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The almost split triangles for perfect complexes over gentle algebras

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ABSTRACT

Throughout the paper k denotes a fixed field. All vector spaces and linear maps are k-vector spaces and k-linear maps, respectively. By $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$, we denote the sets of integers, nonnegative integers, and positive integers, respectively. For $i, j \in \mathbb{Z}$, $[i, j] := \{l \in \mathbb{Z} \mid i \leq l \leq j\}$ (in particular, $[i, j] = \varnothing$ if $i > j$).

0. Introduction

Given an abelian category $\mathcal{A}$ one defines its bounded derived category $\mathcal{D}^b(\mathcal{A})$ [24] (of bounded complexes of objects of $\mathcal{A}$) having a structure of a triangulated category, which is an important homological invariant of $\mathcal{A}$. In particular, given a finite dimensional algebra A one may study the bounded derived category $\mathcal{D}^b(\operatorname{mod} A)$ of the category $\operatorname{mod} A$ of finite dimensional A-modules, which one shortly calls the derived category of A and denotes $\mathcal{D}^b(A)$. Since the observation of Happel [15] (generalized by Cline et al. [11]), which states that derived category is invariant under tilting process, an importance of derived categories in the representation theory of finite dimensional algebras became clear. This observation was supported by results connecting derived categories of finite dimensional algebras with derived categories of coherent sheaves over projective schemes [6,12]. Since that time a lot of results concerning derived categories of finite dimensional algebras were obtained (see for example [1,8,9,13,20]). In particular, Rickard [21] developed the Morita theory for derived categories of finite dimensional algebras. One of the consequences is that the derived categories of two finite dimensional algebras are equivalent as triangulated categories if and only if the subcategories of perfect complexes are equivalent as triangulated categories. Recall that if A is a finite dimensional algebra, then the subcategory of $\mathcal{D}^b(A)$ formed by perfect complexes can be identified with the bounded homotopy category $\mathcal{K}^b(\operatorname{proj} A)$ of (bounded complexes of) projective A-modules.

A class of finite dimensional algebras whose derived categories attract a lot of interest is the class of gentle algebras introduced by Assem and Skowroński [4]. An important feature of this class of algebras is that it is closed under derived equivalence, i.e. if A is a gentle algebra and $\mathcal{D}^b(A)$ is equivalent as a triangulated category to $\mathcal{D}^b(B)$ for a finite dimensional algebra B, then B is also gentle [23]. Next, this class of algebras appears naturally in many classification problems. Namely, the tree gentle algebras are precisely the piecewise hereditary algebras of type $\mathbb{A}$ [2] (i.e. the algebras derived equivalent to hereditary algebras of type $\mathbb{A}$). Further, if A is a derived discrete algebra, then either A is piecewise hereditary of Dynkin type or A is a one-cycle gentle algebra which does not satisfy the clock condition [25]. Moreover, the one-cycle gentle algebras coincide with the piecewise hereditary algebras of type $\tilde{\mathbb{A}}$ [4].

If A is a gentle algebra, then it is possible to investigate $\mathcal{D}^b(A)$ by means of the stable category $\underline{\operatorname{mod}}\,\hat{A}$ of the module category $\operatorname{mod} \hat{A}$ over the repetitive algebra $\hat{A}$ [22] (which is no longer finite dimensional) and the Happel functor [16] $\mathcal{D}^b(A) \to \underline{\operatorname{mod}}\,\hat{A}$. This description is useful, since the description of the indecomposable objects in $\underline{\operatorname{mod}}\,\hat{A}$ is known. Unfortunately, a precise formula for the Happel functor seems to be not known. In [7] Bekkert and Merklen described the indecomposable objects in $\mathcal{D}^b(A)$ without using $\hat{A}$, however they did not describe how the above two descriptions are connected.

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Let A be a gentle algebra. Since gentle algebras are Gorenstein [14], it follows [17] that the almost split triangles in $\mathcal{D}^b(A)$ exist precisely for perfect complexes. The aim of this paper is to describe the almost split triangles in $\mathcal{D}^b(A)$ in terms of the above mentioned description of the indecomposable objects in $\mathcal{D}^b(A)$ due to Bekkert and Merklen. According to the above remark, this is equivalent to describing the almost split triangles in $\mathcal{K}^b(\operatorname{proj} A)$. The precise formulas are given in Section 6. The idea of the proof is to use the Happel functor and the known description of the almost split triangles in $\underline{\operatorname{mod}} \hat{A}$. As a side effect we obtain a link between the two different ways of describing the indecomposable objects in $\mathcal{D}^b(A)$.

