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The Global Goursat Problem and Scattering for Nonlinear Wave Equations

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The Goursat problem for nonlinear scalar equations on the Einstein Universe $\tilde{\mathbf{M}}$, with finite-energy datum, has a unique global solution in the positive-energy, Sobolev-controllable case. Such equations include those of the form $\Box\varphi + H'(\varphi) = 0$, where H denotes a hamiltonian that is a fourth-order polynomial, bounded below, in components of the multicomponent scalar section φ. In particular, the conformally invariant equation $(\Box + 1)\varphi + \lambda\varphi^3 = 0$ $(\lambda \geqslant 0)$ is included. In the higher-dimensional analog $\mathbb{R} \times S^n$ to the Einstein Universe the same result holds under the stronger conditions on H required for Sobolev controllability. Irrespective of energy positivity, there is a unique local-in-time solution for arbitrary finite-energy Goursat datum, for all $n \geqslant 3$, establishing evolution from the given lightcone to any sufficiently close lightcone. These results show the existence of wave operators in the sense of scattering theory, and their continuity in the (Einstein) energy metric, for positive-energy equations of the indicated type. They also permit the comprehensive reduction of scattering theory for conformally invariant wave equations in Minkowski space $\mathbf{M}_0$ to the Goursat problem in $\tilde{\mathbf{M}}$. In particular, any solution of the equation arising from a nonnegative conformally invariant biquadratic interaction Lagrangian on multicomponent scalar sections, having finite Einstein energy at any one time, is asymptotic to solutions of the corresponding multicomponent free wave equation as the Minkowski time $x_0 \to \pm\infty$. Thus given a finite-Einstein-energy solution of the equation $\Box f + \lambda f^3 = 0$ on $\mathbf{M}_0$ $(\lambda \geqslant 0)$ there exist unique solutions $f_\pm$ of the free wave equation which approach f in the Minkowski energy norm as $x_0 \to \pm\infty$, and every finite-Einstein-energy solution of the free wave equation is of the form f_+ (or f_-) for a unique solution f of the nonlinear equation. This generalizes, in part in maximality sharp form, earlier results of Strauss for this equation. © 1990 Academic Press, Inc.

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1. Introduction

During the past three decades, the Cauchy problem for nonlinear wave equations has been studied extensively, with considerable success. Abstractly, this is the initial value problem for an evolutionary equation and naturally plays a fundamental mathematical role. Physically it represents the Newtonian paradigm according to which a system is specified by giving its state throughout space at an initial time, together with the equations of motion.

The Goursat problem, in which a datum is given on the lightcone, has been involved in heuristic field theory for several decades. From a relativistic physical position, it is no less natural a problem than the Cauchy problem, and perhaps more natural, as observed by Dirac and Wigner. Group-theoretically, it also appears more natural than the Cauchy problem, for an appropriately invariant wave equation. Thus the invariance group of a maximal spacelike surface on which Cauchy data are given is at most 6-dimensional, whereas that of a lightcone is 11-dimensional. Indeed, the lightcone in the Einstein Universe $\mathbb{R} \times S^3$ can be regarded as an orbit of the scaling-extended Poincaré group $\mathbf{P}$ on this spacetime. From a hyperbolic differential equations viewpoint, however, the global Goursat problem is less transparent than the Cauchy problem. The evolution of the Goursat datum from one lightcone to another lacks the domain of dependence properties that are familiar in the Cauchy context, and even the infinitesimal generator of such evolution is nonlocal. On the other hand, in the scattering theory of conformally invariant nonlinear wave equations, the formulation of the wave and scattering operators in terms of the Goursat problem eliminates the need for taking limits, as the Minkowski time becomes infinite, and clarifies why scattering takes place.

The present work treats the global Goursat problem from these points of view, and especially in relation to scattering theory. It is more effective to work in the generalized Einstein Universe $\mathbb{R} \times S^n$, which as a conformal manifold we denote as $\tilde{\mathbf{M}}$, rather than in $(n+1)$-dimensional Minkowski space $\mathbf{M}_0$, which is covariantly embedded in $\tilde{\mathbf{M}}$. The boundary of $\mathbf{M}_0$ as thus embedded consists substantially of two lightcones $C_{\pm}$, which represent the limits of spacelike surfaces in $\mathbf{M}_0$ as the Minkowski time approaches $\pm\infty$; each of these lightcones is invariant under the action of $\mathbf{P}$. The wave operators $\Omega_{\pm}$ can be construed as maps that associate to a solution φ of a nonlinear wave equations its Goursat data $\varphi \mid C_{\pm}$. The inverses of these putative operators can correspondingly be construed as the maps from incoming or outgoing free fields to the corresponding solution of the nonlinear wave equation in question (or "interacting field").

We examine nonlinear local perturbations of free wave equations and show that if the perturbation is boundedly Lipschitzian with respect to the natural energy norm in $\tilde{\mathbf{M}}$, called the "Einstein" energy, and if global solutions to the Cauchy problem exist for given smooth data, then the $\Omega_{\pm}$ exist and are continuous in the Einstein energy topology. The inverse problem, concerning the putative $\Omega_{\pm}^{-1}$, is then essentially the Goursat problem. We show that under the boundedly Lipschitzian hypothesis, local-in-time solutions exist to the Goursat problem for arbitrary finite-energy data. In the positive-energy case, these solutions are global.

The Einstein energy is readily expressed in terms of Minkowski space, being the usual relativistic energy plus that of the transform of the solution under inversion. Its finiteness imposes only a weak condition on the decay of Cauchy data near spatial infinity. Although the Einstein energy is not invariant under conformal transformations, the conformal groups acts continuously relative to the Einstein energy topology, which appears as a particularly effective one, even for the study of wave equations purely in $\mathbf{M}_0$. Thus the wave and scattering operators established below are continuous in the Einstein energy topology, but it is doubtful that they are such in the Minkowski energy topology. The greater utility of the Einstein energy has earlier been indicated by its use in eliminating infrared singularities and treating Wick powers of quantized scalar fields.

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2. The Geometry of $\tilde{\mathbf{M}}$

The conformal geometry of the homogeneous space $\tilde{\mathbf{M}}$ and the conformally invariant wave operator on $\tilde{\mathbf{M}}$ have been treated in the literature, most closely in relation to the present work in [1, 2, 4–6]. Here and in the next section we summarize some of the relevant results and establish notation, which in general will be consistent with that of [6].

The conformal compactification $\bar{\mathbf{M}}$ of Minkowski space $\mathbf{M}_0$ may be defined as $SO(2, n+1)/SO(1, n)$, where n ($\geqslant 1$) is the space dimension. $\bar{\mathbf{M}}$ has a unique conformal structure that is preserved by the natural action of $\mathbf{G} = SO(2, n+1)$. As a conformal manifold it is equivalent to $(S^1 \times S^n)/\mathbf{Z}_2$, where $\mathbf{Z}_2$ acts by the product of the antipodal maps on S^1 and S^n, and the usual direct product Lorentzian metric is used on $S^1 \times S^n$. When $n > 1$, $\tilde{\mathbf{M}}$ denotes the universal cover of $\bar{\mathbf{M}}$, given the conformal structure lifted up from that of $\bar{\mathbf{M}}$, which is correspondingly invariant under the natural action of $\tilde{\mathbf{G}}$ on $\tilde{\mathbf{M}}$. $\tilde{\mathbf{M}}$ is conformally equivalent to the "Einstein universe" $\mathbb{R} \times S^n$. In what follows we always assume $n > 1$. We specify points of the manifold S^n as images of pairs (ρ, ω), where $\rho \in [0, \pi]$ and $\omega \in S^{n-1}$, under a function defined as follows. We regard S^{n-1} as imbedded in $\mathbb{R}^n$ as the unit sphere, so that $\omega = (u_1, ..., u_n)$, where $\Sigma_i u_i^2 = 1$. Regarding S^n as correspondingly imbedded in $\mathbb{R}^{n+1}$, we then map $(\rho, \omega) \in [0, \pi] \times S^{n-1}$ into S^n by the map

$$(\rho, \omega) \mapsto (\cos \rho, \sin \rho\omega)$$

The map is C^∞, and when restricted to $(0, \pi) \times S^{n-1}$ is a diffeomorphism onto an open subset of S^n denoted by $S^{n,*}$. We denote by ds^2 the standard Riemannian metric on S^n. The associated volume form v on S^n may be expressed on $S^{n,*}$ as $\sin^{n-1}\rho\, d\rho \wedge d\omega$, where $d\omega$ is the volume form on S^{n-1}. Also associated with the metric ds^2 on S^n is the (negative) Laplace–Beltrami operator Δ_n, which may be expressed on $S^{n,*}$ in terms of ∂_ρ and Δ_{n-1} as

$$\Delta_n = \partial_\rho^2 + (n-1) \cot \rho\, \partial_\rho + \csc^2\rho\, \Delta_{n-1}.$$

We specify points of $\mathbb{R} \times S^n$ by (τ, ρ, ω), where $\tau \in \mathbb{R}$ is called the "Einstein time," and $(\rho, \omega) \in [0, \pi] \times S^{n-1}$ specifies a point of S^n as above. For present purposes, $\tilde{\mathbf{M}}$ may be represented as the manifold $\mathbb{R} \times S^n$ with the conformal structure corresponding to the "Einstein" metric $d\tau^2 - ds^2$.

To treat the Goursat problem, we make use of the cones C_t in $\tilde{\mathbf{M}}$ defined by the equation $\tau - \rho = t$. For all $t \in \mathbb{R}$ these are compact submanifolds of $\tilde{\mathbf{M}}$ smoothly imbedded except at the two points $\{\rho = 0, \pi\}$; the subscript t will be supressed when its value is immaterial. We give C the coordinates (ρ, ω) obtained by restricting the functions (τ, ρ, ω) on $\tilde{\mathbf{M}}$, thus identifying C with S^n; this identification is a homeomorphism and yields (for each t) a map from S^n to $\tilde{\mathbf{M}}$ that is smooth on $S^{n,*}$. We use this identification to transfer to C the Riemannian metric ds^2 and volume form v on S^n.

For arbitrary $t \in \mathbb{R}$, S_t will denote the spacelike surface defined by the equation $\tau = t$; the subscript t will be suppressed when its value is immaterial. The S_t are smooth compact submanifolds of $\tilde{\mathbf{M}}$, each of which is diffeomorphic to S^n; the points of S will be specified by the functions (ρ, ω) indicated above.

We will use the notation $L_{p,q}(X)$ to denote the space of all real distributions f on a compact Riemannian manifold X that are in $L_p(X)$ together with their first q derivatives, and denote the norm in this space as $\|\cdot\|_{p,q}$. For $X = C$ we use the identification of C with S^n to define $L_{p,q}(C)$.

3. The Conformal Wave Equation

The real scalar bundle L_w of "conformal weight w" ($w \in \mathbb{R}$) is derived from the action of $\tilde{\mathbf{G}} = SO^\sim(2, n+1)$ on $\tilde{\mathbf{M}}$ as follows. If $g \in \tilde{\mathbf{G}}$ fixes a point $x \in \tilde{\mathbf{M}}$, the differential dg_x at x differs from an isometry of $T_x\tilde{\mathbf{M}}$ by a constant factor $\alpha(g) > 0$. The map $g \mapsto \alpha(g)^w$ is a one-dimensional representation of the isotropy group $\tilde{\mathbf{G}}_x$, and L_w is the bundle induced from this representation. Unless otherwise specified, "section" of this bundle means "C^∞ section."

