On transitive decompositions of disconnected graphs

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A B S T R A C T
We investigate transitive decompositions of disconnected graphs, and show that these behave very differently from a related class of algebraic graph decompositions, known as homogeneous factorisations. We conclude that although the study of homogeneous factorisations admits a natural reduction to those cases where the graph is connected, the study of transitive decompositions does not.

1. Introduction

By a graph \( \Gamma \) we mean a pair \( (V \Gamma, A \Gamma) \) where \( V \Gamma \) is a set of vertices, and \( A \Gamma \) is a set of ordered pairs of vertices called arcs. Such a graph \( \Gamma \) is called undirected if whenever \( (\alpha, \beta) \) is in \( A \Gamma \), the pair \( (\beta, \alpha) \) is also in \( A \Gamma \); otherwise \( \Gamma \) is called directed (or sometimes a digraph; however we use the term ‘graph’ to include both the cases). A transitive decomposition of a graph is a partition of the arc set such that the parts are preserved and transitively permuted by a group of automorphisms of the graph. Transitive decompositions arise in geometry, algebra and combinatorics; for example, a special subclass corresponds to the line transitive (partial) linear spaces (see [2,3]).

Most studies of transitive decompositions so far have restricted attention to connected graphs. In developing a general theory of transitive decompositions, one is likely to consider at the outset what restrictions or reductions are possible, and in this paper we address the question:

Is it feasible to reduce the study of transitive decompositions to the study of those cases where the graph is connected?

Our short answer to this question is ‘no’. To demonstrate why, we give a number of constructions which show that transitive decompositions of disconnected graphs can sometimes exhibit ‘unexpected’ or ‘wild’ local properties at a connected component. Particularly striking is the difference in behaviour from another closely related class of algebraic graph decompositions known as homogeneous factorisations (see [1]). From the main results of this paper we conclude that although the study of homogeneous factorisations admits a natural reduction to those cases where the graph is connected, an analogous reduction is not possible for the study of transitive decompositions.

2. Definitions, notation and main results

A graph \( \Gamma \) is called connected if for every pair \( \alpha, \beta \) of vertices there is a path of vertices \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_k = \beta \) such that for \( 1 \leq i \leq k - 1 \) we have either \( (\alpha_i, \alpha_{i+1}) \in A \Gamma \) or \( (\alpha_{i+1}, \alpha_i) \in A \Gamma \). Each graph is a vertex-disjoint union of connected graphs called its connected components. For the purposes of this paper, a decomposition of \( \Gamma \) is any partition \( \mathcal{P} = \{P_1, \ldots, P_n\} \) of \( A \Gamma \) where \( n > 1 \). For each \( P \in \mathcal{P} \), we write \( \Gamma_P \) to denote the subgraph of \( \Gamma \) induced by \( P \); that is, \( \Gamma_P \) is the subgraph
with arc set \( P \), whose vertex set is the set of all vertices that belong to arcs in \( P \). Thus we may think of a decomposition of \( \Gamma \) as a set of subgraphs \( \Gamma_1, \ldots, \Gamma_n \) such that each arc of \( \Gamma \) appears in the arc set of exactly one of the \( \Gamma_i \). Note that we do not assume that the \( \Gamma_i \) are spanning subgraphs; that is to say, \( V \Gamma \) may be a proper subset of \( V \Gamma \). If \( \Gamma \) is such that \((\alpha, \beta)\) is an arc whenever \((\beta, \alpha)\) is an arc, then we may identify each pair \((\alpha, \beta)\) with an edge \([\alpha, \beta]\). In such cases, if the partition \( \mathcal{P} \) of \( \Gamma \) is symmetric (that is, if for all \((\alpha, \beta)\) in \( \mathcal{P} \) we have \((\alpha, \beta)\) and \((\beta, \alpha)\) in the same part in \( \mathcal{P} \)) then we may view \( \mathcal{P} \) as a partition of the edge set of \( \Gamma \), and we may describe \( \mathcal{P} \) as an edge-decomposition of \( \Gamma \).

