Order preserving bijections of $C(\mathcal{X}, I)$

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Abstract

Let $\mathcal{X}$ be a compact Hausdorff space which satisfies the first axiom of countability, $I = [0, 1]$ and $C(\mathcal{X}, I)$ the set of all continuous functions from $\mathcal{X}$ to $I$. In the paper we will give a description of bijective maps on $C(\mathcal{X}, I)$ which preserve the order in both directions.

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1. Introduction and statement of the result

Because of important applications in quantum mechanics several authors studied preservers of various operations, relations and quantities on Hilbert space effect algebras (see, for example, [2,4,7,9,11–13]). First, let us recall the following notation. Let $\mathcal{A}$ be a unital $C^*$-algebra. The effects in $\mathcal{A}$ are the positive elements of $\mathcal{A}$ which are less than or equal to the unit of $\mathcal{A}$. The set of all effects in $\mathcal{A}$ is called the effect algebra of $\mathcal{A}$. It is denoted by $E(\mathcal{A})$. If $\mathcal{A}$ equals the algebra $B(\mathcal{H})$ of all bounded linear operators on the complex Hilbert space $\mathcal{H}$, then the corresponding effects are called Hilbert space effects. These objects play very important role in certain parts of quantum mechanics, especially in the quantum theory of measurement where effects represent so-called yes-no measurements which may be unsharp (see [1,3,8]).
The set $E(\mathcal{H})$ of Hilbert space effects can be equipped with several algebraic operations and relations each of them having a serious physical content. For example, there is a partial ordering $\leq$ which comes from the usual ordering between the self-adjoint operators on $\mathcal{H}$ and one can also define the operation of the so-called orthocomplementation by

$$A \mapsto I - A, \quad A \in E(\mathcal{H}).$$

This relation and operation together give the ortho-order structure on $E(\mathcal{H})$. Just as with any other quantum structure or, more generally, algebraic structure it is rather important to explore the corresponding automorphisms of $E(\mathcal{H})$ which, in another language, could also be called symmetries. In fact, it turns out that in the case of ortho-order automorphisms where $\dim \mathcal{H} > 1$ (see [2,9,12]) and also in many other cases (see, for example, [11]), automorphisms in question are exactly the transformations of the form

$$A \mapsto UAU^*, \quad A \in E(\mathcal{H}),$$

where $U$ is an either unitary or antiunitary operator on the Hilbert space $\mathcal{H}$.

Another structure on $E(\mathcal{H})$ comes from the so-called sequential product. Two measurements $a$ and $b$ cannot be preformed simultaneously in general, so they are frequently executed sequentially. We denote $a \circ b$ a sequential measurement in which $a$ is preformed first and $b$ second. We call $a \circ b$ the sequential product of $a$ and $b$. Motivated by such considerations, in their paper [5] Gudder and Nagy introduced the concept of the sequential product between Hilbert space effects (see also [6]). This operation is defined by

$$A \circ B = A^{1/2} BA^{1/2}, \quad A, B \in E(A).$$

In the paper [4], Gudder and Greechie described the general form of the sequential automorphisms of the set of all Hilbert space effects assuming that the underlying Hilbert space is at least three-dimensional. If $A, B$ are unital $C^*$-algebras, then the bijective map $\varphi : E(A) \to E(B)$ is called a sequential isomorphism if it satisfies

$$\varphi(A \circ B) = \varphi(A) \circ \varphi(B), \quad A, B \in E(A).$$

The result of Gudder and Greechie says that for Hilbert space effects every such transformation $\varphi$ is again implemented by an either unitary or antiunitary operator $U$ of the underlying Hilbert space. Recently there has been the first attempt to significantly generalize this result from the case of effects in $B(\mathcal{H})$ to the case of effects in general von Neumann algebras. In the important paper [14] Molnár proved the following result.

Let $A, B$ be von Neumann algebras and let $\varphi : E(A) \to E(B)$ be a sequential isomorphism. Then there are direct decompositions

$$A = A_1 \oplus A_2 \oplus A_3 \quad \text{and} \quad B = B_1 \oplus B_2 \oplus B_3$$

within the category of von Neumann algebras and there are bijective maps

$$\varphi_1 : E(A_1) \to E(B_1), \quad \Phi_2 : A_2 \to B_2, \quad \Phi_3 : A_3 \to B_3,$$

such that

(i) $A_1, B_1$ are commutative von Neumann algebras and the algebras $A_2 \oplus A_3, B_2 \oplus B_3$ have no commutative direct summands;
(ii) \( \varphi_1 \) is a multiplicative bijection, \( \Phi_2 \) is an algebra *-isomorphism, \( \Phi_3 \) is an algebra *-antiisomorphism and \( \varphi = \varphi_1 \oplus \Phi_2 \oplus \Phi_3 \) holds on \( E(A) \).

A lot of information is available on algebra *-isomorphisms and algebra *-antiisomorphisms. For example, we know their general forms in many particular cases. In contrast with this, it seems we must separately study the third factor in the above decomposition, i.e., the bijective, multiplicative maps between the sets of effects in commutative von Neumann algebras or, more generally, in commutative unital \( C^* \)-algebras. In our recent paper [10] we presented a structural result concerning this factor in the above decomposition. It is well known that every commutative \( C^* \)-algebra is isomorphic to the algebra of all continuous complex valued functions on a compact Hausdorff space \( X \). Therefore, it was enough that we considered the structure of all continuous functions from \( X \) to the unit interval \( I \) which we denote by \( C(X, I) \). The main result in [10] describes the general form of all bijective, multiplicative maps of \( C(X, I) \) under the technical condition that \( X \) satisfies the first axiom of countability.

Molnár [13] proved that the ortho-order automorphisms and the sequential automorphisms are subsets of automorphisms that preserve the order \( \leq \) and the zero product in both directions even in the setting of von Neumann algebras. It is clear that a necessary step in understanding the structure of preservers of different types on general von Neumann algebras is to investigate the transformations of \( C(X, I) \).

We will study in our paper the bijective transformations of \( C(X, I) \) which preserve the order \( \leq \) in both directions, i.e., which satisfy \( f \leq g \) if and only if \( \varphi(f) \leq \varphi(g) \) for all \( f, g \in C(X, I) \) \( f \leq g \) if and only if \( f(x) \leq g(x) \) for all \( x \in X \). Let us mention that such maps, as we will show, automatically preserve the zero product in both directions.

**Theorem 1.1.** Let \( X \) be a compact Hausdorff space which satisfies the first axiom of countability and \( I = [0, 1] \). If \( \varphi : C(X, I) \to C(X, I) \) is a bijective map which preserves the order in both directions, then there exists a homeomorphism \( \mu : X \to X \) and for each \( x \in X \) a bijective, increasing map \( m_x : I \to I \) such that

\[
\varphi(f)(x) = m_x \left( f \left( \mu(x) \right) \right), \quad x \in X,
\]

for all \( f \in C(X, I) \). Conversely, suppose \( \mu : X \to X \) is a homeomorphism, let \( m_x : I \to I, x \in X \), be a bijective, increasing map for every \( x \in X \) and let \( m_x(c) : X \to I, x \mapsto m_x(c) \), be a continuous map for every \( c \in I \). Define

\[
\varphi(f)(x) = m_x \left( f \left( \mu(x) \right) \right), \quad x \in X,
\]

for all \( f \in C(X, I) \). Then \( \varphi : C(X, I) \to C(X, I) \) is a bijective map that preserves the order \( \leq \) in both directions.