For $w=(n-1)/2$, the section space of L_w admits a $\tilde{\mathbf{G}}$-invariant subspace whose sections are determined by their restrictions, together with that of their first time derivatives, to any S_τ. These sections satisfy various closely related second-order hyperbolic equations, any one of which may be called a (free) conformal wave equation. In an appropriate parallelization the sections may be represented by real-valued functions on $\tilde{\mathbf{M}}$, in such a way that the equation takes the form

$$(\Box + c_n)\varphi = 0,$$

where $\Box$ denotes the D'Alembertian $\partial_\tau^2 - \Delta_n$ relative to the Einstein metric, and $c_n = ((n-1)/2)^2$. Solutions of the conformal wave equation (putative as well as actual) and its inhomogeneous and nonlinear variants will be denoted by the lower-case Greek letters φ and ψ. The corresponding functions of τ whose values are functions on S will be denoted by the corresponding capital Greek letters; e.g., $\Phi(\tau) = \varphi(\tau, \cdot, \cdot)$ and $\Phi'(\tau) = \partial_\tau \varphi(\tau, \cdot, \cdot)$.

If Δ_n now denotes the usual self-adjoint realization of the Laplacian on $L_2(S)$, and B denotes the positive self-adjoint operator $B = (-\Delta + c_n)^{1/2}$, the Cauchy problem may be formulated in a first-order abstract form as the equation

$$\begin{aligned}\partial_\tau(\Phi'(\tau) \oplus \Phi'(\tau)) &= (\Phi'(\tau) \oplus -B^2\Phi(\tau)) \\ &= A(\Phi(\tau) \oplus \Phi'(\tau)),\end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -B^2 & 0 \end{pmatrix}.$$

When $n > 1$, the operator A is naturally identifiable with a skew-adjoint operator on the real Hilbert space $\mathbf{H}(S) = L_{2,1}(S) \oplus L_2(S)$. More specifically, the inner product in this space is defined by the equation

$$\langle \Phi_1 \oplus \Phi_2, \Psi_1 \oplus \Psi_2 \rangle_E = \langle B\Phi_1, B\Psi_1 \rangle + \langle \Phi_2, \Psi_2 \rangle, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L_2(S)$. The Hilbert space $\mathbf{H}(S)$ is called the "finite energy space," and the norm $\|\cdot\|_E$ is called the "energy norm." We use φ^S to denote the Cauchy datum $\Phi \oplus \Phi' \in \mathbf{H}(S)$. The operator A may be defined as the infinitesimal generator of the one-parameter orthogonal group $V(t)$ on $\mathbf{H}(S)$ of matrices

$$\begin{pmatrix} \cos tB & B^{-1}\sin tB \\ -B\sin tB & \cos tB \end{pmatrix}$$

and then has pure point spectrum bounded away from zero. The domain $D(A^n)$ with the inner product $\langle x, y\rangle_n = \langle A^n x, A^n y\rangle$ $(n = 1, 2, ...)$ will be denoted as $\mathbf{H}_n(S)$, which is thus a Hilbert space. The common part $\bigcap_n \mathbf{H}_n(S)$ of these domains, with the topology of convergence in each $\mathbf{H}_n(S)$, coincides with $C^\infty(S) \oplus C^\infty(S)$ in its usual topology and will be denoted as $\mathbf{H}_\infty(S)$.

We denote by $\mathbf{H}$ the space of solutions φ of the wave equation for which $\Phi(\tau) \oplus \Phi'(\tau) \in \mathbf{H}(S)$ for any (hence all) $\tau \in \mathbb{R}$. The inner product $\langle \cdot, \cdot \rangle_E$ is independent of τ, as noted above, and provides $\mathbf{H}$, thereby, with a Hilbert space structure naturally isomorphic to that of $\mathbf{H}(S)$ for any S. By this means, the above notations regarding objects associated with $\mathbf{H}(S)$ extend to corresponding objects associated with $\mathbf{H}$; in particular, $\mathbf{H}_n$ and $\mathbf{H}_\infty$ denote correspondents to $\mathbf{H}_n(S)$ and $\mathbf{H}_\infty(S)$.

4. The Goursat Format

We will use boldface letters to denote functions on $\tilde{\mathbf{M}}$ in terms of Goursat data, i.e., if $\varphi(\tau, \rho, \omega)$ is a function on $\tilde{\mathbf{M}}$, $\boldsymbol{\varphi}(t, \rho, \omega) = \varphi(t + \rho, \rho, \omega)$; similarly, $\mathbf{\Phi}(t)$ denotes $\boldsymbol{\varphi}(t, \cdot, \cdot)$. We first examine temporal evolution in the Goursat format for the inhomogeneous wave equation.

Proposition 1. *If φ is a C^2 solution of the equation $\Box\varphi = f$ on some neighborhood U of $C_t \subset \tilde{\mathbf{M}}$, where f is a continuous function on U, then for $\rho \neq 0, \pi$,*

$$\partial_t \boldsymbol{\varphi}(t, \rho, \omega) = (2 \sin^{(n-1)/2} \rho)^{-1} \int_0^\rho \sin^{(n-1)/2} \rho' (\Delta_n \boldsymbol{\varphi}(t, \rho', \omega) + \mathbf{f}(t, \rho', \omega))\, d\rho',$$

where $\mathbf{f}(t, \rho, \omega) = f(t + \rho, \rho, \omega)$.

Proof. Writing out the equation satisfied by φ,

$$(\partial_\tau^2 - \partial_\rho^2 - (n-1) \cot \rho\, \partial_\rho - \csc^2 \rho\, \Delta_{n-1}) \varphi = f$$

and noting that $\partial_\tau \varphi = \partial_t \mathbf{\Phi}$ and $\partial_\rho \varphi = (\partial_\rho - \partial_t)\mathbf{\Phi}$, it results that

$$(-\partial_\rho^2 + 2\partial_\rho \partial_t - (n-1) \cot \rho (\partial_\rho - \partial_t) - \csc^2 \rho\, \Delta_{n-1})\, \mathbf{\Phi} = \mathbf{f}.$$

This simplifies to

$$(2\partial_\rho + (n-1) \cot \rho)\, \partial_t \mathbf{\Phi} = \Delta_n \mathbf{\Phi} + \mathbf{f}.$$

Multiplying by the integrating factor $\frac{1}{2} \sin^{(n-1)/2} \rho$ and integrating from $\rho' = 0$ to $\rho' = \rho$, the proposition follows. ∎

Note that given the hypotheses of Proposition 1, $\partial_t \mathbf{\Phi}$ can be evaluated at $\rho = 0$ and π by continuity. For $n > 1$ the fact that $\partial_t \mathbf{\Phi}$ is finite at $\rho = \pi$ implies a certain nonlinear nonlocal constraint is satisfied by the Goursat data:

$$\int_0^\pi \sin^{(n-1)/2} \rho' (\Delta_n \boldsymbol{\varphi}(t', \rho', \omega) + \mathbf{f}(t', \rho', \omega))\, d\rho' = 0$$

for all ω and all t' for which $t - t'$ is sufficiently small. A similar constraint occurs in the case $n = 1$. Thus in the context of smooth solutions, the expression of evolution in the Goursat format for nonlinear wave equations does not take place in a linear space; the Goursat data space is nonlinear.

We next apply Proposition 1 to obtain an expression for the Einstein energy associated with a wave equation in terms of Goursat data. This will yield a description of the space $\mathbf{H}$ in the Goursat format. We first compute the differential of an n-form ε associated with a given function φ on $\tilde{\mathbf{M}}$ and a given function F on $\mathbb{R}$; physically, ε could be interpreted as a component of the energy-momentum tensor associated with φ in an appropriate context.

LEMMA 2. *Let φ be a C^2 function on an open set $U \subset \tilde{\mathbf{M}}$ and let $F: \mathbb{R} \to \mathbb{R}$ be a C^1 function. Set*

$$\varepsilon = \partial_\tau \varphi \wedge * d\varphi + (F \circ \varphi - \tfrac{1}{2} \langle d\varphi, d\varphi \rangle) v,$$

where v here denotes the volume form on S pulled back to an n-form on $\tilde{\mathbf{M}}$. Then

$$d\varepsilon = \partial_\tau \varphi \wedge [\Box \varphi + F' \circ \varphi) V,$$

where $V = d\tau \wedge v$ is the volume form on $\tilde{\mathbf{M}}$.

Proof. Let $\varphi' = \partial_\tau \varphi$, and note that $d = d\tau \wedge \partial_\tau + d_S$, where d_S is the differential on S. Then

$$\begin{aligned} d\varepsilon &= d\tau \wedge \partial_\tau(\varphi' \wedge * d\varphi) + d_S(\varphi' \wedge * d\varphi) \\ &\quad + \partial_\tau (F \circ \varphi - \tfrac{1}{2}(\varphi'^2 + \langle d_S \varphi, d_S \varphi \rangle)) V \\ &= d\tau \wedge (\varphi'' * d\varphi + \varphi' \wedge * d\varphi') + \langle d_S \varphi', d\varphi) V \\ &\quad + \varphi' \wedge d_S * d\varphi + (\varphi'(F' \circ \varphi - \varphi'') - \langle d_S \varphi', d_S \varphi \rangle) V. \end{aligned}$$

Now noting that $\langle d_S \varphi', d_S \phi \rangle = \langle d_S \varphi', d\varphi \rangle$ and $d_S * d\varphi = -\Delta_n \varphi V$ (where Δ_n is the Laplacian on S), it follows that

$$d\varepsilon = (\varphi'' \varphi' + \varphi' \varphi'') V - \varphi' \Delta_n \varphi V + \varphi'(F' \circ \varphi - \varphi'') V$$

or, finally,

$$d\varepsilon = \varphi'(\Box\varphi + F' \circ \varphi) V. \quad \blacksquare$$

PROPOSITION 3. *Let φ be a C^2 function on the contractible open set $U \subset \tilde{\mathbf{M}}$. Let ε be the n-form on U given by*

$$\varepsilon = \partial_\tau \varphi \wedge * d\varphi + (F \circ \varphi - \tfrac{1}{2}\langle d\varphi, d\varphi \rangle) v.$$

Then for all $S_\tau \subset U$,

$$\int_{S_\tau} \varepsilon = \int_{S_\tau} \tfrac{1}{2}(-\Phi\, \Delta_n \Phi + \Phi'^2) + F \circ \Phi$$

and for any $C_t \subset U$,

$$\int_{C_t} \varepsilon = \int_{C_t} -\tfrac{1}{2}\mathbf{\Phi}\, \Delta_n \mathbf{\Phi} + F \circ \mathbf{\Phi}.$$

If in addition φ satisfies the equation $\Box\varphi + F' \circ \varphi = 0$ in U, where F is a C^1 function from $\mathbb{R}$ to $\mathbb{R}$, there is a constant E such that

$$\int_{S_\tau} \varepsilon = \int_{C_t} \varepsilon = E$$

for any S_τ or C_t lying in U.