Let \( \Gamma \) be a graph and let \( \mathcal{P} \) be a decomposition of \( \Gamma \). The automorphism group of \( \Gamma \), denoted by \( \text{Aut} \Gamma \), is the group of all permutations of \( V \Gamma \) that preserve \( \Gamma \). In this paper we investigate situations in which there is a subgroup \( G \) of \( \text{Aut} \Gamma \) which preserves \( \mathcal{P} \); in other words, a subgroup \( G \) such that for all \( g \in G \) and \( P \in \mathcal{P} \), we have \( gP = P \). We call the pair \((\Gamma, \mathcal{P})\) a \( G \)-decomposition. In addition \( G \) induces a transitive action on \( \mathcal{P} \) (so that for all \( P, P' \in \mathcal{P} \), there exists \( g \in G \) with \( gP = P' \)), then we call \((\Gamma, \mathcal{P})\) a \( G \)-transitive decomposition.

A homogeneous factorisation is a \( G \)-transitive decomposition \((\Gamma, \mathcal{P})\) such that the kernel of the \( G \)-action on \( \mathcal{P} \) has a subgroup \( M \) that acts transitively on \( V \Gamma \). Thus the subgroup \( M \) fixes each subgraph in the decomposition and at the same time is transitive on \( V \Gamma \). We call \((\Gamma, \mathcal{P})\) a \((G, M)\)-homogeneous factorisation. Homogeneous factorisations can be thought of as special cases of transitive decompositions.

Given a decomposition of a graph \( \Gamma \), we wish to study the local properties of the induced decomposition at a connected component. To this end, we define a ‘restriction’ of the decomposition to an induced subgraph of \( \Gamma \).

Let \( \mathcal{I} = (\Gamma, \mathcal{P}) \) be a \( G \)-decomposition, and let \( \Delta \) be an induced subgraph of \( \Gamma \) (in other words, \( \Delta \) is a subgraph of \( \Gamma \) whose arc set is the intersection of \( \mathcal{P} \) with the set of all ordered pairs from \( V \Delta \)) and assume that \( \Delta \) has at least one arc. Let \( \mathcal{P}^\Delta \) denote the permutation group induced on \( V \Delta \) by the setwise stabiliser of \( V \Delta \) in \( G \). The restriction of \( \mathcal{I} \) to \( \Delta \), denoted by \( \mathcal{I}_\Delta \), is the \( (G, \mathcal{P}^\Delta) \)-decomposition \((\Delta, \mathcal{P}^\Delta)\) where \( \mathcal{P}^\Delta = \{P \cap \Delta | P \in \mathcal{P}, \Delta \cap \Delta \not= \phi\} \), the partition of \( \Delta \) induced by the partition \( \mathcal{P} \).

We say that a \( G \)-decomposition \((\Gamma, \mathcal{P})\) and a \( \mathcal{G} \)-decomposition \((\Gamma', \mathcal{P}')\) are isomorphic, and write \((G, \Gamma, \mathcal{P}) \cong (G, \Gamma', \mathcal{P}')\), if

(i) there exists a graph isomorphism \( f : \Gamma \rightarrow \Gamma' \) which maps \( \mathcal{P} \) onto \( \mathcal{P}' \), and

(ii) there exists an isomorphism \( \varphi : G 
\rightarrow G' \) such that the pair \((\varphi, f)\) defines a permutational isomorphism between \( G \) acting on \( V \Gamma \) and \( G' \) acting on \( V \Gamma' \); that is to say, for all \( g \in G \) and \( \alpha \in V \Gamma \), we have \((\alpha g) f = (f g) \alpha \).

The first result of the paper demonstrates that restricting a transitive decomposition of a disconnected graph \( \Gamma \) to a connected component \( \Delta \) can produce an arbitrary partition of \( A \Delta \), even if all the connected components of \( \Gamma \) are isomorphic. Moreover, part (b) shows that any collection of transitive decompositions of connected graphs can arise as a subset of the restrictions of a transitive decomposition of a disconnected graph to its connected components.

