \in \Gamma. \) Therefore, \( S(W_{p,m}) \subset C_p \cap \sum_{x \in \Gamma} x^m C_{p+1} \subset W_{p+2,n}. \)

Thus, \( A_{\Gamma, \Psi} \) has a canonical structure of a topological Hopf algebra. Let \( A'_p \) be the image of \( A_{\Gamma, \Psi} \) in \( A_p \) under the canonical homomorphism of \( A_{\Gamma, \Psi} \) into \( A_p. \) From Proposition 5.3.2, \( A_p \) and also \( A'_p \) are discrete and we see that \( A_{\Gamma, \Psi} = \lim A'_p \) has a structure of a Hopf algebra of Shafarevich type.
5.4. Ind-Affine Groups Associated with Integrable Lie Algebras

As we have shown in the previous section, $A_{\gamma\cdot\psi}$ is a Hopf algebra of Shafarevich type; we can define an ind-affine group $G_{\gamma\cdot\psi} = \text{TA}_{\text{Alg}}(A_{\gamma\cdot\psi}, F)$. In this section, we show that $A_{\gamma\cdot\psi}$ is the coordinate ring of $G_{\gamma\cdot\psi}$.

5.4.1. Proposition. The algebra $A_p$ is reduced.

Proof. Note that $A_p \subset C^* \subset (\oplus_{y_1, \ldots, y_p \in \Gamma} U_y \otimes \cdots \otimes U_y)^*$. Since $U_y \otimes \cdots \otimes U_y$ are coalgebras of Birkhoff-Witt type, $A_p$ is reduced.

5.4.2. Theorem.

$$G_{\gamma\cdot\psi} = \text{TA}_{\text{Alg}}(A_{\gamma\cdot\psi}, F) = \lim_{\rightarrow} \text{Alg}_F(A_p', F)$$

has a structure of an ind-affine group whose coordinate ring is $A_{\gamma\cdot\psi}$.

Proof. From Proposition 5.4.1, the algebras $A_p$ and also $A_p'$ are reduced. Thus the theorem follows from Proposition 3.1.2.

5.4.3. Theorem. Let $L$ be an integrable Lie algebra. Then there exists a bijective correspondence between the integrable representations $(\pi, V)$ of $L$ and the right $A_{\gamma\cdot\psi}$-comodules $\rho: V \to V \hat{\otimes} A_{\gamma\cdot\psi}$, where $V$ is a discrete topological vector space.

Proof. Note that $V \hat{\otimes} A_{\gamma\cdot\psi} = \lim_{\rightarrow} (V \otimes A_p)$. Then the theorem follows from the fact that

$$V \otimes A_p = \text{TMod}_F(C_p', V).$$

5.4.4. Corollary. Let $L$ be an integrable Lie algebra. Then for an integrable representation of $L$, there exists one and only one rational representation of the ind-affine group $G_{\gamma\cdot\psi}$ whose differential coincides with the given representation.

We shall state here some open problems.

Problems. (1) Is $G_{\gamma\cdot\psi}$ generated by one parameter subgroups $G_x$ for all $x \in \Gamma$?

(2) Is $G_{\gamma\cdot\psi}$ simply connected or does $G_{\gamma\cdot\psi}$ have a simply connected covering?

(3) Under what condition is $G_{\gamma\cdot\psi}$ ind-affine algebraic or under what condition is the Lie algebra of $G_{\gamma\cdot\psi}$ isomorphic to the given Lie algebra?

(4) If $L$ is defined over $\mathbb{Z}$, can we construct the group scheme $G_{\gamma\cdot\psi}$ whose Lie algebra is isomorphic to $L$?
Remarks. (1) It is easy to show that the dual algebra $A_{\Gamma,\phi}$ of the topological coalgebra $U_{\Gamma,\phi}$ has a canonical structure of a Hopf algebra of Shafarevich type. However, it defines an infinitesimal group, namely $G_{\Gamma,\phi} = \text{TA}_{\Gamma,\phi}(A_{\Gamma,\phi}, F)$ which is the unit group and its coordinate ring is isomorphic to $F$. In fact, since $D_{\rho}$ is a finite dimensional subcoalgebra of $U$, $D_{\rho}$ is irreducible and the dual algebra of $D_{\rho}$ is local.

(2) Let $\Gamma, \Gamma'$ be the integrable systems for $L$. If $\Gamma \subset \Gamma'$, then there exists a canonical continuous homomorphism $U_{\Gamma'} \rightarrow U_{\Gamma}$ which is called dominant if for any integrable system $\Gamma''$ containing $\Gamma$, $U_{\Gamma''}$ is homeomorphic to $U_{\Gamma}$. If $\Gamma$ and $\Gamma'$ are integrable systems, then $\Gamma \cup \Gamma'$ is also an integrable system. Therefore, there exists a maximal integrable system for $L$ but it is not necessarily finite. A dominant system $\Gamma$ is called universal if for any integrable system $\Gamma''$, there exists an integrable system $\Gamma''$ such that $\Gamma'' \supset \Gamma, \Gamma'$. For an integrable Lie algebra $L$, if $L$ has a universal finite integrable system $\Gamma$, then $G_{\Gamma,\phi}$ is universal for the family of groups constructed from finite integrable systems.

REFERENCES

2. HARISH-CHANDRA, On representation of Lie algebras, Ann. of Math. 50 (1940), 900–915.