Proof. By Lemma 2, ε is closed. Thus integrating ε over any S_τ or C_t lying in U gives the same constant E, as these n-submanifolds of $\tilde{\mathbf{M}}$ are all homotopic. Note that as C_t is not a smooth submanifold of $\tilde{\mathbf{M}}$ at the points $\{\rho = 0, \pi\}$, ε does not restrict to a well-defined form on C_t at these points. But these two points are only conical singularities so the integral of ε over C_t is still well defined. One easily calculates that

$$\int_{S_\tau} \varepsilon = \int_{S_\tau} (\tfrac{1}{2}(-\Phi\, \Delta_n \Phi + \Phi'^2) + F \circ \Phi),$$

where the integral on the right is taken with respect to the standard measure on S_τ. Thus to complete the proof it suffices to show that

$$\int_{C_t} \varepsilon = \int_{C_t} -\tfrac{1}{2}\mathbf{\Phi}\, \Delta_n \mathbf{\Phi} + F \circ \mathbf{\Phi}.$$

Strictly speaking, we need to compute the integral of $i^*\varepsilon$ over C_t, where $i: C_t \to \tilde{\mathbf{M}}$ is the inclusion map and the integral on the right is taken with respect to the standard measure on C_t. Again, while $i^*\varepsilon$ is not defined at $\{\rho=0, \pi\}$, this will cause no problems in the computation. For some μ,

$$*d\varphi = (\partial_\tau \varphi \, d\rho + \delta_\rho \varphi \, d\tau) \wedge \sin^{n-1}\rho \, d\omega + \mu \wedge d\tau \wedge d\rho$$

and $i^* \, d\tau = i^* \, d\rho$, so

$$i^*(*d\varphi) = (\partial_\tau + \partial_\rho)\varphi \sin^{n-1}\rho i^* \, d\rho \wedge d\omega.$$

Thus

$$\begin{aligned} i^*\varepsilon &= i^*(\partial_\tau \varphi \wedge *d\varphi + (F\circ\varphi - \tfrac{1}{2}\|d\varphi\|^2) \wedge v) \\ &= \partial_\tau \varphi(\partial_\tau + \partial_\rho)\varphi \sin^{n-1}\rho \, i^*dq \wedge d\omega + (F\circ\varphi - \tfrac{1}{2}\|d\varphi\|^2) \wedge i^*v. \end{aligned}$$

Note that $\sin^{n-1}\rho i^* \, d\rho \wedge d\omega$ and i^*v are both equal to the standard volume form on C_t, so

$$\int i^*\varepsilon = \int \partial_\tau \varphi(\partial_\tau + \partial_\rho)\varphi + (F\circ\varphi\tfrac{1}{2}\langle d\varphi, d\varphi\rangle).$$

Since $\partial_\tau \varphi = \partial_t \mathbf{\Phi}$, $(\partial_\tau + \partial_\rho)\varphi = \partial_\rho \mathbf{\Phi}$, and

$$\begin{aligned} \langle d\varphi, d\varphi\rangle &= (\partial_\tau \varphi)^2 - (\partial_\rho \varphi)^2 - \langle d_\omega \varphi, d_\omega \varphi\rangle \\ &= 2\partial_t \mathbf{\Phi} \, \partial_\rho \mathbf{\Phi} - (\partial_\rho \mathbf{\Phi})^2 - \langle d_\omega \mathbf{\Phi}, d_\omega \mathbf{\Phi}\rangle, \end{aligned}$$

where on S^n we have $df = \partial_\rho f \, d\rho + d_\omega f$, we have

$$\begin{aligned} \int i^*\varepsilon &= \int \frac{1}{2}(\partial_\rho \mathbf{\Phi})^2 + \tfrac{1}{2}\langle d_\omega \mathbf{\Phi}, \partial_\omega \mathbf{\Phi}\rangle + F\circ\mathbf{\Phi} \\ &= \int -\tfrac{1}{2}\mathbf{\Phi} \, \Delta_\eta \mathbf{\Phi} + F\circ\mathbf{\Phi}. \quad \blacksquare \end{aligned}$$

PROPOSITION 4. *If* $\varphi, \psi \in \mathbf{H}_\infty$, *then for any* $t \in \mathbb{R}$,

$$\langle \varphi, \psi\rangle_E = \int_{C_t} \mathbf{\Phi}(-\Delta_n + c_n)\,\mathbf{\Psi}. \tag{2}$$

Proof. If $\varphi \in \mathbf{H}_\infty$ then φ is a C^∞ solution of $\square\varphi + F'\circ\varphi = 0$ on $\tilde{\mathbf{M}}$, where $F(x) = \frac{1}{2}c_n x^2$. With this choice of F, Proposition 3 shows that

$$\langle \varphi, \varphi\rangle_E = \int_{C_t} \mathbf{\Phi}(-\Delta_n + c_n)\,\mathbf{\Phi},$$

taking note of the expression (1) for $\langle \cdot, \cdot\rangle_E$ in terms of Cauchy data. The conclusion then follows by polarization. $\blacksquare$

PROPOSITION 5. *The map $\varphi \mapsto \mathbf{\Phi}$ is one-to-one from $\mathbf{H}_\infty$ to $L_{2,1}(C)$, and is isometric with respect to the inner product in $\mathbf{H}$ and the inner product in $L_{2,1}(C)$ defined by the right side of* (2).

Proof. This is an immediate consequence of Proposition 4. ∎

We now introduce additional notation. We will use U to denote the unique isometry of $\mathbf{H}$ into $L_{2,1}(C)$ specified by Proposition 5. The range of U will be denoted by $\mathbf{H}(C)$. The inner product on $\mathbf{H}(C)$ defined by the right side of (2) will be denoted by $\langle \cdot, \cdot \rangle_E$, and similarly for the norm. We denote $U\mathbf{H}_\infty$ by $\mathbf{H}_\infty(C)$.

PROPOSITION 6. *If $\mathbf{\Phi} \in \mathbf{H}_\infty(C)$ then $\partial_t \mathbf{\Phi} = L_0 \mathbf{\Phi} \in \mathbf{H}_\infty(C)$, where L_0 denotes the operator on $\mathbf{H}_\infty(C)$:*

$$L_0 \mathbf{\Phi} = (2 \sin^{(n-1)/2} \rho)^{-1} \int_0^\rho \sin^{(n-1)/2} \rho' (\Delta_n - c_n)\, \mathbf{\Phi}\, d\rho'$$

(*not defined at $\rho = 0$ and π; cf. the remark following Proposition* 1).

Proof. This follows from Proposition 1, noting that $\mathbf{H}_\infty$ lies in $C^2(\tilde{\mathbf{M}})$ and is invariant under ∂_t. ∎

PROPOSITION 7. *L_0 is an essentially skew-adjoint as an operator on $\mathbf{H}(C)$, and its closure L equals UAU^{-1}.*

Proof. Since U is orthogonal from $\mathbf{H}$ onto $\mathbf{H}(C)$, and since $A_0 = A \mid \mathbf{H}_\infty$ is essentially skew-adjoint, $L_0 = UA_0U^{-1}$ is also essentially skew-adjoint. Since closure is an invariant operation, $L = UAU^{-1}$. ∎

5. THE SPACE OF FINITE-ENERGY GOURSAT DATA

We begin this section by describing the conformal embedding of Minkowski space, $\mathbf{M}_0$, in $\tilde{\mathbf{M}}$ (treated in [6] for the case $n=3$). This embedding gives rise to a representation on $L_{2,1}(C)$ of the group $\mathbf{P}$ of conformal diffeomorphisms of $\mathbf{M}_0$. We then use irreducibility considerations involving this representation to prove that $\mathbf{H}(C) = L_{2,1}(C)$ if $n \geqslant 3$.

Let $\mathbf{M}_0$, "Minkowski space," denote $\mathbb{R} \times \mathbb{R}^n$ with the coordinates $(x_0, \mathbf{x})$, where $\mathbf{x} = (x_1, ..., x_n)$, and the "Minkowski" metric $dx_0^2 - \mathbf{dx}^2$, where $d\mathbf{x}^2 = dx_1^2 + \cdots + dx_n^2$. The coordinate x_0 is called the "Minkowski time." Let $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, and define $\theta \in S^{n-1} \subset \mathbb{R}^n$ by $\mathbf{x} = r\theta$ for $\mathbf{x} \neq 0$. There is a conformal embedding $\iota\colon \mathbf{M}_0 \to \tilde{\mathbf{M}}$ given by

$$\sin \tau|_{\iota(x)} = px_0$$

$$\sin \rho|_{\iota(x)} = pr$$

$$\omega|_{\iota(x)} = \theta,$$

where

$$p = ((1 - (x_0^2 - r^2)/4)^2 + x_0^2)^{-1/2}. \tag{3}$$

We identify $\mathbf{M}_0$ with its image under $\imath$ in $\tilde{\mathbf{M}}$,

$$\imath(\mathbf{M}_0) = \{|\tau| + |\rho| < \pi\} \subset \tilde{\mathbf{M}}$$

and extend p to a smooth function on $\tilde{\mathbf{M}}$ by the equation $p = \frac{1}{2}(\cos\tau + \cos\rho)$. The function p is the conformal factor relating the Einstein and Minkowski metrics, i.e., on $\mathbf{M}_0$,

$$dx_0^2 - d\mathbf{x}^2 = p^{-2}(d\tau^2 - ds^2).$$

We let $\mathbf{P}$ denote the group of conformal diffeomorphisms of $\mathbf{M}_0$, or "scale-extended Poincaré group." The action of $\mathbf{P}$ on $\mathbf{M}_0$ extends uniquely to a conformal action on $\tilde{\mathbf{M}}$, and its Lie algebra is thus identified with a subalgebra of the Lie algebra of the group $\tilde{\mathbf{G}}$. In particular, the vector fields generating scale transformations,

$$S = \sum x_i \,\partial_{x_i},$$

and Minkowski time translations,

$$T_0 = \partial_{x_0},$$

correspond to elements $\mathbf{S}$, $\mathbf{T}_0$ of the Lie algebra of $\tilde{\mathbf{G}}$ and extend to vector fields on $\tilde{\mathbf{M}}$, which we again denote by S and T_0. (The use of S in this section to denote the generator of scale transformations should not be confused with the use of S to denote a spacelike surface in $\tilde{\mathbf{M}}$.) As vector fields on $\tilde{\mathbf{M}}$

$$S = \sin\tau\cos\rho\,\partial_\tau + \cos\tau\sin\rho\,\partial_\rho$$

$$T_0 = \tfrac{1}{2}((1 + \cos\tau\cos\rho)\,\partial_\tau - \sin\tau\sin\rho\,\partial_\rho).$$

Letting U denote the action of $\tilde{\mathbf{G}}$ on sections of L_w, we have

$$dU(\mathbf{S})\varphi = (-S - w\cos\tau\cos\rho)\varphi$$

$$dU(\boldsymbol{T}_0)\varphi = (-T_0 + \tfrac{1}{2}w\sin\tau\cos\rho)\varphi.$$

The boundary of $\mathbf{M}_0$ in $\tilde{\mathbf{M}}$ is the union of two cones, the cone at past infinity, C_-, and the cone at future infinity, C_+, given by

$$C_\pm = \{\pm\tau = \pi - \rho\}.$$

For the rest of this section we take $C = C_-$, and denote $C - \{\rho = 0, \pi\}$ by C^*. We define s to be $-2\cot\rho$; as ρ ranges from 0 to π the variable s is strictly increasing and ranges from $-\infty$ to ∞. Thus there is a diffeomorphism

$$(s, \omega): C^* \to \mathbb{R} \times S^{n-1}$$

by which we identify C^* with $\mathbb{R} \times S^{n-1}$. The group $\mathbf{P}$ preserves C^*, and the vector fields corresponding to Lie algebra elements of $\mathbf{P}$ are tangent to C^*. In particular, as vector fields on C^*, $S = s\,\partial_S$ and $T_0 = \partial_S$.

We also find it convenient to introduce a new trivialization of L_w over C^*, given by

$$\varphi^G(s, \omega) = \sin^w \rho \boldsymbol{\Phi}(\rho, \omega).$$

We reformulate the space $L_{2,1}(C)$ in terms of this new trivialization.

LEMMA 7. *Let $L(C^*)$ be the Hilbert space completion of $C_0^\infty(C^*)$ relative to the inner product $\langle \cdot, \cdot \rangle_{C^*}$ given by*

$$\langle f, g \rangle_{C^*} = \int_{\mathbb{R} \times S} n - 1\{(s^2 + 4)(\partial_s f)(\partial_s g) + fg + \nabla_\omega f \cdot \nabla_\omega g\}\, ds\, d\omega.$$

Given $\varphi^G \in C_0^\infty(C^*)$, define $\boldsymbol{\Phi} \in C^\infty(C)$ by

$$\varphi^G(s, \omega) = \sin^{(n-1)/2} \rho \boldsymbol{\Phi}(\rho, \omega),$$

where $s = -2\cot\rho$. Then the mapping $\varphi^G \mapsto \boldsymbol{\Phi}$ extends to a bounded operator from $L(C^*)$ to $L_{2,1}(C)$ with bounded inverse on its range. If $n \geqslant 3$, its range is all of $L_{2,1}(C)$.