**Theorem 2.1.** (a) Given any connected graph \( \Delta \), subgroup \( H \) of \( \text{Aut} \Delta \), and partition \( \mathcal{Q} \) of \( A \Delta \) such that the action of \( H \) on \( \mathcal{Q} \) is trivial, there exists a \( G \)-transitive decomposition \( \mathcal{I} = (\Gamma, \mathcal{P}) \) such that each connected component \( \Delta' \) of \( \Gamma \) is isomorphic to \( \Delta \), and \((G^{\Delta'}, \Delta', \mathcal{P}^{\Delta'}) \cong (H, \Delta, \mathcal{Q})\).

(b) Given an arbitrary set of \( H \)-transitive decompositions \( \mathcal{I}_i = (\Delta_i, \mathcal{Q}_i) \) with \( \Delta_i \) connected, for \( i \) in some index set \( I \), there exists a \( G \)-transitive decomposition \( \mathcal{I} = (\Gamma, \mathcal{P}) \) such that for all \( i \in I \), \( \Gamma \) has a connected component \( \Delta'_i \) with \((G^{\Delta_i}, \Delta'_i, \mathcal{P}^{\Delta'_i}) \cong (H_i, \Delta_i, \mathcal{Q}_i)\).

We show in Theorem 2.2 that analogous results do not hold for homogeneous factorisations. In particular, the restriction of a homogeneous factorisation of a disconnected graph to a connected component is itself a homogeneous factorisation, and moreover, restrictions to different connected components are isomorphic to one another (as transitive decompositions). Note that the proof of Theorem 2.2 (which appears in Section 4) relies on Theorem 2.3.

**Theorem 2.2.** Let \( \mathcal{I} = (\Gamma, \mathcal{P}) \) be a \( G \)-transitive decomposition such that for some \( M \leq G \), the tuple \((\Gamma, \mathcal{P})\) is a \((G, M)\)-homogeneous factorisation.

(a) Let \( \Delta \) be a connected component of \( \Gamma \). Then there exists a subgroup \( M' \) of \( G^\Delta \) such that \((\Delta, \mathcal{P}^\Delta) \) is a \((G^{\Delta'}, M')\)-homogeneous factorisation.

(b) If \( \Gamma \) is a disconnected graph, and \( \Delta_1 \) and \( \Delta_2 \) are connected components of \( \Gamma \), then \((G^{\Delta_1}, \Delta_1, \mathcal{P}^{\Delta_1}) \cong (G^{\Delta_2}, \Delta_2, \mathcal{P}^{\Delta_2})\).

Finally, we give a simple characterisation of transitive decompositions of disconnected graphs which are ‘well-behaved’ locally at a connected component. Observe that if \( \Gamma \) is a graph and \( G \) is a subgroup of \( \text{Aut} \Gamma \), then \( G \) induces a group of permutations on the set of connected components of \( \Gamma \). We denote the image of a subgraph \( \Delta \) under the action of an element \( g \in G \) by \( \Delta^g \), and set \( \Delta^G = \{\Delta^g | g \in G \} \).

**Theorem 2.3.** Let \( \mathcal{I} = (\Gamma, \mathcal{P}) \) be a \( G \)-transitive decomposition and let \( \Delta \) be a connected component of \( \Gamma \). If \(|\mathcal{P}^\Delta| > 1\), then the restriction \( \mathcal{I}_\Delta \) is a \( \mathcal{G}^\Delta \)-transitive decomposition if and only if \( G \) acts transitively on the set of all pairs \((P, \Delta')\) such that \( P \in \mathcal{P}, \Delta' \in \Delta^G \), and \( A \Delta \cap P \not= \phi \).
For Construction 3.1, we define a graph $G$ acting on the element $T$. From the definition of $G$, we prove $P$ by permuting the connected components cyclically, and furthermore $G$ preserves and acts transitively on $P$; so $(G, P)$ is a $G$-transitive decomposition.