The paper is organized as follows. In Section 1 we introduce the language of quivers and their representations, and in Section 2 we present notions of strings and bands. Next, in Section 3 we present a description of the indecomposable perfect complexes over gentle algebras due to Bekker and Merklen, while in Section 4 we collect necessary information about the repetitive algebras of gentle algebras. Finally, in Section 5 we describe the correspondence between the indecomposable perfect complexes over a gentle algebra and the indecomposable modules over its repetitive algebras, and in Section 6 we use this correspondence to describe the almost split sequences in $\mathcal{K}^b(\operatorname{proj} A)$.

For basic background on the representation theory of algebras (in particular, on the tilting theory) we refer to [3].

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1. Preliminaries on quivers and their representations

By a quiver Γ we mean a set Γ_0 of vertices and a set Γ_1 of arrows together with two maps $s, t : \Gamma_1 \to \Gamma_0$ which assign to $\alpha \in \Gamma_1$ the starting vertex $s\alpha$ and the terminating vertex $t\alpha$, respectively. We assume that all considered quivers Γ are locally finite, i.e. for each $x \in \Gamma_0$ there is only a finite number of $\alpha \in \Gamma_1$ such that either $s\alpha = x$ or $t\alpha = x$. A quiver Γ is called finite if Γ_0 (and, consequently, also Γ_1) is a finite set. For technical reasons we assume that all considered quivers Γ have no isolated vertices, i.e. there is no $x \in \Gamma_0$ such that $s\alpha \neq x \neq t\alpha$ for each $\alpha \in \Gamma_1$.

Let Γ be a quiver. If $l \in \mathbb{N}_+$, then by a path in Γ of length l we mean $\sigma = \alpha_1 \cdots \alpha_l$ such that $\alpha_i \in \Gamma_1$ for each $i \in [1, l]$ and $s\alpha_i = t\alpha_{i+1}$ for each $i \in [1, l-1]$. In the above situation we put $s\sigma := s\alpha_l$ and $t\sigma := t\alpha_1$. Moreover, we put $\alpha_i(\sigma) := \alpha_i$ for $i \in [1, l]$. Observe that each $\alpha \in \Gamma$ is a path in Γ of length 1. Moreover, for each $x \in \Gamma_0$ we introduce the path $\mathbf{1}_x$ in Γ of length 0 such that $s\mathbf{1}_x := x =: t\mathbf{1}_x$. We denote the length of a path σ in Γ by $\ell(\sigma)$. If σ' and σ'' are two paths in Γ such that $s\sigma' = t\sigma''$, then we define the composition $\sigma'\sigma''$ of σ' and σ'', which is a path in Γ of length $\ell(\sigma') + \ell(\sigma'')$, in the obvious way (in particular, $\sigma \mathbf{1}_{s\sigma} = \sigma = \mathbf{1}_{t\sigma}\sigma$ for each path σ). In order to increase clarity we sometimes write $\sigma' \cdot \sigma''$ instead of $\sigma'\sigma''$ in the above situation. If σ is a path such that $s\sigma = t\sigma$, then for $n \in \mathbb{N}_+$ we denote by σ^n the n-fold composition of σ with itself.

Let Γ be a quiver. We define the double quiver $\overline{\Gamma}$ of Γ in the following way: $\overline{\Gamma}_0 := \Gamma_0$, $\overline{\Gamma}_1 := \Gamma_1 \cup \Gamma_1^{-1}$, where $\Gamma_1^{-1} := \{\alpha^{-1} \mid \alpha \in \Gamma_1\}$, and $s\alpha^{-1} := t\alpha$ and $t\alpha^{-1} := s\alpha$ for $\alpha \in \Gamma_1$. By $\approx$ we denote the equivalence relation in $\overline{\Gamma}_1$ whose residue classes are Γ_1 and Γ_1^{-1}. We put $(\alpha^{-1})^{-1} := \alpha$ for $\alpha \in \Gamma_1$ and extend the operation $(-)^{-1}$ to the paths in $\overline{\Gamma}$ of positive length in such a way that $(\omega'\omega'')^{-1} = \omega''^{-1}\omega'^{-1}$ for all paths ω' and ω'' in $\overline{\Gamma}$ of positive length such that $s\omega' = t\omega''$. If ω is a path in $\overline{\Gamma}$ of positive length and $i \in [1, \ell(\omega)]$, then $\alpha_i^{-1}(\omega) := (\alpha_i(\omega))^{-1}$. For a set Σ of paths in $\overline{\Gamma}$ of positive length we put $\Sigma^{-1} := \{\sigma^{-1} \mid \sigma \in \Sigma\}$.