Proof. Suppose $\varphi^G \in C_0^\infty(C^*)$ and $\boldsymbol{\Phi}$ is defined as above. Then

$$\|\boldsymbol{\Phi}\|_E^2 = \int \{(\partial_\rho \boldsymbol{\Phi})^2 + \csc^2\rho\, \|\nabla_\omega \boldsymbol{\Phi}\|^2 + c_n \boldsymbol{\Phi}^2\} \sin^{n-1}\rho\, d\rho\, d\omega.$$

Writing $f(\rho, \omega) = \sin^{(n-1)/2} \rho \boldsymbol{\Phi}(\rho, \omega)$,

$$\begin{aligned}
&\{\partial_\rho \boldsymbol{\Phi})^2 + \csc^2\rho\, \|\nabla_\omega \boldsymbol{\Phi}\|^2 + c_n \boldsymbol{\Phi}^2\} \sin^{n-1}\rho \\
&\quad = (\partial_\rho f)^2 + (1-n)\cot\rho\, \partial_\rho f + \tfrac{1}{4}(1-n)^2 \cot^2 \rho f^2 + \csc^2\rho \|\nabla_\omega f\|^2 \\
&\quad = (\partial_\rho f)^2 + \tfrac{1}{2}(1-n)\, \partial_\rho(\cot \rho f^2) + \{\tfrac{1}{2}(n-1)\csc^2\rho \\
&\qquad + \tfrac{1}{4}(1-n)^2 \cot^2\rho + c_n)\, f^2 + \csc^2\rho\, \|\nabla_\omega f\|^2.
\end{aligned}$$

Thus the $\|\cdot\|_E$ norm of $\mathbf{\Phi}$ is equivalent to the norm whose square is

$$\int \{(\partial_\rho f)^2 + \csc^2 \rho f^2 + \csc^2\rho \, \|\nabla_\omega f\|^2\} \, d\rho \, d\omega.$$

Using the fact that $\partial_\rho s = 2 \csc^2 \rho$, this equals

$$\tfrac{1}{2} \int \{(s^2+4)(\partial_s \varphi^G)^2 + (\varphi^G)^2 + \|\nabla_\omega \varphi^G\|^2\} \, ds \, d\omega.$$

The equivalence of the norms implies the mapping $\varphi^G \mapsto \mathbf{\Phi}$ extends to a bounded operator from $L(C^*)$ to $L_{2,1}(C)$ with bounded inverse on its range. If $n \geqslant 3$, an elementary cutoff argument shows that $C_0^\infty(C - \{\rho = 0, \pi\})$ is dense in $L_{2,1}(C)$, so the range of the above operator is all of $L_{2,1}(C)$. ∎

We have the following actions of $\mathbf{S}$ and $\mathbf{T}_0$ on sections of L_w:

$$(dU(\mathbf{S})\varphi)^G = -s \, \partial_s \varphi^G$$
$$(dU(\mathbf{T}_0)\varphi)^G = -\partial_s \varphi^G.$$

The group $\mathbf{P}$ also has $SO(n)$ as a subgroup, where $g \in SO(n)$ acts on $\tilde{\mathbf{M}}$ by $g\colon (\tau, \rho, \omega) \mapsto (\tau, \rho, g\omega)$. If φ is a section of L_w and $g \in SO(n)$, then

$$(U(g)\varphi)^G(s, \omega) = \varphi^G(s, g^{-1}\omega).$$

Let $\mathbf{Q}$ denote the subgroup of $\mathbf{P}$ generated by this subgroup $SO(n)$ and the elements $\exp(t\mathbf{T}_0)$, $\exp(t\mathbf{S})$. As a group acting on $C^* = \mathbb{R} \times S^{n-1}$, $\mathbf{Q}$ is the direct product of the group of transformations $\{s \mapsto as + b \colon a > 0\}$ with $SO(n)$.

We now make use of the action of $\mathbf{Q}$ on sections of $L_{(n-1)/2}$ to prove that $\mathbf{H}(C) = L_{2,1}(C)$ if $n \geqslant 3$.

THEOREM 8.[1] *If* $n \geqslant 3$, $\mathbf{H}(C) = L_{2,1}(C)$.

Proof. The space C^*, and the two points of C with $\rho = 0$, π are invariant under $\mathbf{P}$. Observe that the action of $\mathbf{Q}$ on C^* extends uniquely to a C^∞ action on C. This is evident in the case of the $SO(n)$ subgroup of $\mathbf{Q}$. In the case of scaling and Minkowski time translations, this follows by the relation to stereographic projection. Accordingly all Sobolev spaces on C are invariant and acted on continuously by $\mathbf{Q}$. In particular, the action α of $\mathbf{Q}$ on $L_{2,1}(C)$ is well defined and continuous. On any smooth section of $L_{(n-1)/2}$, the action U of $\mathbf{Q}$, when restricted to C, coincides with this action α. Accordingly, the closed subspace $\mathbf{H}(C) \subseteq L_{2,1}(C)$ is invariant under α.

By the equivalence of $L_{2,1}(C)$ with $L(C^*)$ given in Lemma 7, $\mathbf{Q}$ has a

[1] A more general result has been shown by L. Hörmander (1990) by differential equation arguments.

strongly continuous representation on $L(C^*)$, which we denote as β. We identify $\mathbf{H}(C)$ with a closed $\mathbf{Q}$-invariant subspace of $L(C^*)$ which we denote by $\mathbf{H}(C^*)$. Let $P_j: L(C^*) \to L(C^*)$ be the self-adjoint projection onto the jth eigenspace of Δ_ω, realized as a self-adjoint operator on $L(C^*)$. The operators P_j commute with the action of $\mathbf{Q}$ on $L(C^*)$. We have

$$\mathbf{H}(C^*) = \bigoplus_j P_j \mathbf{H}(C^*),$$

where the summands are $\mathbf{Q}$-invariant closed subspaces of $L(C^*)$. We claim that for each j, $P_j\mathbf{H}(C^*)$ equals either 0 or $P_jL(C^*)$.

By Lemma 7, if $f \in P_jL(C^*)$ is compactly supported and C^∞ then

$$\|f\|_{C^*}^2 = \int \{(s^2+4)(\partial_s f)^2 + (1-\lambda_j) f^2\}\, ds\, d\omega, \tag{4}$$

where λ_j is the jth eigenvalue of Δ_ω. Let V_j be the jth eigenspace of Δ_ω, realized as a self-adjoint operator on $L_2(S^{n-1})$. By Lemma 7, $P_jL(C^*)$ is the completion of $C_0^\infty(\mathbb{R}, V_j)$, the space of compactly supported C^∞ V_j-valued functions on $\mathbb{R}$, with respect to the norm given by (4).

This implies that if $f \in P_jL(C^*)$ then f lies in $L_2(\mathbb{R}, V_j)$. Thus the Fourier transform of f in the s variable, $f^\wedge$, exists as an element of $L_2(\mathbb{R}, V_j)$. Let μ be a finite signed Borel measure on $\mathbf{Q}$. Since β is a strongly continuous representation of $\mathbf{Q}$ on $P_j\mathbf{H}(C^*)$, for any $f \in P_j\mathbf{H}(C)$ we have

$$\beta(\mu) f = \int_{\mathbf{Q}} \beta(g) f\, d\mu(g) \in P_j\mathbf{H}(C^*). \tag{5}$$

The formulas for the action of $\mathbf{S}$, $\mathbf{T}_0$, and $SO(n)$ on $L(C^*)$ imply that

$$(\beta(\mu) f)^\wedge(\sigma) = (2\pi)^{-1/2} \int_{\mathbb{R}^+ \times \mathbb{R} \times SO(n)} a e^{-ib\sigma} g f^\wedge(a\sigma)\, d\mu(a, b, g).$$

Here $gf^\wedge(a\sigma)$ stands for the result of acting on $f^\wedge(a\sigma) \in V_j$ by $g \in SO(n)$.

We may choose μ so that its Fourier transform in the b variable is any compactly supported C^∞ function on $\mathbb{R}^2 \times SO(n)$, say $h(a, \sigma, g)$, that satisfies $h(a, -\sigma, g) = h(a, \sigma, g)^*$. (This last constraint is necessary since μ is real.) Then we have

$$(\beta(\mu) f)^\wedge(\sigma) = \int_{\mathbb{R}^+ \times SO(n)} a h(a, \sigma, g)\, g f^\wedge(a\sigma)\, da\, dg.$$

Now suppose there is a nonzero vector $f_0 \in P_j\mathbf{H}(C^*)$. With an appropriate choice of h we obtain $\beta(\mu) f_0 = f \in P_j\mathbf{H}(C^*)$ such that $f^\wedge$ is a nonzero element of $C^\infty(\mathbb{R}, V_j)$ compactly supported on $\mathbb{R} - \{0\}$. Write $f = \sum f_k e_k$, where $\{e_k\}$ is a basis for V_j and $f_k: \mathbb{R} \to \mathbb{R}$. Each $(f_k)^\wedge$ lies in

$C^\infty(\mathbb{R})$ and is compactly supported on $\mathbb{R}-\{0\}$. Choose i such that $f_i \neq 0$. Since V_j is an irreducible representation of $SO(n)$, the projection $\sum f_k e_k \mapsto f_i e_i$ is of the form $\beta(\mu)$ for some choice of μ in (5). Thus $f_i e_i \in P_j \mathbf{H}(C^*)$.

By (5), if $h \in C_0^\infty(\mathbb{R}^2)$ satisfies $h(a, -\sigma) = h(\alpha, \sigma)^*$ and we define f by

$$f^\wedge(\sigma) = \int ah(a, \sigma)(f_i)^\wedge(a\sigma)\, da,$$

then $fe_i \in P_j\mathbf{H}(C^*)$. By an appropriate choice of h this implies that for any $f^\wedge \in C^\infty(\mathbb{R})$ compactly supported on $\mathbb{R}-\{0\}$, $fe_i \in P_j\mathbf{H}(C^*)$. Acting on such functions fe_i by elements of $SO(n) \subseteq \mathbf{Q}$ and summing, we see that any function in $C^\infty(\mathbb{R}, V_j)$ compactly supported on $\mathbb{R}-\{0\}$ is the Fourier transform of an element of $P_j\mathbf{H}(C^*)$.

We claim that such elements are dense in $P_jL(C^*)$; this will imply that $P_j\mathbf{H}(C^*) = P_jL(C^*)$. By Lemma 7 it suffices to show that for any $g \in C_0^\infty(\mathbb{R}, V_j)$ there are functions g_n with $(g_n)^\wedge \in C_0^\infty(\mathbb{R}-\{0\}, V_j)$ and

$$\|g - g_n\|_{C^*} \to 0$$

Let

$$(g_n)^\wedge(\sigma, \omega) = (1 - \psi(\sigma))\, g^\wedge(\sigma, \omega),$$

where $\psi_n \in C^\infty(\mathbb{R})$ has $\psi_n(\sigma) = 1$ if $|\sigma| \leqslant n^{-1}$ or $|\sigma| \geqslant 2n$, $\psi_n(\sigma) = 0$ if $2n^{-1} \leqslant |\sigma| \leqslant n$, and $|\partial_\sigma \psi_n| \leqslant n$. A calculation using (5) shows that

$$\|f\|_{C^*} \leqslant 2 \int \{\sigma^2 |\partial_\sigma f^\wedge|^2 + (4\sigma^2 + 2 - \lambda_j)|f^\wedge|^2\}\, d\sigma.$$

Thus we have

$$\|g - g_n\|_{C^*} \leqslant 2 \int \{\sigma^2 |\partial_\sigma(\psi_n g^\wedge)|^2 + (4\sigma^2 + 2 - \lambda_j)|(\psi_n g^\wedge)|^2\}\, d\sigma.$$

The second term approaches zero as $n \to \infty$, since $g^\wedge$ is rapidly decreasing; the first is less than or equal to

$$4 \int \sigma^2 |\psi_n' g^\wedge|^2 + \psi_n^2 |(g^\wedge)'|^2.$$

Here the second term approaches zero since $(g^\wedge)'$ is rapidly decreasing. Regarding the first, note that $|\psi_n'| \leqslant n$ and ψ_n' is supported on $\{|\sigma| \leqslant 2n^{-1}\} \cup \{|\sigma| \geqslant n\}$, so

$$\int \sigma^2 |\psi_n' g^\wedge|^2 \leqslant \int_{|\sigma| \leqslant 2/n} |g^\wedge|^2 + \int_{|\sigma| \geqslant n} n^2 |(g^\wedge)'|^2.$$

The right side approaches zero as $n \to \infty$, so $\|g - g_n\|_{C^*} \to 0$.