**Remark 1.2.** Let \( 1_X(x) = 1 \) for all \( x \in X \). It follows immediately from Theorem 1.1 that in the case of the ortho-order automorphisms, if we assume additionally that \( \varphi(1_X - f) = 1_X - \varphi(f) \), we get the same result, with each map \( m_x : I \to I, x \in X \), satisfying \( m_x(1 - c) = 1 - m_x(c) \) for all \( c \in I \).

We believe that the same results hold also without the countability assumption and it would be interesting to find a proof of this conjecture.
2. Proof of the theorem

Let $0_{\mathcal{X}}(x) = 0$ for all $x \in \mathcal{X}$. If $\varphi: \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ is a surjective map which preserves the order $\leq$ we obtain

$$\varphi(0_{\mathcal{X}}) = 0_{\mathcal{X}} \quad \text{and} \quad \varphi(1_{\mathcal{X}}) = 1_{\mathcal{X}}.$$

**Lemma 2.1.** Suppose $\varphi: \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ is a bijective map where $f \leq g$ if and only if $\varphi(f) \leq \varphi(g)$. Then

$$fg = 0_{\mathcal{X}} \quad \text{if and only if} \quad \varphi(f)\varphi(g) = 0_{\mathcal{X}}.$$

**Proof.** Let $fg = 0_{\mathcal{X}}$. If $f(x) \neq 0$ then $g(x) = 0$ and if $g(x) \neq 0$ then $f(x) = 0$. So,

$$\min\{f, g\} = 0_{\mathcal{X}}.$$

This yields, if $h \in \mathcal{C}(\mathcal{X}, I)$ such that $h \leq f$ and $h \leq g$ then $h = 0_{\mathcal{X}}$. Let $\varphi(f)\varphi(g) \neq 0_{\mathcal{X}}$. Then there exists $x \in \mathcal{X}$ such that $\varphi(f)(x) \neq 0$ and $\varphi(g)(x) \neq 0$. Hence,

$$\min\{\varphi(f), \varphi(g)\} \neq 0_{\mathcal{X}}. \quad \text{(2.1)}$$

By the surjectivity of $\varphi$ there exists $h_1 \in \mathcal{C}(\mathcal{X}, I)$ such that $\varphi(h_1) = \min\{\varphi(f), \varphi(g)\}$. By (2.1) we establish that $\varphi(h_1) \neq 0_{\mathcal{X}}$ and therefore $h_1 \neq 0_{\mathcal{X}}$. Also, since $\varphi(h_1) \leq \varphi(f)$ and $\varphi(h_1) \leq \varphi(g)$ we get $h_1 \leq f$ and $h_1 \leq g$. But then $h_1 = 0_{\mathcal{X}}$, a contradiction. Similarly we prove the reverse implication. \(\square\)

Throughout the proof we will need the notion of the so-called zero-maximal and one-maximal open sets of the function in $\mathcal{C}(\mathcal{X}, I)$. Let $W$ be a closed subset of $\mathcal{X}$ with $W \neq \mathcal{X}$ and $\text{Int} W = U \neq \emptyset$. If there exists $f \in \mathcal{C}(\mathcal{X}, I)$ such that $f^{-1}(0) = W$ then $U$ will be called the zero-maximal open set of the function $f$. Similarly, if there exists $f \in \mathcal{C}(\mathcal{X}, I)$ such that $f^{-1}(1) = W$ then $U$ will be called the one-maximal open set of the function $f$.

From now on let $\varphi: \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ be a bijective map which preserves the order in both directions.

**Lemma 2.2.** Let $U$ be an open nonempty subset of $\mathcal{X}$ where $\overline{U} \neq \mathcal{X}$. Then there exists $f \in \mathcal{C}(\mathcal{X}, I)$, $f \neq 0_{\mathcal{X}}$, such that $f(\overline{U}) = \{0\}$. Furthermore, for every $f \in \mathcal{C}(\mathcal{X}, I)$, $f \neq 0_{\mathcal{X}}$, with $f(\overline{U}) = \{0\}$ there exists the zero-maximal open set $V$ of the function $\varphi(f)$.

**Proof.** Let $U$ be a nonempty open subset of $\mathcal{X}$ with $\overline{U} \neq \mathcal{X}$. Every compact Hausdorff space is normal, so by Urysohn’s lemma there exists a continuous function $f$ from $\mathcal{X}$ to $I$ such that $f(x) = 0$ for all $x \in \overline{U}$ and $f \neq 0_{\mathcal{X}}$.

Let $f \in \mathcal{C}(\mathcal{X}, I)$, $f \neq 0_{\mathcal{X}}$ and $f(\overline{U}) = \{0\}$. Again by Urysohn’s lemma there exists a continuous function $g: \mathcal{X} \to I$ such that $g(x) = 0$ for all $x \in U^c$ and $g \neq 0_{\mathcal{X}}$. Then $fg = 0_{\mathcal{X}}$ and therefore by Lemma 2.1

$$\varphi(f)\varphi(g) = 0_{\mathcal{X}}.$$

This implies that if $\varphi(g)(x) \neq 0$ then $\varphi(f)(x) = 0$ and if $\varphi(f)(x) \neq 0$ then $\varphi(g)(x) = 0$ for $x \in \mathcal{X}$. Let us define $V = \text{Int}(\varphi(f)^{-1}(0))$. The set $(\varphi(g)^{-1}(0))^c$ is a nonempty open
set since \( g \neq 0_X \), \( \varphi \) is injective and \( \varphi(0_X) = 0_X \). Also, \((\varphi(g)^{-1}(0))^{c} \subset \varphi(f)^{-1}(0)\) and therefore by the definition of \( V \), \( V \neq \emptyset \). Observe that \( \varphi(f)^{-1}(0) \neq X \) since \( f \neq 0_X \) and \( \varphi \) is injective. This yields that \( V \) is the zero-maximal open set of the function \( \varphi(f) \).

**Lemma 2.3.** Let \( U \) be an open nonempty subset of \( X \) where \( \overline{U} \neq X \). Then there exists \( f \in C(X, I) \), \( f \neq 1_X \), such that \( f(\overline{U}) = \{1\} \). Furthermore, for every \( f \in C(X, I) \), \( f \neq 1_X \), with \( f(\overline{U}) = \{1\} \) there exists the one-maximal open set \( V \) of the function \( \varphi(f) \).

**Proof.** Similarly as in the proof of Lemma 2.2 there exists a continuous function \( f \) from \( X \) to \( I \) such that \( f(x) = 1 \) for all \( x \in \overline{U} \) and \( f \neq 1_X \).

Let \( f \in C(X, I) \), \( f \neq 1_X \) and \( f(\overline{U}) = \{1\} \). By Urysohn’s lemma there exists a continuous function \( g : X \rightarrow I \) and an open set \( U_1 \subset U \) such that \( U_1 \) is the zero-maximal open set of the function \( g \) and \( g(x) = 1 \) for all \( x \in U^c \). By Lemma 2.2 there exists the zero-maximal open set \( V_1 \) of the function \( \varphi(g) \). If \( h \) is a function where \( f \leq h \) and \( g \leq h \) then \( h = 1_X \). Suppose there does not exist a nonempty open set \( V \) such that \( \varphi(f)(x) = 1 \) for all \( x \in V \). Then there exists \( x \in V_1 \) such that \( \varphi(f)(x) \neq 1 \) and therefore we conclude that \( \max\{\varphi(f), \varphi(g)\} \neq 1_X \). By the surjectivity of \( \varphi \) there exists the function \( h_1 \) such that \( \varphi(h_1) = \max\{\varphi(f), \varphi(g)\} \). Since \( \varphi(f), \varphi(g) \leq \varphi(h_1) \) we establish that \( f, g \leq h_1 \). But then \( h_1 = 1_X \), a contradiction since \( \varphi(h_1) \neq 1_X \). If we define \( V = \text{Int}(\varphi(f)^{-1}(1)) \) then it follows that \( V \) is the one-maximal open set of the function \( \varphi(f) \).