3. Constructions for Theorem 2.1

We prove Theorem 2.1 by constructing examples for each of the two parts of the statement. Construction 3.1 is illustrated with an example in Fig. 1.

Construction 3.1. Let $H$, $\Delta$ and $Q$ be as in the statement of Theorem 2.1(a), and suppose that $Q = \{Q_0, \ldots, Q_{k-1}\}$ has size $k$. We define a graph $\Gamma$, a partition $\mathcal{P}$ of $Aut \Gamma$ and a subgroup $G$ of $Aut \Gamma$ as follows.

(i) Let $\alpha$ be the union of $k$ disjoint copies of $\Delta$, which we formalise as the graph with vertex set $V \Delta \times \mathbb{Z}_k$, and arcs $((\alpha, i), (\beta, j))$ whenever $i = j$ and $(\alpha, \beta)$ is an arc of $\Delta$.

(ii) For $0 \leq j \leq k - 1$, let $P_j$ be the subset of $Aut \Gamma$ satisfying $((\alpha, i), (\beta, j)) \in P_j$ if and only if $(\alpha, \beta) \in Q_{j+i}$, and let $\mathcal{P} = \{P_j \mid 0 \leq j \leq k-1\}$.

(iii) Let $G = H \times \mathbb{Z}_k$, and define an action of $G$ on $V \Gamma$ by $(\alpha, i) \sim (\alpha, i + z)$, for all $(h, z) \in G$.

Let $\mathcal{T} = (\Gamma, \mathcal{P})$.

Lemma 3.1. Let $H$, $\Delta$, $Q$ and $\mathcal{T} = (\Gamma, \mathcal{P})$ be as in Construction 3.1. Then

(a) each connected component of $\Gamma$ is isomorphic to $\Delta$, and
(b) $\mathcal{T}$ is a $G$-transitive decomposition, and for each connected component $\Delta'$ of $\Gamma$, $(G^{\Delta'}, \Delta', \mathcal{P}^{\Delta'}) \cong (H, \Delta, Q)$.

Proof. (a) Let $\Delta'$ be a connected component of $\Gamma$. Then for some $i$ with $0 \leq i \leq k - 1$, the component $\Delta'$ has vertex set $\{(\alpha, i) \mid \alpha \in V \Delta\}$ and arc set $\{((\alpha, i), (\beta, i)) \mid (\alpha, \beta) \in A \Delta\}$. The map $f : (\alpha, i) \mapsto \alpha$ is an isomorphism from $\Delta'$ to $\Delta$.

(b) Every arc of $\Gamma$ is contained in some $P_j \in \mathcal{P}$, and since the elements of $Q$ are pairwise disjoint, it follows that the elements of $\mathcal{P}$ are pairwise disjoint; so $\mathcal{P}$ is a partition of $Aut \Gamma$. Now, let $P_j \in \mathcal{P}$, and let $(h, z) \in G$. By definition, $((\alpha, i), (\beta, i)) \in P_j$ whenever $(\alpha, \beta) \in Q_{j+i}$, and since $H$ acts trivially on $Q$, we have that $(\alpha, \beta) \in Q_{j+i}$ if and only if $(\alpha^h, \beta^h) \in Q_{j+i}$. So $((\alpha^h, i + z), (\beta^h, i + z)) \in P_{j+1}$ if and only if $(\alpha^h, \beta^h) \in Q_{j+i}$, and it follows that $P_j \sim (h, z) = P_{j+1}$. Hence $\mathcal{P}$ is $G$-invariant, and $G$ acts transitively on $\mathcal{P}$. Thus $\mathcal{T}$ is a $G$-transitive decomposition.