It suffices now to show that $P_j\mathbf{H}(C^*)$ is nonzero for all j. The map from $\mathbf{H}(S)$ to $\mathbf{H}(C^*)$ given by $\varphi^S \mapsto \varphi^G$, where $\varphi \in \mathbf{H}$, intertwines the action of $SO(n) \subseteq \tilde{\mathbf{G}}$ on these spaces. Taking elements $\Phi \oplus 0 \in \mathbf{H}(S)$, where $\Phi: S^n \to \mathbb{R}$ is a spherical harmonic, we obtain nonzero elements φ^G in each of the $P_j\mathbf{H}(C^*)$. ∎

This characterization of $\mathbf{H}(C)$ should be compared with the translation representation of solutions of the wave equation on $\mathbf{M}_0$ as described by Lax and Phillips in the case of $n \geqslant 3$ odd. In particular, note Theorem IV.2.4, [3], which relates the translation representation of a sufficiently regular solution f of the wave equation on $\mathbf{M}_0$ to limits along rays of the form $(x_0, (x_0 - s)\theta)$. Considering $\mathbf{M}_0$ as embedded in $\tilde{\mathbf{M}}$, the points along such a ray converge to a point in C^* as $x_0 \to -\infty$.[2]

6. The Map from Cauchy Data to Goursat Data

For nonlinear or inhomogeneous wave equations which extend from $\mathbf{M}_0$ to $\tilde{\mathbf{M}}$ the wave operator can be construed as a map from Cauchy data to Goursat data [8]. Namely, the wave operator $\Omega_\pm$ maps Cauchy data on S_0 to Goursat data on the cone

$$C_\pm = \{\pm\tau = \pi - \rho\}.$$

Here we prove the continuity of these maps with respect to energy norms for certain cases of the nonlinear equation

$$(\Box + c_n)\varphi + F' \circ \varphi = 0.$$

For convenience we deal below only with Ω_-, because C_- is one of our standard cones C_t, namely $C_{-\pi}$. The cone C_+ is the image of C_- under the isometry of $\tilde{\mathbf{M}}$ given by $(\tau, \rho, \omega) \mapsto (-\tau, \rho, \omega)$, which fixes S_0. Thus the theorem below extends to the case of Ω_+ by symmetry considerations.

Theorem 9. *Let $F: \mathbb{R} \to \mathbb{R}$ be a C^1 function, bounded below, such that the map $\Phi \mapsto F' \circ \Phi$ is boundedly Lipschitzian from $L_{2,1}(S)$ to $L_2(S)$, i.e., such that*

$$\|F' \circ \Phi_1 - F' \circ \Phi_2\|_2 \leqslant g(\|\Phi_1\|_{2,1}, \|\Phi_2\|_{2,1}) \|\Phi_1 - \Phi_2\|_{2,1} \tag{6}$$

[2] In work that came to our attention after this paper was completed, F. G. Friedlander (1980) treated the temporal asymptotics of linear wave equations in $\mathbf{M}_0$ using its imbedding in $\tilde{\mathbf{M}}$.

for some function $g: \mathbb{R}^2 \to \mathbb{R}$ that is bounded on bounded sets. Suppose further that for any $\varphi^S \in \mathbf{H}_\infty(S)$ there is a unique C^2 solution φ on $\tilde{\mathbf{M}}$ of the equation

$$(\square + c_n)\varphi + F' \circ \varphi = 0$$

with φ^S as its $\tau = 0$ Cauchy data.

Let $\mathbf{\Phi} = \Phi \mid C_-$. Then the map $\Omega_0 : \mathbf{H}_\infty(S) \to L_{2,1}(C)$ given by

$$\Omega_0(\varphi^S) = \mathbf{\Phi}$$

extends uniquely to a continuous map $\Omega_- : \mathbf{H}(S) \to L_{2,1}(C)$. Moreover, Ω_- is boundedly Lipschitzian.

Proof. Suppose $\varphi_i^S \in \mathbf{H}_\infty(S)$, where $i = 1, 2$. Let φ_i be the corresponding C^2 solution of $[\square + c_n)\varphi_i + f \circ \varphi_i = 0$ on $\tilde{\mathbf{M}}$. Let $\psi = \varphi_1 - \varphi_2$, $\psi^S = \varphi_1^S - \varphi_2^S$, and $\mathbf{\Psi} = (\varphi_1 - \varphi_2) \mid C_-$. Then by the boundedly Lipschitzian character of the nonlinear time evolution in $\mathbf{H}(S)$, for all $\tau \in [-\pi, 0]$ we have

$$\|\Psi(\tau)\|_{L_{2,1}(S)}, \|\partial_\tau \Psi(\tau)\|_{L_2(S)} \leqslant h(\|\varphi_1^S\|_E, \|\varphi_2^S\|_E) \|\psi^S\|_E \tag{7}$$

where $h: \mathbb{R}^2 \to \mathbb{R}$ is bounded on bounded sets [7].

Define the n-form ε on $\tilde{\mathbf{M}}$ by

$$\varepsilon = \psi' \wedge * d\psi + \tfrac{1}{2}(c_n \psi^2 - \langle d\psi, d\psi \rangle) v,$$

where $\psi' = \partial_\tau \psi$ and v is defined as in Lemma 2. As in the proof of Proposition 3,

$$\int_{\tau=0} \varepsilon = \tfrac{1}{2} \|\psi^S\|_E^2$$

and

$$\int_C \varepsilon = \tfrac{1}{2} \|\mathbf{\Psi}\|_E^2.$$

By Lemma 2,

$$\begin{aligned} d\varepsilon &= \psi'(\square + c_n)\psi V \\ &= \psi'(F' \circ \varphi_2 - F' \circ \varphi_1) V \end{aligned}$$

and if R is the region of $\tilde{\mathbf{M}}$ bounded by $\{\tau=0\}$ and C_-, we have

$$\|\psi^S\|_E^2 - \|\mathbf{\Psi}\|_E^2 = 2\int_{\partial R} \varepsilon = 2\int_R d\varepsilon$$

$$= 2\int_R \psi'(F'\circ\varphi_2 - F'\circ\varphi_1)V,$$

so

$$\|\mathbf{\Psi}\|_E^2 \leqslant \|\psi^S\|_E^2 + 2\int_{[-\pi,0]\times S^n} |\psi'(F'\circ\varphi_2 - F'\circ\varphi_1)|\, V.$$

We have

$$2\int_{[-\pi,0]\times S^n} |\psi'(F'\circ\varphi_2 - F'\circ\varphi_1)|\, V$$

$$\leqslant 2\pi \sup_{\tau\in[-\pi,0]} \|\Psi'(\tau)\|_2\, \|F'\circ\Phi_2(\tau) - F'\circ\Phi_1(\tau)\|_2$$

and by (6) and (7), the latter is less than or equal to

$$k(\|\varphi_1^S\|_E, \|\varphi_2^S\|_E)\, \|\psi^S\|_E^2$$

for some function $k\colon \mathbb{R}^2 \to \mathbb{R}$ that is bounded on bounded sets. It follows that

$$\|\mathbf{\Psi}\|_E \leqslant (k(\|\varphi_1^S\|_E, \|\varphi_2^S\|_E) + 1)^{1/2}\, \|\psi^S\|_E.$$

Thus the map $\Omega_0\colon \mathbf{H}_\infty(S) \to L_{2,1}(C)$ is boundedly Lipschitzian with respect to the energy norm on $\mathbf{H}_\infty(S)$. Since $\mathbf{H}_\infty(S)$ is dense in $\mathbf{H}(S)$, this implies that Ω_0 extends uniquely to a continuous map $\Omega_-\colon \mathbf{H}(S) \to L_{2,1}(C)$, which is also boundedly Lipschitzian. ∎

7. Existence of Solutions to the Goursat Problem

In this section we prove theorems on local and global existence of solutions of nonlinear versions of the conformal wave equation given Goursat data in the finite energy space $\mathbf{H}(C)$. We begin with a theorem about the inhomogeneous equation. We introduce the notation $D(t, t')$ for the subset of $\tilde{\mathbf{M}}$ defined by $\{\rho + t \leqslant \tau \leqslant \rho + t'\}$, where $t < t'$; this subset has as its boundary the cones C_t and $C_{t'}$.

LEMMA 10. *Let $f \in C^\infty(\tilde{\mathbf{M}})$ be a function supported in the interior of $D(t, t')$. If $\mathbf{\Phi}_0 \in \mathbf{H}_\infty(C)$, there exists a unique C^∞ function $\varphi: \tilde{\mathbf{M}} \to \mathbb{R}$ such that*

$$\begin{aligned} (\Box + c_n)\varphi &= f \\ \mathbf{\Phi}(t) &= \mathbf{\Phi}_0. \end{aligned} \tag{8}$$

The map $K_f(t, t'): \mathbf{H}_\infty(C) \to \mathbf{H}_\infty(C)$ defined by

$$K_f(t, t')\, \mathbf{\Phi}(t) = \mathbf{\Phi}(t')$$

extends to an isometric affine map from $\mathbf{H}(C)$ to itself, and $K_f(t, t')\, \mathbf{\Phi}(t)$ is continuous in f with respect to the $L_2(D(t, t'))$ topology. Moreover,

$$\|\mathbf{\Phi}(t') - e^{(t'-t)L}\mathbf{\Phi}(t)\|_E \leqslant c((t'-t)+\pi)^{1/2}\, \|f\|_2, \tag{9}$$

where c is a constant depending only on n.