**Lemma 2.4.** Let \( U_1, U_2, \ldots, U_n \) be the zero-maximal open sets of the functions \( f_1, f_2, \ldots, f_n \), respectively. Then

\[ U_1 \cap U_2 \cap \cdots \cap U_n \neq \emptyset \]

if and only if

\[ V_1 \cap V_2 \cap \cdots \cap V_n \neq \emptyset, \]

where \( V_1, V_2, \ldots, V_n \) are the zero-maximal open sets of the functions \( \varphi(f_1), \varphi(f_2), \ldots, \varphi(f_n) \), respectively.

**Proof.** Suppose \( U_1 \cap U_2 \cap \cdots \cap U_n \neq \emptyset \). The finite intersection of open sets is an open set, so by Urysohn’s lemma there exist a function \( h \in C(X, I) \) and an open set \( U \subset U_1 \cap U_2 \cap \cdots \cap U_n \), where \( h(x) = 1 \) for every \( x \in (U_1 \cap U_2 \cap \cdots \cap U_n)^c \) and \( U \) is the zero-maximal open set of \( h \). This yields

\[ f_i \leq h \quad \text{for } i = 1, 2, \ldots, n \]

and therefore

\[ \varphi(f_i) \leq \varphi(h) \quad \text{for } i = 1, 2, \ldots, n. \] (2.2)

By Lemma 2.2 there exist the zero-maximal open set \( V \) of the function \( \varphi(h) \) and the zero-maximal open sets \( V_1, V_2, \ldots, V_n \) of the functions \( \varphi(f_1), \varphi(f_2), \ldots, \varphi(f_n) \), respectively. From (2.2) we may conclude that

\[ \emptyset \neq V \subset V_1 \cap V_2 \cap \cdots \cap V_n. \]
This implication is also true in the converse direction since \( \varphi^{-1} \) has the same properties as \( \varphi \). □

**Lemma 2.5.** Let \( U_1 \) be the zero-maximal open set of the function \( f_1 \) and \( U_2 \) the one-maximal open set of the function \( f_2 \). Then

\[ U_1 \cap U_2 \neq \emptyset \]

if and only if

\[ V_1 \cap V_2 \neq \emptyset, \]

where \( V_1 \) is the zero-maximal open set of the function \( \varphi(f_1) \) and \( V_2 \) is the one-maximal open set of the function \( \varphi(f_2) \).

**Proof.** Suppose \( U_1 \cap U_2 \neq \emptyset \). Similarly as in the proof of Lemma 2.4 there exist a function \( h \in C(X, I) \) and an open set \( U \subset U_1 \cap U_2 \), where \( h(x) = 0 \) for every \( x \in (U_1 \cap U_2)^c \) and \( U \) is the one-maximal open set of \( h \). Then \( hf_1 = 0 \) and therefore by Lemma 2.1 \( \varphi(h)\varphi(f_1) = 0 \). Let \( V \) be the one-maximal open set of the function \( \varphi(h) \). Then \( V \subset V_1 \). Since \( h \leq f_2 \) and therefore \( \varphi(h) \leq \varphi(f_2) \) we obtain \( V \subset V_2 \). So,

\[ \emptyset \neq V \subset V_1 \cap V_2. \]

This implication is also true in the converse direction since \( \varphi^{-1} \) has the same properties as \( \varphi \). □

In the next step we will construct the homeomorphism \( \mu : \mathcal{X} \to \mathcal{X} \). The construction is similar to one in [10]. For the sake of completeness we will present it in its entirety.

Almost to the end of the proof we will assume that \( |\mathcal{X}| > 1 \). For the point \( x_0 \in \mathcal{X} \) let \( A_{x_0} \), \( A_{x_0} \neq \mathcal{X} \), be an arbitrary open neighbourhood of \( x_0 \in \mathcal{X} \). By Urysohn’s lemma there exist a function \( f \in C(\mathcal{X}, I) \) and an open neighbourhood \( U \) of the point \( x_0 \), \( U \subset A_{x_0} \), where \( U \) is the zero-maximal open set of \( f \). Let \( U_{A_{x_0}} \) be the family of all pairs \((U, f)\) where \( U \) is the zero-maximal open set of \( f \). Let \( x_1 \in \mathcal{X} \), \( x_1 \neq x_0 \). Then there exist open sets \( A_1, A_2 \) such that \( A_1 \cap A_2 = \emptyset \) and \( x_0 \in A_1, x_1 \in A_2 \). Again by Urysohn’s lemma there exists \((U, f)\) \( U \subset A_1 \cap A_{x_0} \). So, \( U \cap A_2 = \emptyset \) and hence \( x_1 \notin \bigcap_{(U, f) \in U_{A_{x_0}}} U \). This gives us

\[
\bigcap_{(U, f) \in U_{A_{x_0}}} U = \{x_0\}.
\]

By Lemma 2.2 there exists for every \((U, f) \in U_{A_{x_0}}\) the zero-maximal open set \( V \) of the function \( \varphi(f) \). Let \( V_{A_{x_0}} \) be the family of all \((V, \varphi(f))\) where \((U, f) \in U_{A_{x_0}}\) and \( V \) is the zero-maximal open set of \( \varphi(f) \). We will next show that there exists a point \( x_1 \in \mathcal{X} \) such that

\[
\bigcap_{(V, \varphi(f)) \in V_{A_{x_0}}} V = \{x_1\}.
\]
Let us first assume that \( \bigcap_{(V, \varphi(f)) \in \mathcal{V}_{x_0}} V = \emptyset \). Since \( \mathcal{X} \) is a compact space there exist \( (V_1, \varphi(f_1)), (V_2, \varphi(f_2)), \ldots, (V_n, \varphi(f_n)) \in \mathcal{V}_{x_0} \) such that \( V_1 \cap V_2 \cap \cdots \cap V_n = \emptyset \). Let \((U_1, f_1), (U_2, f_2), \ldots, (U_n, f_n) \in \mathcal{U}_{x_0}\). Since \( U_1 \cap U_2 \cap \cdots \cap U_n \neq \emptyset \) we obtain by Lemma 2.4 that \( V_1 \cap V_2 \cap \cdots \cap V_n \neq \emptyset \), a contradiction. So,

\[
\bigcap_{(V, \varphi(f)) \in \mathcal{V}_{x_0}} V \neq \emptyset.
\]