From the definition of $\mathcal{P}$ it follows that for $P_j \in \mathcal{P}$, $f$ maps $P_j \cap A \Delta'$ to the part $Q_{j+i} \in Q$, and hence $f$ induces a bijection from $P^{\Delta}$ to $Q$. Furthermore, the setwise stabiliser of $V \Delta'$ in $G$ is $H \times \{0\}$, and the map $\varphi : (h, 0) \mapsto h$ induces an isomorphism from $G^{\Delta'}$ to $H$. For any $(\alpha, i) \in V \Delta'$ and $(h, 0) \in G^{\Delta'}$, we have $((\alpha, i)^{(h, 0)})f = (\alpha^h, (\alpha, i)^{(h, 0)})$. Hence $G^{\Delta'}$ acting on $V \Delta'$ is permutationally isomorphic to $H$ acting on $V \Delta$, and so we obtain the result, that $(G^{\Delta'}, \Delta', \mathcal{P}^{\Delta'}) \cong (H, \Delta, Q)$.

Construction 3.2. For $i = 0, 1$, let $\mathcal{T}_i = (\Sigma, \mathcal{Q}_i)$ be an $H_i$-transitive decomposition, (where $\Sigma_0$ and $\Sigma_1$ are not necessarily connected). We define a graph $\Gamma$, a partition $\mathcal{P}$ of $Aut \Gamma$ and a subgroup $G$ of $Aut \Gamma$ as follows.

(i) Let $\Gamma$ be the graph consisting of $\mid \mathcal{Q}_0 \mid$ vertex-disjoint copies of $\Sigma$ and $\mid \mathcal{Q}_1 \mid$ vertex-disjoint copies of $\Sigma$; formally $\Gamma$ has vertex set $(V \Sigma) \cup (V \Sigma \times \mathcal{Q}_1)$ and arcs $((\alpha, Q), (\beta, Q'))$ whenever $(\alpha, \beta) \in A \Sigma$ and $Q = Q' \in \mathcal{Q}_1 \cup \mathcal{Q}_0$ for $i = 1, 0$. Fig. 1. At the top, a graph $\Delta$ decomposed into three non-isomorphic subgraphs; and underneath, a graph $\Gamma$ with three connected components, each isomorphic to $\Delta$, and with a partition $\mathcal{P}$ of $\Gamma$ obtained from Construction 3.1. If we take $H$ to be the trivial subgroup of $Aut \Delta$, then the group $G := H \times \mathbb{Z}_3$ acts as a group of automorphisms of $\Gamma$ by permuting the connected components cyclically, and furthermore $G$ preserves and acts transitively on $\mathcal{P}$; so $(\Gamma, \mathcal{P})$ is a $G$-transitive decomposition.
Let $G = H_0 \times H_1$, and define an action of $G$ on $V \Gamma'$ as follows. For $g = (h_0, h_1) \in G$ and $(\alpha, Q) \in V \Gamma'$

$$(\alpha, Q)^g = \begin{cases} (\alpha h_0, Q h_1) & \text{if } (\alpha, Q) \in V \Delta_0 \times Q_1 \\ (\alpha h_1, Q h_0) & \text{if } (\alpha, Q) \in V \Delta_1 \times Q_0. \end{cases}$$

Then $G \leq \text{Aut}\Gamma'$.

(iii) For each pair of parts $Q, Q'$ (where $Q \in Q_0$ and $Q' \in Q_1$), set

$$P(Q, Q') = \{(\alpha, \beta) \mid (\alpha, \beta) \in Q \cup \{((\alpha, Q), (\beta, Q)) \mid (\alpha, \beta) \in Q' \}. $$

Thus $P(Q, Q')$ contains the arcs from the copy of $Q$ in the $(Q')$th copy of $\Sigma_0$, and the arcs from the copy of $Q'$ in the $Q$th copy of $\Sigma_1$. Finally, set

$$\mathcal{P} = \{P(Q, Q') \mid Q \in Q_0, Q' \in Q_1\}.$$

Set $T = (\Gamma', \mathcal{P})$.