Proof. Given $\mathbf{\Phi}_0 \in \mathbf{H}_\infty(C)$, there exists a solution ψ of the free equation

$$\begin{aligned} (\Box + c_n)\psi &= 0 \\ \mathbf{\Psi}(t) &= \mathbf{\Phi}_0, \end{aligned}$$

having $\psi^S(t) \in \mathbf{H}_\infty(S)$. Consider next the Cauchy problem:

$$\begin{aligned} (\Box + c_n)\varphi &= f \\ \varphi^S(0) &= \varphi^S(0). \end{aligned}$$

This has a unique global C^∞ solution, and by the domain of dependence properties of the wave equation, $\mathbf{\Phi}(t) = \mathbf{\Phi}_0$, so φ satisfies (8) and is the unique such function. Let $U: \mathbf{H}(S_{t'+\pi}) \to \mathbf{H}(C_{t'})$ and $V: \mathbf{H}(C_t) \to \mathbf{H}(S_t)$ be orthogonal operators as given by Proposition 5. Since $(\Box + c_n)\varphi = 0$ in the regions $\{\tau \leqslant \rho + t\}$ and $\{\tau \geqslant \rho + t'\}$, the integral equation form of the Cauchy problem implies

$$\begin{aligned} K_f(t, t')\, \mathbf{\Phi}(t) &= U e^{(t'+\pi-t)A} V \mathbf{\Phi}_t + U \int_t^{t'+\pi} e^{(t'+\pi-\tau)A}(0 \oplus f \mid S_\tau)\, d\tau \\ &= e^{(t'-t)L}\mathbf{\Phi}_t + U \int_t^{t'+\pi} e^{(t'+\pi-\tau)A}(0 \oplus f \mid S_\tau)\, d\tau. \end{aligned}$$

It follows immediately that $K_f(t, t')$ extends to an isometric affine map from $\mathbf{H}(C)$ to itself. Since

$$\begin{aligned} \left\| U \int_t^{t'+\pi} e^{A(t'+\pi-\tau)}(0 \oplus f \mid S_\tau)\, d\tau \right\|_E &\leqslant c \int_t^{t'+\pi} \|f \mid S_\tau\|_2\, d\tau \\ &\leqslant c((t'-t)+\pi)^{1/2}\, \|f\|_2, \end{aligned}$$

where the constant c depends only on n, (9) holds, and $K_f(t, t')\,\mathbf{\Phi}(t)$ is continuous in f with respect to the $L_2(D(t, t'))$ topology. ∎

THEOREM 11. *Let $f \in L_2(\tilde{\mathbf{M}})$ and $\mathbf{\Phi}_0 \in \mathbf{H}(C)$. Then there is a unique continuous $\mathbf{\Phi}\colon [0, \infty) \to \mathbf{H}(C)$ such that $\mathbf{\Phi}(0) = \mathbf{\Phi}_0$ and the associated function $\varphi\colon \tilde{\mathbf{M}} \to \mathbb{R}$ satisfies $(\Box + c_n)\varphi = f$ in the distributional sense. Moreover,*

$$\|\mathbf{\Phi}(t) - e^{tL}\mathbf{\Phi}(0)\|_E \leqslant c(t+\pi)^{1/2}\,\|f \,|\, D(0, t)\|_2,$$

where c is a constant depending only on n.

Proof. Given $f \in L_2(\tilde{\mathbf{M}})$ and $\mathbf{\Phi}_0 \in \mathbf{H}(C)$ we construct $\mathbf{\Phi}(t)$ as follows. Choose any sequence of $f_j \in C^\infty(\tilde{\mathbf{M}})$ supported in $D(0, t)$ and converging to $f \,|\, D(0, t)$ in $L_2(D(0, t))$, and any sequence of $\mathbf{\Psi}_j \in \mathbf{H}_\infty(C)$ converging to $\mathbf{\Phi}_0$ in $\mathbf{H}(C)$. Let

$$\mathbf{\Psi}_j(t) = K_{f_j}(0, t)\,\mathbf{\Psi}_j$$

as in Lemma 10. Then by Lemma 10 the $\mathbf{\Psi}_j(t)$ converge to some $\mathbf{\Phi}(t) \in \mathbf{H}(C)$ independent of the choice of approximating sequences, and by (9),

$$\|\mathbf{\Phi}(t) - e^{tL}\mathbf{\Phi}(0)\|_E \leqslant c(t+\pi)^{1/2}\,\|f \,|\, D(0, t)\|_2.$$

Choosing an approximating sequence of f_j as above that vanishes on C_t and $C_{t-\varepsilon}$, (9) implies that $\mathbf{\Phi}(t)$ is continuous in t.

Lemma 10 implies that the C^∞ functions ψ_j associated to the $\mathbf{\Psi}_j(t)$ satisfy

$$(\Box + c_n)\,\psi_j = f_j.$$

By (9) $\psi_j \to \varphi$ in $L_2(D(0, t))$, and by definition $f_j \to f$ in $L_2(D(0, t))$. Thus $(\Box + c_n)\varphi = f$ in the distributional sense on $D(0, t)$. Since $t > 0$ was arbitrary the theorem is proved except for the uniqueness of φ.

Any φ satisfying the conclusions of the theorem extends to a distributional solution of the Cauchy problem

$$(\Box + c_n)\varphi + \chi(D(0, \infty))f = 0$$
$$\varphi^S(0) = V\mathbf{\Phi}_0,$$

where for any set A, $\chi(A)$ denotes its characteristic function, and $V\colon \mathbf{H}(C_0) \to \mathbf{H}(S_0)$ is the orthogonal operator given by Proposition 5. Thus the uniqueness of φ follows from the uniqueness of the distributional solution to such a Cauchy problem. ∎

COROLLARY 12. *Given the hypotheses of the previous theorem, if* $t \leqslant \frac{1}{2}$ *then*

$$\|\mathbf{\Phi}(t)\|_E \leqslant \|\mathbf{\Phi}(0)\|_E + 3c\left(\int_0^t \|f \mid C_s\|_2^2\, ds\right)^{1/2}$$

with the same constant c.

Proof. Since e^{tL} is orthogonal the inequality proved in the previous theorem implies that

$$\|\mathbf{\Phi}(t)\|_E \leqslant \|\mathbf{\Phi}(0)\|_E + 2c\,\|f \mid D(0, t)\|_2$$

if $t \leqslant \frac{1}{2}$. Since the measure on C_s induced from its inclusion in $\tilde{\mathbf{M}}$ is $\sqrt{2}$ times the standard volume form v,

$$2c\,\|f \mid D(0, t)\|_2 \leqslant 3c\left(\int_0^t \|f \mid C_s\|_2^2\, ds\right)^{1/2}. \quad \blacksquare$$

The above yields a local existence and uniqueness theorem for the non-linear equation.

THEOREM 13. *Let* $F\colon \mathbb{R} \to \mathbb{R}$ *be a* C^1 *function such that the map* $\mathbf{\Phi} \mapsto F' \circ \mathbf{\Phi}$ *is boundedly Lipschitzian from* $L_{2,1}(S)$ *to* $L_2(S)$. *Let* $\mathbf{\Phi}_0 \in \mathbf{H}(C)$. *Then for some* $t_0 > 0$ *depending only on F and* $\|\mathbf{\Phi}_0\|_E$ *there exists a unique continuous function* $\mathbf{\Phi}\colon [0, t_0] \to \mathbf{H}(C)$ *such that* $\mathbf{\Phi}(0) = \mathbf{\Phi}_0$ *and the associated function* $\varphi\colon D(0, t_0) \to \mathbb{R}$ *satisfies*

$$(\square + c_n)\varphi + F' \circ \varphi = 0$$

in the distributional sense.

Proof. The hypothesis that $\Phi \mapsto F' \circ \Phi$ is boundedly Lipschitzian from $L_{2,1}(S)$ to $L_2(S)$ implies that there is an increasing continuous function $g\colon \mathbb{R} \to \mathbb{R}$ with $g(0) > 1$ such that

$$\|F' \circ \Phi_1 - F' \circ \Phi_2\|_2 \leqslant g(M)\,\|\Phi_1 - \Phi_2\|_{2,1}$$

if $\|\Phi_1\|_{2,1}$, $\|\Phi_2\|_{2,1} \leqslant M$.

We use Theorem 11 and an iteration procedure to prove the theorem, as follows. Choose $M > \|\mathbf{\Phi}_0\|_E + 1$, let $k = g(M)$, and choose c_0 such that $\|F' \circ \mathbf{\Psi}\|_2 \leqslant c_0$ if $\|\mathbf{\Psi}\|_{2,1} \leqslant M$. Choose a positive number $t_0 < \frac{1}{2}$ such that

$$3cc_0 t_0^{1/2} \leqslant 2^{-3} \tag{10}$$

and

$$3c(2k^2+4c_0^2)^{1/2}t_0^{1/2} \leqslant \tfrac{1}{2}, \tag{11}$$

where c is as in Theorem 11.

Let $\mathbf{\Psi}_j \in \mathbf{H}_\infty(C)$ be a sequence converging to $\mathbf{\Phi}_0$ in $\mathbf{H}(C)$ with $\|\mathbf{\Psi}_j\|_E \leqslant \|\mathbf{\Phi}_0\|_E$ and $\|\mathbf{\Psi}_{j+1}-\mathbf{\Psi}_j\|_E \leqslant 2^{-(j+1)}$. Let $D_j \in C^\infty(\tilde{\mathbf{M}})$ be a sequence of functions with $0 \leqslant D_j \leqslant 1$, $D_j = 0$ outside $D(0, t_0)$, and $D_j = 1$ in $D(2^{-2j}t_0, (1-2^{-2j})\, t_0)$. Let φ_1 satisfy

$$(\Box + c_n)\, \varphi_1 = 0$$
$$\mathbf{\Phi}_1(0) = \mathbf{\Psi}_1$$

and for $j > 1$ let φ_j satisfy

$$(\Box + c_n)\, \varphi_j + D_j F' \circ \varphi_{j-1} = 0$$
$$\mathbf{\Phi}_j(0) = \mathbf{\Psi}_j;$$

Theorem 11 assures us that this can be done in any region $D(0, t)$.

We claim that

$$\|\mathbf{\Phi}_j(t)\|_E \leqslant M$$

holds for all j and all $t \in [0, t_0]$, and that $\{\mathbf{\Phi}_j(t)\}$ is uniformly convergent in $\mathbf{H}(C)$ in the interval $[0, t_0]$. In fact, it is enough to show that

$$\|\mathbf{\Phi}_j(t) - \mathbf{\Phi}_{j-1}(t)\|_E \leqslant 2^{-j} \qquad \text{for} \quad t \in [0, t_0]. \tag{12}$$

We prove this by induction. Since $(\Box + c_n)(\varphi_2 - \varphi_1) = -D_2 F' \circ \varphi_1$, Corollary 12 implies that if $t \in [0, t_0]$ then

$$\begin{aligned}\|\mathbf{\Phi}_2(t) - \mathbf{\Phi}_1(t)\|_E &\leqslant \|\mathbf{\Psi}_2 - \mathbf{\Psi}_1\|_E + 3c\left(\int_0^t \|D_2 F' \circ \mathbf{\Phi}_1(s)\|_2^2\, ds\right)^{1/2} \\ &\leqslant 2^{-3} + 3cc_0 t_0^{1/2}.\end{aligned}$$

By (10) this implies that (12) holds for $j = 2$. If $j > 2$, Corollary 12 implies that if $t \in [0, t_0]$ then

$$\begin{aligned}&\|\mathbf{\Phi}_j(t) - \mathbf{\Phi}_{j-1}(t)\|_E \\ &\quad \leqslant 2^{-(j+1)} + 3c\left(\int_0^t \|D_j F' \circ \mathbf{\Phi}_{j-1}(s) - D_{j-1} F' \circ \mathbf{\Phi}_{j-2}(s)\|_2^2\, ds\right)^{1/2}.\end{aligned}$$

Suppose that (12) holds for $j-1$. Using the properties of the functions D_j it results that

$$\begin{aligned}
&\int_0^t \|D_j F' \circ \mathbf{\Phi}_{j-1}(s) - D_{j-1} F' \circ \mathbf{\Phi}_{j-2}(s)\|_2^2 \, ds \\
&\quad \leqslant 2 \int_0^t \|(F' \circ \mathbf{\Phi}_{j-1}(s) - F' \circ \mathbf{\Phi}_{j-2}(s))\|_2^2 + \|(D_j - D_{j-1}) \, F' \circ \mathbf{\Phi}_{j-2}(s)\|_2^2 \, ds \\
&\quad \leqslant 2 \int_0^t k^2 \|\mathbf{\Phi}_{j-1}(s) - \mathbf{\Phi}_{j-2}(s))\|_E^2 \, ds + 4 \cdot 2^{-2j} t_0 c_0^2 \\
&\quad \leqslant 2^{-2j}(2k^2 + 4c_0^2) \, t_0.
\end{aligned}$$

Thus

$$\|\mathbf{\Phi}_j(t) - \mathbf{\Phi}_{j-1}(t)\|_E \leqslant 2^{-(j+1)} + 2^{-j} \cdot 3c(2k^2 + 4c_0^2)^{1/2} t_0^{1/2}$$

and applying (11) it follows that (12) holds.