Let us next assume that \( \bigcap_{(V, \varphi(f)) \in \mathcal{V}_{x_0}} V = \emptyset \). Then there exist \( x_\lambda \in \bigcap_{(V, \varphi(f)) \in \mathcal{V}_{x_0}} \overline{V} \) and \((V_\lambda, \varphi(f_\lambda)) \in \mathcal{V}_{x_0}\), where \( x_\lambda \in \overline{V_\lambda} \) and \( x_\lambda \notin \overline{\mathcal{X}}_\lambda \). Let \((U_\lambda, f_\lambda) \in \mathcal{U}_{x_0}\). By Urysohn’s lemma there exist a function \( h \) and open sets \( U' \) and \( U'' \) where \( x_0 \in U'' \), \( U'' \subset U' \), \( U' \) is the zero-maximal open set of \( h \) and \( h(U'') = \{1\} \). Let \( V' \) be the zero-maximal open set of the function \( \varphi(h) \) and \( V'' \) the one-maximal open set of \( \varphi(h) \). Since \( hf_\lambda = 0_{\mathcal{X}} \) we obtain by Lemma 2.1

\[
\varphi(h)(f_\lambda) = 0_{\mathcal{X}}.
\]

If \( \varphi(h)(x) \neq 0 \) then \( \varphi(f_\lambda)(x) = 0 \), \( x \in \mathcal{X} \). Let us assume that \( x_\lambda \notin \overline{V} \). Then there exists by the normality of \( \mathcal{X} \) an open set \( A_\lambda \) such that \( x_\lambda \in A_\lambda \) and \( \overline{A_\lambda} \cap \overline{V'} = \emptyset \). We may conclude, since \( x_\lambda \in \overline{V_\lambda} \setminus \overline{A_\lambda} \), that \( A_\lambda \subset \overline{V_\lambda} \setminus \overline{A_\lambda} \neq \emptyset \). So, there exists \( x_\lambda \in A_\lambda \setminus \overline{V_\lambda} \subset \overline{V_\lambda \setminus \overline{A_\lambda}} \). Since \( A_\lambda \subset \overline{V_\lambda} \) is an open nonempty set and \( V' \) is the zero-maximal open set of \( \varphi(h) \), we may without loss of generality assume that \( \varphi(h)(x_\lambda) \neq 0 \). Also, since \( \varphi(h) \) is continuous there exists an open neighbourhood \( A_{x_\lambda} \) of \( x_\lambda \) where \( \varphi(h)(x) \neq 0 \) for all \( x \in A_{x_\lambda} \). Again, by the normality of \( \mathcal{X} \) there exists an open neighbourhood \( A_1 \) of \( x_\lambda \) where \( \overline{A_1} \cap \overline{V_\lambda} = \emptyset \). Then \( \varphi(h)(x) \neq 0 \) for all \( x \in A_1 \cap A_{x_\lambda} \) and therefore

\[
\varphi(f_\lambda)(x) = 0 \quad \text{for all } x \in A_1 \cap A_{x_\lambda}.
\]

But \( A_1 \cap A_{x_\lambda} \subset \overline{V_\lambda} \), a contradiction, since \( V_\lambda \) is the zero-maximal open set of \( \varphi(f_\lambda) \). So, our assumption was wrong and therefore

\[
x_\lambda \in \overline{V}'.
\]

There exist a function \( f_\mu \) and an open set \( U_\mu, \overline{U_\mu} \subset U'' \) where \( (U_\mu, f_\mu) \in \mathcal{U}_{x_0} \) and \( f_\mu(U''') = \{1\} \). Let \( V_\mu \) be the zero-maximal open set of \( \varphi(f_\mu) \). Since \( \max\{h, f_\mu\} = 1_{\mathcal{X}} \) and \( \varphi \) preserves the ordering \( \leq \) we conclude that \( V_\mu \subset V'' \). So, \( \overline{V_\mu} \cap \overline{V'} = \emptyset \) and therefore \( x_\lambda \notin \overline{V_\mu} \). A contradiction, since \( (V_\mu, \varphi(f_\mu)) \in \mathcal{V}_{x_0} \) and \( x_\lambda \in \bigcap_{(V, \varphi(f)) \in \mathcal{V}_{x_0}} V \). We have proven that

\[
\bigcap_{(V, \varphi(f)) \in \mathcal{V}_{x_0}} V \neq \emptyset.
\]

Let us now assume that there exist the points \( x_1, x_2 \in \mathcal{X} \), \( x_1 \neq x_2 \), such that \( \{x_1, x_2\} \subset \bigcap_{(V, \varphi(f)) \in \mathcal{V}_{x_0}} V \). Let \( V' \) and \( V'' \) be disjoint open neighbourhoods of the points \( x_1 \) and \( x_2 \), respectively. There exists by Urysohn’s lemma and the surjectivity of \( \varphi \) a function \( \varphi(h_1) \) with one-maximal open set \( V_1 \) where \( \overline{V_1} \subset V' \), \( x_1 \in V_1 \), and \( \varphi(h_1)(V'') = \{0\} \). Similarly,
there exists a function \( \varphi(h_2) \) with one-maximal open set \( V_2 \) where \( \overline{V}_2 \subset V'' \), \( x_2 \in V_2 \), and \( \varphi(h_2)(V'') = \emptyset \). Clearly, \( \overline{V}_1 \cap \overline{V}_2 = \emptyset \), \( \varphi(h_1)\varphi(h_2) = 0 \), and hence by Lemma 2.1
\[
h_1h_2 = 0. \tag{2.3}
\]

By Lemma 2.3 and (2.3) there exist \( U_1 \) and \( U_2 \) which are one-maximal open sets of \( h_1 \) and \( h_2 \) such that \( \overline{U}_1 \cap \overline{U}_2 = \emptyset \). Without loss of generality we may assume that \( x_0 \notin U_2 \). Since \( \mathcal{X} \) is normal we may find an open neighbourhood \( U_3 \), \( \overline{U}_3 \subset A_{x_0} \), of the point \( x_0 \), such that
\[
U_2 \cap U_3 = \emptyset.
\]

By Urysohn’s lemma there exist a function \( h_3 \) and an open set \( U_4 \) such that \( \overline{U}_4 \subset U_3 \), where \( U_4 \) is the zero-maximal open set of \( h_3 \) and \( x_0 \in U_4 \). So, \((U_4, h_3) \in U_{A_{x_0}} \). This yields that there exists \( V_3 \) where \((V_3, \varphi(h_3)) \in V_{A_{x_0}} \). Then on the one hand we establish that
\[
x_2 \in V_3 \cap V_2
\]
and on the other hand, since \( U_4 \cap U_2 = \emptyset \), we obtain by Lemma 2.5
\[
V_3 \cap V_2 = \emptyset,
\]
which is a contradiction. Therefore
\[
\bigcap_{(V, \varphi(f)) \in V_{A_{x_0}}} V = \{x_1\}.
\]

Next, we will prove that this intersection is independent of the selection of an open neighbourhood of the point \( x_0 \). Let now \( B_{x_0} \), \( \overline{B}_{x_0} \neq \mathcal{X}, B_{x_0} \neq A_{x_0} \), be an open neighbourhood of \( x_0 \) and let \( \bigcap_{(V, \varphi(f)) \in V_{B_{x_0}}} V = \{x_2\} \). Suppose \( C_{x_0} = A_{x_0} \cap B_{x_0} \) and let \( \bigcap_{(V, \varphi(f)) \in V_{C_{x_0}}} V = \{x_3\} \). Clearly, if \((U, f) \in U_{C_{x_0}} \) then \((U, f) \in U_{A_{x_0}} \) and \((U, f) \in U_{B_{x_0}} \). Hence \((V, \varphi(f)) \in V_{A_{x_0}} \) and \((V, \varphi(f)) \in V_{B_{x_0}} \) for all \((V, \varphi(f)) \in V_{C_{x_0}} \). This yields
\[
x_1 = x_3 = x_2.
\]