**Lemma 3.2.** Let $T_0, T_1, T, H_0, H_1$ and $G$ be as in Construction 3.2. Then $T$ is a $G$-transitive decomposition, and furthermore $\Gamma'$ has subgraphs $\Sigma'_0$ and $\Sigma'_1$ such that $(G^V_0, \Sigma'_0, \mathcal{P}^V_0) \cong (H_0, Q_0, \mathcal{P}_0)$ and $(G^V_1, \Sigma'_1, \mathcal{P}^V_1) \cong (H_1, \Sigma_1, Q_1)$.

**Proof.**

Since $Q_0$ is a partition of $A \Sigma_0$ and $Q_1$ is a partition of $A \Sigma_1$, it follows that $Q$ is a partition of $A \Gamma$. Now, let $(h_0, h_1) \in G$, and let $P(Q_0, Q_1) \in \mathcal{P}$. It follows from the definition of $P(Q_0, Q_1)$ that $P(Q_0, Q_1)(h_0, h_1) = P(Q_0, Q_1)$, and hence that $\mathcal{P}$ is $G$-invariant and $G$ acts transitively on $\mathcal{P}$. Set $T = (\Gamma', \mathcal{P})$.

Let $\mathcal{P} = \{P(Q, Q') \mid Q \in Q_0, Q' \in Q_1\}$.

Set $T = (\Gamma', \mathcal{P})$.

**Lemma 3.2.** Let $T_0, T_1, T, H_0, H_1$ and $G$ be as in Construction 3.2. Then $T$ is a $G$-transitive decomposition, and furthermore $\Gamma'$ has subgraphs $\Sigma'_0$ and $\Sigma'_1$ such that $(G^V_0, \Sigma'_0, \mathcal{P}^V_0) \cong (H_0, Q_0, \mathcal{P}_0)$ and $(G^V_1, \Sigma'_1, \mathcal{P}^V_1) \cong (H_1, \Sigma_1, Q_1)$.

**Proof.**

Since $Q_0$ is a partition of $A \Sigma_0$ and $Q_1$ is a partition of $A \Sigma_1$, it follows that $Q$ is a partition of $A \Gamma$. Now, let $(h_0, h_1) \in G$, and let $P(Q_0, Q_1) \in \mathcal{P}$. It follows from the definition of $P(Q_0, Q_1)$ that $P(Q_0, Q_1)(h_0, h_1) = P(Q_0, Q_1)$, and hence that $\mathcal{P}$ is $G$-invariant and $G$ acts transitively on $\mathcal{P}$. Set $T = (\Gamma', \mathcal{P})$.

Now, let $Q' \in Q_1$. Then the subgraph $\Sigma'_0$ of $\Gamma'$ induced by the set of vertices $\{(\alpha, Q') \mid \alpha \in \Sigma_0\}$ is a subgraph of $\Gamma$ isomorphic to $\Sigma_0$ by the isomorphism $\alpha \mapsto \alpha$. For any part $Q \in Q_0$, the intersection of the part $P(Q, Q') \in \mathcal{P}$ with $A \Sigma'_0$ is the set $\{(\alpha, Q'), (\beta, Q') \mid (\alpha, \beta) \in Q\}$, and it follows that $\alpha$ maps $\mathcal{P}^V_0$ to $Q_0$. The setwise stabiliser in $G$ of $V \Sigma'_0$ is $H_0 \times (H_1)_Q$, and so $G^V_0 \cong H_0$. Thus the map $\varphi : G^V_0 \mapsto H_0 : (h_0, h_1) \mapsto h_0$ is an isomorphism, and the pair $(\varphi, f)$ defines a permutable isomorphism between $G^V_0$ acting on $V \Sigma'_0$ and $H_0$ acting on $V \Sigma_0$. Hence $(G^V_0, \Sigma'_0, \mathcal{P}^V_0) \cong (H_0, Q_0, \mathcal{P}_0)$. Similarly, for $Q \in Q_0$, the subgraph induced on $\{(\alpha, Q) \mid \alpha \in \Sigma_1\}$ is a subgraph of $\Gamma'$ such that $(G^V_1, \Sigma'_1, \mathcal{P}^V_1) \cong (H_1, \Sigma_1, Q_1)$. □

**4. Proofs of the main Theorems**

First we prove Theorem 2.1 using the constructions from the previous section.