Since the $\mathbf{\Phi}_j$: $[0, t_0] \to \mathbf{H}(C)$ are continuous and uniformly convergent functions with $\mathbf{\Phi}_j(0)$ converging to $\mathbf{\Phi}_0$ in $\mathbf{H}(C)$, the limit $\mathbf{\Phi}(t)$: $[0, t_0] \to \mathbf{H}(C)$ is continuous and $\mathbf{\Phi}(0) = \mathbf{\Phi}_0$. The uniform convergence of the continuous functions $\mathbf{\Phi}_j(t)$ implies that the φ_j converges in $L_2(D(0, t_0))$ to a function φ. Since $\mathbf{\Phi} \mapsto F' \circ \mathbf{\Phi}$ is boundedly Lipschitzian from $L_{2,1}(C)$ to $L_2(C)$ the functions $F' \circ \varphi_j$ converge in $L_2(D(0, t_0))$ to $F' \circ \varphi$. The formula

$$(\Box + c_n) \, \varphi_j + F' \circ \varphi_{j-1} = 0$$

thus implies that

$$(\Box + c_n)\varphi + F' \circ \varphi = 0$$

holds in the distributional sense on $D(0, t_0)$.

It remains to show that φ is unique. Note that any φ satisfying the conclusions of the theorem extends to a (locally) finite-energy solution of the Cauchy problem

$$(\Box + c_n)\varphi + \chi(D(0, \infty))F' \circ \varphi = 0$$
$$\varphi^S(0) = V\mathbf{\Phi}_0$$

on $\{0 \leqslant \tau \leqslant t_0 + \rho\}$. The uniqueness of finite-energy solutions to such a Cauchy problem implies that φ is unique. ∎

In order to prove that global solutions exist if the function F is bounded below and sufficiently regular, we first prove a lemma extending Proposition 3.

LEMMA 14. *Let $F: \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that the map $\mathbf{\Phi} \mapsto F' \circ \mathbf{\Phi}$ is boundedly Lipschitzian from $L_{2,1}(S)$ to $L_2(S)$. Let $\mathbf{\Phi}_0 \in \mathbf{H}(C)$. Let $\varphi: D(0, t_0) \to \mathbb{R}$ be the function with $\mathbf{\Phi}(0) = \mathbf{\Phi}_0$ and*

$$(\Box + c_n)\varphi + F' \circ \varphi = 0$$

as in Theorem 15. If $t \in [0, t_0]$, then

$$\tfrac{1}{2}\|\mathbf{\Phi}(t)\|_E^2 + \int_C F \circ \mathbf{\Phi}(t) = \tfrac{1}{2}\|\mathbf{\Phi}(0)\|_E^2 + \int_C F \circ \mathbf{\Phi}(0).$$

Proof. First we note that if $\mathbf{\Phi} \mapsto F' \circ \mathbf{\Phi}$ is boundedly Lipschitzian from $L_{2,1}(S)$ to $L_2(S)$, then $\mathbf{\Phi} \mapsto F \circ \mathbf{\Phi}$ is continuous from $L_{2,1}(S)$ to $L_1(S)$, because

$$\begin{aligned}\|F \circ \mathbf{\Phi}_1 - F \circ \mathbf{\Phi}_2\|_1 &\leqslant \int_0^1 \|(\mathbf{\Phi}_2 - \mathbf{\Phi}_1)\, F' \circ (\mathbf{\Phi}_1 + t(\mathbf{\Phi}_2 - \mathbf{\Phi}_1))\|_1 \, dt \\ &\leqslant \int_0^1 \|\mathbf{\Phi}_2 - \mathbf{\Phi}_1\|_2 \, \|F' \circ (\mathbf{\Phi}_1 + t(\mathbf{\Phi}_2 - \mathbf{\Phi}_1))\|_2 \, dt \\ &\leqslant g(\|\mathbf{\Phi}_1\|, \|\mathbf{\Phi}_2\|)\, \|\mathbf{\Phi}_2 - \mathbf{\Phi}_1\|_{2,1},\end{aligned}$$

where g is bounded on bounded sets.

Next, let $\varphi_j: D(0, t_0) \to \mathbb{R}$ be the approximating functions constructed in Theorem 13. By Lemma 10 these extend to C^∞ functions on $\tilde{\mathbf{M}}$ satisfying

$$\begin{aligned}(\Box + c_n)\, \varphi_1 &= 0 \\ \mathbf{\Phi}_1(0) &= \mathbf{Y}_1\end{aligned}$$

and for $j > 1$

$$\begin{aligned}(\Box + c_n)\, \varphi_j + D_j F' \circ \varphi_{j-1} &= 0 \\ \mathbf{\Phi}_j(0) &= \mathbf{\Psi}_j.\end{aligned}$$

By Lemma 2 and Proposition 3 we have

$$\begin{aligned}&\tfrac{1}{2}\|\mathbf{\Phi}_j(t)\|_E^2 + \int_C F \circ \mathbf{\Phi}_j(t) \\ &\quad = \tfrac{1}{2}\|\mathbf{\Phi}_j(0)\|_E^2 + \int_C F \circ \mathbf{\Phi}_j(0) + \int_{D(0,t)} \partial_\tau \varphi_j (F' \circ \varphi_j - D_j F' \circ \varphi_{j-1}).\end{aligned}$$

Since the $\mathbf{\Phi}_j(s)$ converge uniformly in $\mathbf{H}(C)$ for $s \in [0, t]$ and F is continuous from $\mathbf{H}(C)$ to $L_1(C)$, it is enough to prove that

$$I_j = \int_{D(0,t)} \partial_\tau \varphi_j (F' \circ \varphi_j - D_j F' \circ \varphi_{j-1}) \to 0.$$

Note that

$$|I_j| \leqslant \|\partial_\tau \varphi_j \mid [0, t+\pi]\, xS\|_2 \, \|(F' \circ \varphi_j - D_j F' \circ \varphi_{j-1}) \mid D(0, t)\|_2.$$

Since the $\varphi_j^S(\tau)$ converge uniformly in $\mathbf{H}(S)$ to a solution of

$$(\square + c_n)\varphi + \chi(D(0, t_0))\, F' \circ \varphi = 0,$$

the norms $\|\partial_\tau \varphi_j \mid [0, t+\pi]\, xS\|_2$ are bounded uniformly in j. Since the $\mathbf{\Phi}_j(s)$ converge uniformly in $\mathbf{H}(C)$ for $s \in [0, t]$ and F' is boundedly Lipschitzian from $\mathbf{H}(C)$ to $L_2(C)$,

$$\|(F' \circ \varphi_j - D_j F' \circ \varphi_{j-1}) \mid D(0, t)\|_2 \to 0,$$

proving the lemma. ▮

THEOREM 15. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function, bounded below, such that the map $\mathbf{\Phi} \mapsto F' \circ \mathbf{\Phi}$ is boundedly Lipschitzian from $L_{2,1}(S)$ to $L_2(S)$. Let $\mathbf{\Phi}_0 \in \mathbf{H}(C)$. Then there is a unique continuous function $\mathbf{\Phi}\colon [0, \infty) \to \mathbf{H}(C)$ such that $\mathbf{\Phi}(0) = \mathbf{\Phi}_0$ and the associated function $\varphi\colon D(0, \infty) \to \mathbb{R}$ satisfies*

$$(\square + c_n)\varphi + F' \circ \varphi = 0$$

in the distributional sense.

Proof. This follows straightforwardly from Theorem 13 and Lemma 14, by the method used for the Cauchy problem [7]. ▮

The global solutions of the Goursat problem obtained above allow us to invert the wave operators $\Omega_\pm : \mathbf{H}(S) \to L_{2,1}(C)$ described in Section 6 if in fact $\mathbf{H}(C) = L_{2,1}(C)$.

THEOREM 16. *Suppose $n \geqslant 3$. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function, bounded below, such that the map $\mathbf{\Phi} \mapsto F' \circ \mathbf{\Phi}$ is boundedly Lipschitzian from $L_{2,1}(S)$ to $L_2(S)$. Suppose further that for any $\varphi^S \in \mathbf{H}_\infty(S)$ there is a unique C^2 solution φ on $\tilde{\mathbf{M}}$ of the equation*

$$(\square + c_n)\varphi + F' \circ \varphi = 0$$

with φ^S as its $\tau = 0$ Cauchy data. Then the map Ω_-; $\mathbf{H}(S) \to \mathbf{H}(C)$ given by Theorems 8 and 9 has a boundedly Lipschitzian inverse.

Proof. By Theorem 15, given $\mathbf{\Phi} \in \mathbf{H}(C)$ there is a unique continuous $\mathbf{\Phi}: [-\pi, \infty) \to \mathbf{H}(C)$ such that $\mathbf{\Phi}(0) = \mathbf{\Phi}$ and the associated function $\varphi: D(-\pi, \infty) \to \mathbb{R}$ satisfies

$$(\Box + c_n)\varphi + F' \circ \varphi = 0$$

in the distributional sense. As in the proof of Theorem 13 we note that φ extends to a global finite-energy solution of the Cauchy problem

$$\begin{aligned}(\Box + c_n)\varphi + \chi(D(0, \infty))\, F' \circ \varphi &= 0\\ \varphi^S(-\pi) &= V\mathbf{\Phi},\end{aligned}$$

where $V: \mathbf{H}(C) \to \mathbf{H}(S)$ is orthogonal. The theory of the Cauchy problem implies that the map $W: \mathbf{H}(S) \to \mathbf{H}(S)$ given by

$$W(\varphi^S(-\pi)) = \varphi^S(0)$$

is boundedly Lipschitzian. Consequently the map $WV: \mathbf{H}(C) \to \mathbf{H}(S)$ is boundedly Lipschitzian; we claim it is the inverse of $\Omega_-: \mathbf{H}(S) \to \mathbf{H}(C)$.

It suffices to show $WV\Omega_-$ is the identity on $\mathbf{H}_\infty(S)$, since $\mathbf{H}_\infty(S)$ is dense in $\mathbf{H}(S)$. Let $\varphi^S \in \mathbf{H}_\infty(S)$ and let φ be the C^2 solution on $D(-\pi, \infty)$ of the equation

$$(\Box + c_n)\varphi + F' \circ \varphi = 0$$

with φ^S as its $\tau = 0$ Cauchy data. Then by definition $\Omega_- \varphi^S = \varphi \mid C_{-\pi}$. As above, solving the Goursat problem with $\Omega_- \varphi^S$ as Goursat data, we obtain a solution $\psi: \tilde{\mathbf{M}} \to \mathbb{R}$ of

$$\begin{aligned}(\Box + c_n)\psi + \chi(D(0, \infty))\, F' \circ \psi &= 0\\ \psi^S(-\pi) &= V\Omega_- \varphi^S.\end{aligned}$$

By unit propagation speed we have $\psi = \varphi$ on $D(0, \infty)$, hence by the definition of W we have $WV\Omega_- \varphi^S = \varphi^S$. ∎

8. Relativistic Scattering

The covariance of the wave operator $(\Box + c_n)$ with respect to conformal transformations makes possible a simple transformation of equations on $\mathbf{M}_0$ into equations on $\tilde{\mathbf{M}}$ that is particularly illuminating as regards temporal asymptotics. This in turn facilitates the development of differential geometric aspects of the solution variety of nonlinear wave equations in $\mathbf{M}_0$. In this section we illustrate the foregoing results by application to the simplest non-trivial case of conformally invariant scalar equations of positive energy. These cases appear to be prototypical from certain physical standpoints and of considerable interest in their own right.

To avoid inessential complications of a notational rather than analytical character we treat the scalar equation

$$\Box_0 f + \lambda f^3 = 0 \tag{13}$$

with $\lambda \geqslant 0$ and a single unknown function $f: \mathbf{M}_0 \to \mathbb{R}$, where $n = 3$. The more general case in which f has several components, i.e., the equation

$$\Box_0 f_j + (\partial_j P) \circ f = 0; \qquad 1 \leqslant j \leqslant n,$$

where $f: \mathbf{M}_0 \to \mathbb{R}^n$ and P is non-negative homogeneous polynomial of degree four in n variables, follows by straightforward adaptation of the methods given here for the one-component case.