Let now \( \psi : \mathcal{X} \to \mathcal{X} \) be the function which \( x_0 \mapsto x_1 \). We will prove that \( \psi \) is a homeomorphism. Let \( x_a \neq x_b, x_a, x_b \in \mathcal{X} \). Then there exist functions \( f_a \) and \( f_b \) and disjoint zero-maximal open sets \( U_a \) and \( U_b \) of \( f_a, f_b \), respectively, that are neighbourhoods of the points \( x_a, x_b \). Let \( V_a, V_b \) be the corresponding zero-maximal open sets of the functions \( \varphi(f_a), \varphi(f_b) \). Therefore \((V_a, \varphi(f_a)) \in V_{A_{x_a}} \) and \((V_b, \varphi(f_b)) \in V_{B_{x_b}} \). Since \( U_a \cap U_b = \emptyset \) we get by Lemma 2.4 \( V_a \cap V_b = \emptyset \). This yields
\[
\left\{ \psi(x_a) \right\} = \bigcap_{(V, \varphi(f)) \in V_{A_{x_a}}} \neq \bigcap_{(V, \varphi(f)) \in V_{B_{x_b}}} V = \left\{ \psi(x_b) \right\}.
\]
So, \( \psi(x_a) \neq \psi(x_b) \) which means that \( \psi \) is injective. Since \( \varphi^{-1} \) has the same properties as \( \varphi \) it follows that \( \psi \) is also surjective.

Let now \( V \subset \mathcal{X} \) be the zero-maximal open set of some function \( \varphi(f) \) and let \( x_v \) be an arbitrary point in \( V \). Let \( U \) be the corresponding zero-maximal open set of the function \( f \). The set \( U \) is then a neighbourhood of the point \( \psi^{-1}(x_v) \). So, \( \psi^{-1}(V) \subset U \). Similarly, for each \( x_u \in U \) we can conclude that \( \psi(x_u) \in V \) which yields \( \psi(U) \subset V \) and therefore
\[
\psi^{-1}(V) = U.
\]
We have proven that the inverse image of any zero-maximal open set of some function \( \varphi(f) \) is the zero-maximal open set of the function \( f \).

Let now \( A \subset \mathcal{X} \) be any nonempty open set. We may find, by Urysohn’s lemma, for every \( a \in A \) a function \( f_a \) and the zero-maximal open set \( V_a \) of \( f_a \), \( \overline{V_a} \subset A \), such that \( V_a \) is a neighbourhood of the point \( a \). Therefore \( A = \bigcup_{a \in A} V_a \) which gives us

\[
\psi^{-1}(A) = \bigcup_{a \in A} \psi^{-1}(V_a),
\]

hence \( \psi^{-1}(A) \) is an open set. This yields that \( \psi \) is a continuous function. Since \( \mathcal{X} \) is a compact Hausdorff space and \( \psi \) is a continuous bijection we can conclude that \( \psi \) is a homeomorphism. Let us denote the homeomorphism

\[
\mu = \psi^{-1}.
\]

In the conclusion of the proof of the theorem we will need another auxiliary result. We will show that the ordering \( \leq \) is valid also locally.

**Lemma 2.6.** Let \( f, g \in \mathcal{C}(\mathcal{X}, I) \) such that \( f(\mu(x_1)) > g(\mu(x_1)), x_1 \in \mathcal{X} \). Then \( \varphi(f)(x_1) > \varphi(g)(x_1) \).

**Proof.** By the continuity of the functions \( f \) and \( g \) there exists an open neighbourhood \( U \) of \( \mu(x_1) \) where \( f(x) > g(x) \) for all \( x \in U \). Let us assume that \( \varphi(f)(x_1) < \varphi(g)(x_1) \). Then by the continuity of \( \varphi(f) \) and \( \varphi(g) \) there exists an open neighbourhood \( V \) of \( x_1 \) where \( \varphi(f)(x) < \varphi(g)(x) \) for all \( x \in V \). By Urysohn’s lemma and the surjectivity of \( \varphi \) there exist an open set \( V_2 \) and a function \( \varphi(h_2) \) where \( x_1 \in V_2 \subset V \) and \( V_2 \) is the zero-maximal open set of \( \varphi(h_2) \). By Lemma 2.2 and since \( \varphi^{-1} \) has the same properties as \( \varphi \) there exists the zero-maximal open set \( U_2 \) of \( h_2 \). Note that \( \mu(x_1) \in U_2 \cup U \). The set \( U_2 \) is generally not necessarily a subset of \( U \). We may find in any case by Urysohn’s lemma a function \( h^2_2 \in \mathcal{C}(\mathcal{X}, I) \) and an open set \( U^2_2 \) where \( h^2_2(\mathcal{X}^c) = 1 \), \( U^2_2 \) is the zero-maximal open set of \( h^2_2 \) and \( \mu(x_1) \in U^2_2 \). Let \( h^2_2 = \max\{h_2, h^2_2\} \). We notice that the zero-maximal open set of \( h^2_2 \) is \( U_2 \cup U^2_2 \), \( \mu(x_1) \in U_2 \cup U^2_2 \) and \( U_2 \cup U^2_2 \subset U \). By Lemma 2.2 there exists the zero-maximal open set \( V^2_2 \) of \( \varphi(h^2_2) \) where \( x_1 \in V^2_2 \). Since \( h_2 \leq h^2_2 \) we obtain \( \varphi(h_2) \leq \varphi(h^2_2) \) and therefore \( V^2_2 \subset V_2 \subset V \). So, without loss of generality we may assume that the closure of \( U_2 \) is a subset of \( U \). Again, by Urysohn’s lemma and the surjectivity of \( \varphi \) there exist open sets \( V_1 \) and \( V_3 \) and a function \( \varphi(h_1) \) where \( x_1 \in V_1 \), \( V_1 \subset V_2 \), \( V^2_2 \subset V_3 \), \( V_1 \) is the one-maximal open set of \( \varphi(h_1) \) and \( V_3 \) is the zero-maximal open set of \( \varphi(h_1) \).

By the surjectivity of \( \varphi \) there exists the function \( h_3 \) such that \( \varphi(h_3) = \min\{\varphi(g), \varphi(h_1)\} \). Then \( V_3 \) is the zero-maximal open set of \( \varphi(h_3) \). Also, \( \varphi(h_3) \leq \varphi(g) \) and \( \varphi(h_3) \leq \varphi(f) \) and therefore

\[
h_3 \leq g \quad \text{and} \quad h_3 \leq f.
\]

By Lemma 2.2 there exists the zero-maximal open set \( U_1 \) of \( h_3 \). Also, \( \varphi(h_2)\varphi(h_3) = 0_{\mathcal{X}} \) and therefore by Lemma 2.1 \( h_2 h_3 = 0_{\mathcal{X}} \). Suppose \( \overline{U_1} \cup \overline{U_2} \neq \mathcal{X} \) and let \( x \in \mathcal{X} \setminus (\overline{U_1} \cup \overline{U_2}) \).

By the normality of \( \mathcal{X} \) there exists an open neighbourhood \( C \) of \( x \), \( C \cap (\overline{U_1} \cup \overline{U_2}) = \emptyset \).