**Proof of Theorem 2.1.** Part (a) follows immediately from Lemma 3.1.

For part (b) there is nothing to prove if $m = |I| = 1$. Assume $n \geq 1$ and that part (b) holds for $m = n$, and suppose that $T_1 = (\Delta_i, \mathcal{Q}_i)$ is an $H_i$-transitive decomposition for $i = 1, \ldots, n+1$. By assumption there exists a $G$-transitive decomposition $\tilde{T} = (\tilde{\Gamma}, \tilde{\mathcal{P}})$ such that $\tilde{T}$ has, for all $i$ with $1 \leq i \leq n$, a connected component $\Delta'_i$ such that $(G^\Delta_i, \Delta'_i, \tilde{\mathcal{P}}_i) \cong (H_i, \Delta_i, \mathcal{Q}_i)$. Let $T = (\Gamma', \mathcal{P})$ be the $G$-transitive decomposition obtained from $\tilde{T}$ and $T_{n+1}$ by Construction 3.2. Then it follows from Lemma 3.2 that $\Gamma'$ has, for all $i$ with $1 \leq i \leq n+1$, a connected component $\Delta'_i$ such that $(G^\Delta_i, \Delta'_i, \mathcal{P}'_i) \cong (H_i, \Delta_i, \mathcal{Q}_i)$. Hence, by induction, the result holds. □

The proof of Theorem 2.2 relies on Theorem 2.3, and so we will prove Theorem 2.3 first. To make the proof simpler, we will rephrase the statement of Theorem 2.3 using slightly different terminology, which we now explain.

Let $T = (\Gamma', \mathcal{P})$ be a $G$-transitive decomposition. Let $\Delta$ be a connected component of $\Gamma'$, and let $\Gamma = \{\Delta^g \mid g \in G\}$. Let $F_0 \times \mathcal{P}$ denote the graph with vertex set $F \cup \mathcal{P}$ and arcs $(\alpha, \beta) \times \mathcal{P}$ whenever $\Delta^\alpha \cap \mathcal{P} \neq \emptyset$ and observe that $\Gamma$ in its actions on $\Delta^\alpha$ and $\mathcal{P}$ is a subgroup of $\text{Aut}(F_0 \times \mathcal{P})$. We will rephrase the statement of Theorem 2.3 in terms of $\Delta^\mathcal{P}$. For the sake of completeness, we state Theorem 2.3 again.

**Theorem 2.3.** Let $T = (\Gamma', \mathcal{P})$ be a $G$-transitive decomposition, let $\Delta$ be a connected component of $\Gamma'$, and let $\Delta^\mathcal{P}$ be the orbit of $\Delta$ under $G$. If $|\mathcal{P}^\Delta| > 1$ then the restriction $T_\Delta$ is a $G^\Delta$-transitive decomposition if and only if $G$ acts arc-transitively on $\Delta^\mathcal{P}$.

**Proof.** Assume first that $T_\Delta$ is a transitive decomposition. Then the induced group $G^\Delta$ acts transitively on $\mathcal{P}^\Delta$, and hence $G^\Delta$ transitively permutes the parts of $\mathcal{P}$ whose intersection with $\Delta$ is non-empty. This means that the stabiliser in $G$ of the vertex $\Delta$ of $\Delta^\mathcal{P}$ acts transitively on all arcs in $\Delta^\mathcal{P}$ with initial vertex $\Delta$. Since $G$ is also transitive on $\Delta^\mathcal{P}$, $G$ acts transitively on the arcs of $\Delta^\mathcal{P}$. On the other hand, if $\Delta^\mathcal{P}$ is $G$-arc-transitive, then $G^\Delta$ transitively permutes the parts of $\mathcal{P}$ whose intersection with $\Delta$ is non-trivial. It follows that $G^\Delta$ is $G$-transitive on $\mathcal{P}^\Delta$, and that if $|\mathcal{P}^\Delta| > 1$, then $T_\Delta$ is a transitive decomposition. □