In particular, the coupled conformally invariant equations

$$\Box f + fg^2 + \lambda f^3 = 0$$
$$\Box g + f^2 g = 0$$

are covered: in relation to f, g here plays the role of a mass, which in suitable contexts it will approximate.

If $\varphi: \tilde{\mathbf{M}} \to \mathbb{R}$ is a solution of the equation

$$(\Box + 1)\varphi + \lambda\varphi^3 = 0$$

and $f: \mathbf{M}_0 \to \mathbb{R}$ is defined by $f = p\varphi \mid \mathbf{M}_0$, where the function p is given by Eq. (3), then f is a solution of (13). Given $f: \mathbf{M}_0 \to \mathbb{R}$, we define its Cauchy datum at time x_0 to be $(f_1 \oplus f_2)(x_0) = f(x_0. \cdot) \oplus T_0 f(x_0, \cdot)$. A sufficiently regular Cauchy datum $f_1 \oplus f_2$ determines a Cauchy datum $\Phi \oplus \Phi' \in \mathbf{H}(S)$; the explicit formulae in the case $x_0 = 0$ are

$$\begin{aligned} f_1(0, 2\tan(\rho/2)\omega) &= p\Phi(0, \rho, \omega) \\ f_2(0, 2\tan(\rho/2)\omega) &= p^2\Phi'(0, \rho, \omega). \end{aligned} \tag{14}$$

A Cauchy datum $f_1 \oplus f_2$ is said to have finite Einstein energy if $\varphi^S = \Phi \oplus \Phi'$, as determined by the above formulae, lies in $\mathbf{H}(S)$.

PROPOSITION 16. *A Cauchy datum $f_1 \oplus f_2$ has finite Einstein energy if and only if any one of the following holds*:

(1) *The solution of Eq.* (13) *with the given Cauchy datum has finite Einstein energy Cauchy data at all times and in all Lorentz frames.*

(2) *There exists a unique solution* $\varphi : \tilde{\mathbf{M}} \to \mathbb{R}$ *of the equation*

$$(\Box + 1)\varphi + \lambda\varphi^3 = 0$$

such that $\varphi^S \in \mathbf{H}(S)$ *and* $f = p\varphi \,|\, \mathbf{M}_0$, *where* p *is defined by Eq.* (3) *and* $\mathbf{M}_0$ *is identified with its image under the embedding* $\iota : \mathbf{M}_0 \to \tilde{\mathbf{M}}$.

(3) *The expression*

$$\int_{\mathbb{R}^3} \{(1 + r^2/4)((\nabla f_1)^2 + f_2^2) - \tfrac{1}{2} f_1^2\} \, d^3\mathbf{x}$$

is absolutely convergent and finite.

Proof. The only non-trivial point, apart from ones readily derivable by methods earlier indicated and the expression in Theorem 5.6(iv) of [6], used in (3), is the $\tilde{\mathbf{G}}$-invariance of the solution variety. Invariance under Einstein time evolution follows from basic differential equation existence theory [7]. Invariance under transformations in $\tilde{\mathbf{G}}$ leaving the surface S_0 fixed follows from invariance of the Sobolev spaces on S^3 under diffeomorphisms, together with the computation of the action of such on the first time derivatives. But this subgroup of $\tilde{\mathbf{G}}$ together with Einstein time evolution generates the full group, since it includes altogether both the cover of the maximal compact $\mathbf{K}$ [6], which is a maximal subgroup, and Euclidean space translations. ∎

The basic connection with relativistic scattering may be formulated as follows. The results of Strauss [9], who first treated scattering for this equation and applied conformal invariance to show the existence of temporal asymptotes at $\pm\infty$, are thereby extended and given a certain completeness.

Let B_0 denote $(-\Delta_0)^{1/2}$, Δ_0 the (negative) self-adjoint Laplacian on $L_2(\mathbb{R}^3)$, and let $\mathbf{D}(B_0)$ denote the Hilbert space completion of the domain of B_0 in $L_2(\mathbb{R}^3)$ with respect to the inner product $\langle B_0 \cdot, B_0 \cdot \rangle_2$. Let $\mathbf{H}_{\mathrm{Mink}}$ denote $\mathbf{D}(B_0) \oplus L_2(\mathbb{R}^3)$, the space of Cauchy data having finite Minkowski energy. Let $W(t)$ denote the strongly continuous one-parameter orthogonal group on $\mathbf{H}_{\mathrm{Mink}}$ describing time evolution for the free wave equation on $\mathbf{M}_0$, namely

$$W(t) = \begin{pmatrix} \cos tB_0 & B_0^{-1} \sin tB_0 \\ -B_0 \sin tB_0 & \cos tB_0 \end{pmatrix}.$$

Then:

THEOREM 17. *Let* f *be any solution of* (13) *having finite Einstein energy, and let* φ *be the solution of* $(\Box + 1)\varphi + \lambda\varphi^3 = 0$ *such that* $f = p\varphi \,|\, \mathbf{M}_0$.

Define $f_\pm$ (so-called "in" and "out" fields) by

$$f_\pm = p\varphi_\pm \,|\, \mathbf{M}_0,$$

where $\varphi_\pm$ are the solutions of the free conformal wave equation on $\tilde{\mathbf{M}}$ having the Goursat data $\varphi \,|\, C_\pm$. Let $u(x_0) = f(x_0, \cdot) \otimes T_0 f(x_0, \cdot)$ and $u_\pm(x_0) = f_\pm(x_0, \cdot) \oplus T_0 f_\pm(x_0, \cdot)$. Then the $u_\pm$ satisfy the equations

$$u_+(t) = u(t) + \int_t^\infty W(t-s)(0 \oplus \lambda f(s, \cdot)^3)\, ds$$

$$u_-(t) = u(t) - \int_{-\infty}^t W(t-s)(0 \oplus \lambda f(s, \cdot)^3)\, ds,$$

where the integrals converge absolutely in $\mathbf{H}_{\text{Mink}}$.

Proof. That the restrictions $\varphi \,|\, C_\pm$ are well-defined elements of $\mathbf{H}(C)$ follows from Theorem 9, in which these are given by $\Omega_\pm \varphi^S(0)$. The only point not covered by earlier results or methods is the convergence of the integrals, which is implied by the convergence of

$$\int_{-\infty}^\infty \|\sin(t-s)\, B_0 f(s, \cdot)^3\|_2\, ds, \qquad \int_{-\infty}^\infty \|\cos(t-s)\, B_0 f(s, \cdot)^3\|_2\, ds.$$

Since $|\sin x|$, $|\cos x| \leqslant 1$, it suffices to show that

$$\int \left(\int f(s, \mathbf{x})^6\, d^3\mathbf{x} \right)^{1/2} ds < \infty.$$

Let $M(s)$ be the submanifold of $\tilde{\mathbf{M}}$ defined by the equation $\{x_0 = s\}$, and identify $M(s)$ with $S^3 - \{\rho = \pi\}$ by means of the coordinates (ρ, ω). Then a calculation, as in [1], shows that on $M(s)$,

$$d^3\mathbf{x} = (\cos\rho + \cos\tau)(1 + \cos\rho\cos\tau)^{-1} p^{-3} \sin^2\rho\, d\rho \wedge d\omega.$$

Noting that on $\mathbf{M}_0 \subseteq \tilde{\mathbf{M}}$,

$$(\cos\rho + \cos\tau)(1 + \cos\rho\cos\tau)^{-1} \leqslant 1,$$

it is enough to show that

$$\int \left(\int_{M(s)} p^3 \varphi^6 \right)^{1/2} ds < \infty,$$

where the measure on $M(s)$ is the restriction of the standard volume form on S^3. By formula (3) defining the function p, for some $c>0$ the supremum of p on $M(s)$ is less than $c(|s|+1)^{-1}$, so it suffices to show

$$\int \left(\int_{M(s)} \varphi^6\right)^{1/2} (|s|+1)^{-3/2}\, ds < \infty.$$

In fact we will show that $\|\varphi \mid M(s)\|_{2,1}$, hence by Sobolev's inequality $\|\varphi \mid M(s)\|_6$, is bounded as a function of s.

First suppose that φ is C^2. Then Lemma 3 shows that if $i\colon M(s) \to \tilde{\mathbf{M}}$ is the inclusion map then the integral

$$E(\varphi) = \int_{M(s)} i^*(\partial_\tau \varphi \wedge *d\varphi + (\tfrac{1}{2}\varphi^2 + \tfrac{1}{4}\varphi^4 - \tfrac{1}{2}\langle d\varphi, d\varphi \rangle) v$$

is independent of s. If $M(s)$ is given by $\{\tau = h(\rho),\ \rho < \pi\}$, then

$$i^*\, d\tau = h'(\rho)\, i^*\, d\rho$$

and a calculation like that in the proof of Proposition 4 shows that

$$E(\varphi) = \int_{M(s)} \tfrac{1}{2}\{\varphi^2 - \varphi\, \Delta\varphi + (1-h'^2)(\partial_\tau \varphi)^2) + \tfrac{1}{4}\varphi^4\},$$

where Δ is the Laplacian on S^3 transferred to $M(s)$. Since $M(s)$ is a Cauchy surface, $|h'| \leqslant 1$ and

$$E(\varphi) \geqslant \|\varphi \mid M(s)\|_{2,1}^2.$$

Using the Sobolev inequality, a calculation of the Cauchy surface $\{\tau = 0\}$ shows $E(\varphi)$ is bounded by $g(\|\varphi^S(0)\|_E)$ for some continuous function g. Thus

$$\begin{aligned} \|\varphi \mid M(s)\|_{2,1}^2 &\leqslant kE(\varphi) \\ &\leqslant kg(\|\varphi^S(0)\|_E). \end{aligned}$$

Next, an arbitrary solution ψ of (17) with finite Einstein energy can be approximated by C^2 solutions to conclude that

$$\|\psi \mid M(s)\|_{2,1}^2 \leqslant kg(\|\psi^S(0)\|_E). \quad \blacksquare$$

The above theorem makes use of the wave operators $\Omega_\pm$ to describe temporal asymptotes of solutions of (13) with finite Einstein energy. The existence of solutions to the Goursat problem allows us to invert the $\Omega_\pm$, obtaining a continuous scattering operator.

THEOREM 18. *There is a boundedly Lipschitzian map* $\Sigma: \mathbf{H} \to \mathbf{H}$ *such that if* f *is a solution of* (15) *having finite Einstein energy, and* $\varphi_{\pm} \in \mathbf{H}$ *are defined as in Theorem* 17, *then* $\Sigma\varphi_{-} = \varphi_{+}$. *Moreover,* Σ *has a boundedly Lipschitzian inverse.*

Proof. As in Theorem 17 we have $\varphi_{\pm} \mid C_{\pm} = \Omega_{\pm}\varphi^{S}(0) \in \mathbf{H}$. Let $U_{\pm}$: $\mathbf{H}(S_0) \to \mathbf{H}(C_{\pm})$ be the orthogonal maps given by restricting solutions of the free wave equation with given Cauchy data at $\tau = 0$ to the cones $C_{\pm}$ (as in Proposition 5). Then we have

$$U_{\pm}\varphi_{\pm} = \Omega_{\pm}\varphi^{S}(0)$$

so by Theorem 16 the map $\Sigma = U_{+}^{-1}\Omega_{+}\Omega_{-}^{-1}U_{-}$ is well defined and boundedly Lipschitzian from $\mathbf{H}$ to itself, and $\Sigma\varphi_{-} = \varphi_{+}$. The same arguments apply to $\Sigma^{-1} = U_{-}^{-1}\Omega_{-}\Omega_{+}^{-1}U_{+}$. ∎

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