Let \( y \in C \). Then \( h_2(y) = 0 \) or \( h_3(y) = 0 \). Let us assume the former. Since \( U_2 \) is the

where on the one hand $\phi(f)(x) = 0$ and therefore, since $h_3$ is continuous, $h_3(y) = \lim_{x \to y} h_3(y) = 0$. For every $y \in C$ we obtain $h_2(y) = h_3(y) = 0$, which is a contradiction since $U_1$ and $U_2$ are zero-maximal open sets of $h_2$ and $h_3$, respectively. So, $\overline{U_1} \cup \overline{U_2} = \mathcal{X}$ and therefore $h_3(x) \neq 0$ only if $x \in U_2$. But $\overline{U_2} \subset U$ and therefore since $h_3 \leq g$ we obtain

$h_3 \leq f$,

a contradiction. So, our assumption was wrong and therefore $\phi(f)(x_1) \geq \phi(g)(x_1)$.

Let $x_0 \in \mathcal{X}$ and let $m_{x_0} : I \to I$ be the function defined in the following way:

$$m_{x_0}(c) = \phi(c)(x_0),$$

where on the one hand $c \in I$ and on the other hand $c \in \mathcal{C}(\mathcal{X}, I)$ is a constant function. We will prove that $m_{x_0}$ is a bijective and increasing map. Let $c_i \leq c_j$ for $c_i, c_j \in I$. Then $\phi(c_i)(x_0) \leq \phi(c_j)(x_0)$ and therefore $m_{x_0}(c_i) \leq m_{x_0}(c_j)$. So, $m_{x_0}$ is a monotone increasing map. Also, $m_{x_0}(0) = 0$ and $m_{x_0}(1) = 1$. Notice that then $m_{x_0}$ is surjective if and only if it is continuous. Let us assume that there exists $c_0 \in I$ where $m_{x_0}$ is not continuous. We will discuss different options. Suppose first $m_{x_0}(c_0) \in (0, 1)$. Then $c_0 \in (0, 1)$. There exists $\epsilon > 0$ such that $m_{x_0}(c_0) - \epsilon > m_{x_0}(c)$ for all $c < c_0$ or $m_{x_0}(c_0) + \epsilon < m_{x_0}(c)$ for all $c > c_0$. Without loss of generality we may assume the latter. Define $\epsilon_m = \sup\{\epsilon; m_{x_0}(c) - m_{x_0}(c_0) > \epsilon$ for all $c > c_0\}$. By the surjectivity of $\phi$ there exists $g \in \mathcal{C}(\mathcal{X}, I)$ such that

$$\phi(g)(x_0) = \phi(c_0)(x_0) + \frac{\epsilon_m}2 \quad \text{and} \quad \phi(g) \geq \phi(c_0).$$

By Lemma 2.6 since $\phi^{-1}$ has the same properties as $\phi$ and $\phi(g)(x_0) < \phi(c)(x_0)$ for all $c > c_0$ we obtain that $g(\mu(x_0)) \leq c_0$ and therefore $g(\mu(x_0)) = c_0$. Suppose $\{x_0\}$ is not an open set. Then, since $\mathcal{X}$ satisfies the first axiom of countability, there exists a sequence $\{x_i, i \in \mathbb{N}\}$ where $x_0 = \lim_{i \to \infty} x_i \not\equiv 0$ and $x_0 \neq x_i \neq x_j$ for all $i \neq j, i, j \in \mathbb{N}$. Since $\mu$ is continuous we obtain $\mu(x_0) = \lim_{i, j \to \infty} \mu(x_i)$. Also, since $g$ is continuous there exists $i_0 \in \mathbb{N}$ such that $c_0 - 1/j \in (0, 1)$ and $g(\mu(x_j)) + 1/j \in (0, 1)$ for all $j \geq i_0$. Let now $h_r : \{\mu(x_j), j \in \mathbb{N}, j \geq i_0\} \cup \{\mu(x_0)\} \to [0, 1]$ be the function defined in the following way:

$$h_r(\mu(x_j)) = \begin{cases} g(\mu(x_j)) + \frac{1}{j}, & j \geq i_0 \text{ and } j \text{ odd}, \\ c_0 - \frac{1}{j}, & j \geq i_0 \text{ and } j \text{ even}, \\ c_0, & j = 0. \end{cases}$$

Notice that $h_r$ is continuous. The space $\mathcal{X}$ is first countable therefore for an arbitrary $A \subset \mathcal{X}$, $\overline{A} = \{x \in \mathcal{X}; x \text{ is a limit of a sequence from } A\}$. This yields that $\{\mu(x_j), j \in \mathbb{N}, j \geq i_0\} \cup \{\mu(x_0)\}$ is a closed subset of $\mathcal{X}$ and therefore by Tietze theorem there exists a continuous extension $h : \mathcal{X} \to I$. By Lemma 2.6 we obtain for $j \geq i_0$:

$$\phi(h)(x_j) \geq \phi(g)(x_j) \quad \text{if } j \text{ is odd} \quad \text{and} \quad \phi(h)(x_j) \leq \phi(c_0)(x_j) \quad \text{if } j \text{ is even}.$$
On the one hand we obtain
\[ \varphi(h)(x_0) = \lim_{i \to \infty} \varphi(h)(x_{2^i-1}) \geq \varphi(c_0)(x_0) + \frac{\epsilon_m}{2}, \]
but on the other hand we get
\[ \varphi(h)(x_0) = \lim_{i \to \infty} \varphi(h)(x_{2^i}) \leq \varphi(c_0)(x_0), \]
a contradiction. Let now \( \{x_0\} \) be an open set. By the surjectivity of \( \varphi \) there exists a function \( h_1 \in \mathcal{C}(\mathcal{X}, I) \) such that
\[ \varphi(h_1)(x) = \begin{cases} \varphi(c_0)(x_0) + \frac{\epsilon_m}{2}, & x = x_0, \\ 0, & x \neq x_0. \end{cases} \]
So, \( \varphi(h_1) \leq \varphi(c_0) \) and \( \varphi(h_1)(x_0) < \varphi(c)(x_0) \) for all \( c > c_0 \). By Lemmas 2.2 and 2.6 and since \( \varphi^{-1} \) has the same properties as \( \varphi \) we obtain
\[ h_1(x) = \begin{cases} c_0, & x = \mu(x_0), \\ 0, & x \neq \mu(x_0). \end{cases} \]
Therefore \( h_1 \leq c_0 \) which is a contradiction.