Theorem 2.3 follows immediately from Theorem 4.1 and the definition of $\Delta^\mathcal{P}$. In order to prove Theorem 2.2 we will also need the following result.

**Lemma 1.** Let $T = (\Gamma', \mathcal{P})$ be a $G$-transitive decomposition and assume that $G^V_0$ acts vertex-transitively on $\Gamma_p$ for some $P \in \mathcal{P}$. Let $\Delta$ be a connected component of $\Gamma$ with $\Delta \cap P \neq \emptyset$. If $|\mathcal{P}^\Delta| > 1$, then $T_\Delta$ is a transitive decomposition.
Proof. Let $\Delta^G = \{ \Delta^g \mid g \in G \}$ as above. Since $G^\Gamma_P$ acts vertex-transitively on $\Gamma_P$, $G$ transitively permutes the connected components of $\Gamma$ that intersect $\Gamma_P$ non-trivially, and these are the components whose arc sets have a non-empty intersection with $P$. This implies that the stabiliser in $G$ of the vertex $P$ of $\Delta^G \circ \Diamond_P$ acts transitively on the arcs of $\Delta^G \circ \Diamond_P$ with second entry $P$. Now $G$ is also transitive on $\Delta$, and so the graph $\Delta^G \circ \Diamond_P$ is $G$-arc-transitive. Hence, by Theorem 4.1, if $|\Delta| > 1$, $T_{\Delta}$ is a transitive decomposition. □

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. (a) Let $P \in \Delta$. By the definition of a homogeneous factorisation, $V\Gamma_P = V\Gamma$, which implies that $\Delta \cap P \neq \emptyset$. Moreover, $M$ fixes $P$ and acts transitively on $V\Gamma$. Hence $G^\Gamma_P$ acts transitively on $V\Gamma$, and thus also on the connected components of $\Gamma$. Observe therefore that $|\Delta| > 1$, since otherwise we would have $|\Delta| = 1$. It follows from Lemma 4.1 that $T_{\Delta}$ is a $G$-transitive decomposition. Now, $M$ acts transitively on $V\Delta$, and since $\Delta$ is a connected component of $\Gamma$, $V\Delta$ is a block of imprimitivity for $M$, and so $M_{V\Delta}$ acts transitively on $V\Delta$. Since $M$ is a subgroup of the kernel of the $G$-action on $\Delta$, the group $M_{V\Delta}$ is contained in the kernel of the $G^\Gamma_{\Delta}$-action on $\Delta$. Thus, setting $M' = M_{V\Delta}$, we have that $(\Delta, \Delta, P_{\Delta})$ is a $(G\Delta, M')$-homogeneous factorisation.

(b) Since $M$ acts vertex-transitively on $V\Gamma$, there exists $g \in G$ such that $(V\Delta_1)^g = V\Delta_2$, and $g$ induces an isomorphism $f_g$ from $\Delta_1$ to $\Delta_2$. Furthermore, since $g$ preserves $\Delta$, it follows that $(\Delta_1)^g = \Delta_2$. The element $g$ also induces (by conjugation) an isomorphism $\varphi_g$ from $G\Delta_1$ to $G\Delta_2$, and the pair $(f_g, \varphi_g)$ defines a permutational isomorphism between $G\Delta_1$ acting on $V\Delta_1$ and $G\Delta_2$ acting on $V\Delta_2$. Hence $(G\Delta_1, \Delta_1, P_{\Delta_1}) \cong (G\Delta_2, \Delta_2, P_{\Delta_2})$. □

References