Next, if \( m_{x_0}(c_0) = 0 \) and \( c_0 \in (0, 1) \) or \( m_{x_0}(c_0) = 1 \) and \( c_0 \in (0, 1) \) we get a contradiction using similar arguments. Suppose now \( c_0 = 0 \). Let us define \( \varphi(g) \) in the same way as before. Then \( g(\mu(x_0)) = 0 \). Supposing first that \( \{x_0\} \) is not an open set there exists as before a sequence \( \{x_i, i \in \mathbb{N}\} \) where \( x_0 = \lim_{i \to \infty} x_i \) and where \( x_0 \neq x_i \neq x_j \) for all \( i \neq j \), \( i, j \in \mathbb{N} \). Observe, since \( \mu \) is injective, that \( \mu(x_0) \neq \mu(x_i) \neq \mu(x_j) \) and since \( \mu \) is continuous \( \mu(x_0) = \lim_{i \to \infty} \mu(x_i) \). There exists \( i_0 \in \mathbb{N} \) such that \( g(\mu(x_j)) + 1/j \in (0, 1) \) for all \( j \geq i_0 \). Since \( \mathcal{X} \) satisfies the first axiom of countability there exists at most countable family \( \{H_n, n \in \mathbb{N}\} \) of open neighbourhoods of \( \mu(x_0) \) where for each open set \( G, \mu(x_0) \in G \), there is some \( H_n \subset G \). Let
\[ G_k = \bigcap_{i=1}^{k} H_i. \]
Then \( \mu(x_0) \in G_k \) and \( G_1 \supset G_2 \supset G_3 \supset \cdots \). There exists for each \( n \in \mathbb{N} \) a \( k_n \in \mathbb{N} \) such that \( \{\mu(x_k), k \geq k_n\} \subset G_n \). Without loss of generality we may assume that \( k_i < k_j \) if \( i < j \) and that \( k_1 \geq i_0 \). By the normality of \( \mathcal{X} \) there exist open neighbourhoods \( U_{\mu(x_{k_i})} \) of \( \mu(x_{k_i}) \) where
\[ U_{\mu(x_{k_i})} \subset G_i, \quad i \in \mathbb{N} \quad \text{and} \quad U_{\mu(x_{k_i})} \cap U_{\mu(x_{k_j})} = \emptyset \quad \text{for every} \; i, j \in \mathbb{N}, \; i \neq j. \]
Let
\[ f_r : \{\mu(x_{k_{2l-1}}), l \in \mathbb{N}\} \cup \{U_{\mu(x_{k_{2l}})}, l \in \mathbb{N}\} \cup \{\mu(x_0)\} \to [0, 1] \]
be the function defined in the following way:
\[ f_r(\mu(x)) = \begin{cases} g(\mu(x_{k_{2l-1}})) + \frac{1}{k_{2l-1}}, & x = x_{k_{2l-1}}, \; l \in \mathbb{N}, \\ 0, & \mu(x) \in \{\mu(x_0)\} \cup \{U_{\mu(x_{k_{2l}})}, l \in \mathbb{N}\}. \end{cases} \]
As before, by the argument that the closure of the domain of $f_r$ is the set of all $x \in \mathcal{X}$, where $x$ is a limit of a sequence from the domain of $f_r$, we establish that

$$\{\mu(x_{k_{2l-1}}), \ l \in \mathbb{N}\} \cup \{U_{\mu(x_{k_{2l}})}, \ l \in \mathbb{N}\} \cup \{\mu(x_0)\}$$

is a closed subset of $\mathcal{X}$. So, by Tietze theorem since $f_r$ is continuous there exists a continuous extension $f : \mathcal{X} \to I$. By Lemma 2.6 we get

$$\varphi(f)(x_{k_{2l-1}}) \geq \varphi(g)(x_{k_{2l-1}}), \ l \in \mathbb{N}.$$ 

By Lemma 2.2 and the definition of $\mu$ we obtain

$$\varphi(f)(x_{k_{2l}}) = 0, \ l \in \mathbb{N}.$$ 

Since $\varphi(g)$ is continuous there exists $k_0 \in \mathbb{N}$ such that $\varphi(g)(x_k) > \varepsilon_m/4$ for all $k \geq k_0$. So, on the one hand we obtain

$$\varphi(f)(x_0) = \lim_{l \to \infty} \varphi(f)(x_{k_{2l-1}}) \geq \frac{\varepsilon_m}{4},$$

but on the other hand we get

$$\varphi(f)(x_0) = \lim_{l \to \infty} \varphi(f)(x_{k_{2l}}) = 0,$$

a contradiction. Assuming that $\{x_0\}$ is an open set we get a contradiction in the same way as in the previous step.

Finally, if we assume that $c_0 = 1$ we obtain a contradiction in a similar way using Lemmas 2.3 and 2.6.

We have proven that $m_x : I \to I, \ x \in \mathcal{X}$, is a surjective map. We already know that $m_x, \ x \in \mathcal{X}$, is monotone increasing. By using Lemmas 2.2, 2.3, 2.6 and Tietze theorem similarly as in the proof of surjectivity of $m_x$ and applying that $\varphi^{-1}$ has the same properties as $\varphi$ we prove that $m_x$ is strictly monotone increasing, that is $c_i < c_j$ if and only if $m_x(c_i) < m_x(c_j)$ for $c_i, c_j \in I$ and $x \in \mathcal{X}$. We established that $m_x : I \to I, \ x \in \mathcal{X}$, is a bijective, increasing function. Observe that then $m_x, \ x \in \mathcal{X}$, is also continuous. Let us prove that

$$\varphi(f)(x) = m_x(f(\mu(x)))$$

for every $f \in C(\mathcal{X}, I)$. Let $f \in C(\mathcal{X}, I)$ be any function for which $f(\mu(x)) \in (0, 1), \ x \in \mathcal{X}$. Then there exists $\varepsilon > 0$ such that $(f(\mu(x)) - \varepsilon, f(\mu(x)) + \varepsilon) \subset (0, 1)$. Since

$$f(\mu(x)) - \varepsilon < f(\mu(x)) < f(\mu(x)) + \varepsilon$$

we obtain by Lemma 2.6

$$\varphi(f(\mu(x)) - \varepsilon)(x) \leq \varphi(f)(x) \leq \varphi(f(\mu(x)) + \varepsilon)(x).$$

By (2.4) we have

$$\varphi(f(\mu(x)) + \varepsilon)(x) = m_x(f(\mu(x)) + \varepsilon) \quad \text{and} \quad \varphi(f(\mu(x)) - \varepsilon)(x) = m_x(f(\mu(x)) - \varepsilon).$$

So,

$$m_x(f(\mu(x)) - \varepsilon) \leq \varphi(f)(x) \leq m_x(f(\mu(x)) + \varepsilon)$$
for every \( \varepsilon > 0 \) for which \( (f(\mu(x)) - \varepsilon, f(\mu(x)) + \varepsilon) \subset (0, 1) \). This yields by the continuity of \( m_x \) that

\[
\varphi(f)(x) = m_x\left(f(\mu(x))\right)
\]

for every \( f \in \mathcal{C}(\mathcal{X}, I) \) where \( f(\mu(x)) \in (0, 1) \).

Let \( f(\mu(x)) = 1 \). Then \( \varphi(f(\mu(x)) - \varepsilon)(x) = m_x(f(\mu(x)) - \varepsilon), \varepsilon \in (0, 1) \). By Lemma 2.6 we obtain

\[
m_x\left(f(\mu(x)) - \varepsilon\right) = m_x(1 - \varepsilon) \leq \varphi(f)(x),
\]

therefore by the continuity of \( m_x \) and since \( m_x(1) = 1 \) we get \( \varphi(f)(x) = 1 \).

Similarly, if \( f(\mu(x)) = 0 \) we obtain \( \varphi(f)(x) = 0 \).

Let \( |\mathcal{X}| = 1 \) and \( \varphi(f)(x) = m(f) \) where \( m : I \to I \). Then \( m \) is a bijective and increasing function.

Finally, let us prove the second part of the theorem. Suppose \( \mu : \mathcal{X} \to \mathcal{X} \) is a homeomorphism, let \( m_x : I \to I, x \in \mathcal{X} \), be a bijective, increasing map for every \( x \in \mathcal{X} \) and let \( m_x(c) : \mathcal{X} \to I, x \mapsto m_x(c) \), be a continuous map for every \( c \in I \). Let

\[
h(x) = m_x(f(\mu(x))), \quad x \in \mathcal{X},
\]

for \( f \in \mathcal{C}(\mathcal{X}, I) \). Then \( h : \mathcal{X} \to I \). Let us prove that \( h \in \mathcal{C}(\mathcal{X}, I) \). Assume that \( h \notin \mathcal{C}(\mathcal{X}, I) \). Also, suppose \( h \) is not continuous at \( x_0 \). Then there exists a sequence \( \{x_i\} \to x_0 \) where \( \{h(x_i)\} \to h(x_0) \). Assume that \( h(x_0) \in (0, 1) \). There exist an open interval \( (a, b) \subset I \) where \( h(x_0) \in (a, b) \) and a subsequence \( \{y_k, k \in \mathbb{N}\} \subset \{x_i, i \in \mathbb{N}\} \) where \( h(y_k) \in I \setminus (a, b) \) for all \( k \). There also exists a subsequence \( \{s_j, j \in \mathbb{N}\} \subset \{y_k, k \in \mathbb{N}\} \) such that \( h(s_j) \geq b \) for all \( j \) or there exists a subsequence \( \{v_l, l \in \mathbb{N}\} \subset \{y_k, k \in \mathbb{N}\} \) such that \( h(v_l) \leq a \) for all \( l \).

Without loss of generality we may assume the former. Let \( c_0 \in \mathcal{C}(\mathcal{X}, I) \), \( c_0 \neq f(\mu(x_0)) \), be any constant function. Then \( m_x(c_0) : \mathcal{X} \to I, x \mapsto m_x(c_0) \), is continuous and therefore since \( \{s_j\} \to x_0 \) we obtain \( \lim_{j \to \infty} m_{s_j}(c_0) = m_{x_0}(c_0) \). If \( c_0 < f(\mu(x_0)) \) we obtain, since \( m_{x_0} \) is strictly monotone increasing, that \( m_{x_0}(c_0) < h(x_0) \). Let \( c_0 > f(\mu(x_0)) \). Then, since \( f \) and \( \mu \) are continuous, there exists \( \varepsilon > 0 \) and \( j_0 \in \mathbb{N} \) such that \( c_0 > f(\mu(x_0)) + \varepsilon \) and \( f(\mu(s_j)) < f(\mu(x_0)) + \varepsilon \) for all \( j \geq j_0 \). So, since \( m_x \) are strictly monotone increasing for all \( x \in \mathcal{X} \), we obtain \( m_{s_j}(f(\mu(s_j))) < m_{s_j}(c_0) \) for all \( j \geq j_0 \). But then, since \( m_{s_j}(f(\mu(s_j))) = h(s_j) \geq b \) for all \( j \in \mathbb{N} \), we obtain

\[
m_{s_j}(c_0) > b \quad \text{for all } j \geq j_0.
\]

This yields that

\[
\lim_{j \to \infty} m_{s_j}(c_0) = m_{x_0}(c_0) \geq b.
\]

Therefore there does not exist \( c \in I \) and \( c \neq f(\mu(x_0)) \) such that \( m_{x_0}(c) \in (h(x_0), b) \). This is a contradiction since \( m_{x_0} : I \to I \) is surjective. If \( h(x_0) = 0 \) or \( h(x_0) = 1 \) we get a contradiction in a similar way. So, \( h \in \mathcal{C}(\mathcal{X}, I) \).

Define

\[
\varphi(f)(x) = m_x\left(f(\mu(x))\right), \quad x \in \mathcal{X},
\]

for all \( f \in \mathcal{C}(\mathcal{X}, I) \). Then \( \varphi : \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I) \). Let us prove that \( \varphi \) is a bijective map that preserves the order in both directions. We will first prove that \( \varphi \) is injective. Suppose
Then there exists $x_0 \in \mathcal{X}$ such that $f(\mu(x_0)) \neq g(\mu(x_0))$. Since $m_{x_0}$ is injective we get $m_{x_0}(f(\mu(x_0))) = m_{x_0}(g(\mu(x_0)))$ and therefore $\varphi(f) \neq \varphi(g)$. So, $\varphi$ is injective. Let $g \in \mathcal{C}(\mathcal{X}, I)$. Function $m_x : I \to I$, $x \in \mathcal{X}$, is bijective and continuous, so, since $I$ is a compact space we establish that $m_x$ is a homeomorphism where $m_x^{-1}$ is also increasing. Let us prove that for a constant $c_0 \in I$, $m_x^{-1}(c_0) : \mathcal{X} \to I$, $x \mapsto m_x^{-1}(c_0)$, is a continuous map. Let $h(x) = m_x^{-1}(c_0)$. So, $c_0 = m_x(h(x))$. Let $h \notin \mathcal{C}(\mathcal{X}, I)$ and suppose $h$ is not continuous at $x_0$. Then there exists a sequence $\{x_i\} \to x_0$ where $\{h(x_i)\} \not\to h(x_0)$. Note that $m_x^{-1}(c_0) = 0$ if and only if $c_0 = 0$ and $m_x^{-1}(c_0) = 1$ if and only if $c_0 = 1$. Therefore, if $h(x_0) = 0$ we obtain $h = 0_{\mathcal{X}}$ and if $h(x_0) = 1$ we get $h = 1_{\mathcal{X}}$. So, $h(x_0) \in (0, 1)$. There exist an open interval $(a, b) \subset I$ where $h(x_0) \in (a, b)$ and a subsequence $\{y_k\}, k \in \mathbb{N} \subset \{x_i, i \in \mathbb{N}\}$ where $h(y_k) \in I \setminus (a, b)$ for all $k \in \mathbb{N}$. This yields since $m_x : I \to I$, $x \in \mathcal{X}$, is strictly monotone increasing that

$$c_0 = m_{y_k}(h(y_k)) \in I \setminus (m_{x_0}(a), m_{x_0}(b))$$

for all $k \in \mathbb{N}$. Since $m_x(c) : \mathcal{X} \to I$, $x \mapsto m_x(c)$, $c \in I$, is continuous and $\{y_k\} \to x_0$, we obtain that $\lim_{k \to \infty} m_{y_k}(a) = m_{x_0}(a)$ and $\lim_{k \to \infty} m_{y_k}(b) = m_{x_0}(b)$. This yields, on the one hand, that $c_0 \in I \setminus (m_{x_0}(a), m_{x_0}(b))$. But on the other hand, since $h(x_0) \in (a, b)$, we obtain $c_0 \in (m_{x_0}(a), m_{x_0}(b))$, a contradiction.

Let now $f(x) = m_x^{-1}(g(\mu^{-1}(x)))$. Then $f \in \mathcal{C}(\mathcal{X}, I)$ and

$$\varphi(f)(x) = m_x(m_x^{-1}(g(\mu(\mu^{-1}(x))))) = g(x)$$

which yields that $\varphi$ is surjective.

Finally, let $f \leq g$, $f, g \in \mathcal{C}(\mathcal{X}, I)$. Let $x_0 \in \mathcal{X}$. Then $f(\mu(x_0)) \leq g(\mu(x_0))$ and since $m_{x_0}$ is monotone increasing we obtain $\varphi(f)(x_0) \leq \varphi(g)(x_0)$. So, $\varphi(f) \leq \varphi(g)$. Using the same argument we prove the reverse implication. So, $\varphi$ preserves the ordering $\leq$ in both directions.

References