Let Γ be a quiver. We define the path category $k\Gamma$ of Γ as follows. The objects of $k\Gamma$ are the vertices of Γ. If $x', x'' \in \Gamma_0$, then the homomorphism space $k\Gamma(x', x'')$ consists of the formal k-linear combinations of paths starting at x' and terminating at x''. The composition of maps in $k\Gamma$ is induced by the composition of paths in Γ. For a set R of morphisms in $k\Gamma$ we denote by $\langle R\rangle$ the ideal in $k\Gamma$ generated by R. A morphism ϱ in Γ is called a relation if $\varrho \in \langle \Gamma_1\rangle^2$. A set R of relations in $k\Gamma$ is called admissible if there exists $n \in \mathbb{N}_+$ such that $\langle \Gamma_1\rangle^n \subset \langle R\rangle$.

By an (admissible) bound quiver we mean a pair (Γ, R) consisting of a quiver Γ and an (admissible, respectively) set of relations in $k\Gamma$. For a bound quiver $\mathbf{\Gamma} = (\Gamma, R)$ we denote by $k\mathbf{\Gamma}$ the corresponding factor category $k\Gamma/\langle R\rangle$. If $\mathbf{\Gamma} = (\Gamma, R)$ is a bound quiver and $\varrho \in k\mathbf{\Gamma}(x', x'')$ for $x', x'' \in \Gamma_0$, then we put $s\varrho := x'$ and $t\varrho := x''$. A bound quiver (Γ, R) is called monomial if R consists of paths.

Let $\mathbf{\Gamma} = (\Gamma, R)$ be a monomial bound quiver. By a path in $\mathbf{\Gamma}$ we mean a path in Γ which does not belong to $\langle R\rangle$. If $x', x'' \in \Gamma_0$, then we identify $k\mathbf{\Gamma}(x', x'')$ with the subspace of $k\Gamma(x', x'')$ spanned by the paths in $\mathbf{\Gamma}$ starting at x' and terminating at x''. A path σ in $\mathbf{\Gamma}$ is said to be maximal in $\mathbf{\Gamma}$ if there are no paths σ' and σ'' in $\mathbf{\Gamma}$ such that $s\sigma' = t\sigma$, $t\sigma'' = s\sigma'$, $\sigma'\sigma\sigma''$ is a path in $\mathbf{\Gamma}$, and $\ell(\sigma') + \ell(\sigma'') > 0$. The lack of isolated vertices in Γ implies that $\ell(\sigma) > 0$ for each maximal path σ in $\mathbf{\Gamma}$.

For the rest of the section we assume that $\mathbf{\Gamma} = (\Gamma, R)$ is an admissible bound quiver.

By a representation of $\mathbf{\Gamma}$ we mean a functor $M : k\mathbf{\Gamma} \to \operatorname{mod} k$, where $\operatorname{mod} k$ denotes the category of finite dimensional vector spaces over k, such that $M(x) \neq 0$ only for a finite number of $x \in \Gamma_0$. Observe that a representation M of $\mathbf{\Gamma}$ is uniquely determined by the collection $(M(x))_{x\in\Gamma_0}$ of vector spaces and the collection $(M(\alpha))_{\alpha\in\Gamma_1}$ of linear maps. On the other hand, a pair of such collections determines a representation of $\mathbf{\Gamma}$ if and only if the induced map $M(\varrho)$ vanishes for all $\varrho \in R$. If M and N are two representations of $\mathbf{\Gamma}$, then the morphism space $\operatorname{Hom}_{\mathbf{\Gamma}}(M, N)$ consists of the natural transformations of the corresponding functors. We denote the category of representations of $\mathbf{\Gamma}$ by $\operatorname{rep} \mathbf{\Gamma}$. It is well known that $\operatorname{rep} \mathbf{\Gamma}$ is an abelian category which possesses almost split sequences. We denote by $\tau_{\mathbf{\Gamma}}$ the Auslander–Reiten translation in $\operatorname{rep} \mathbf{\Gamma}$. We remark that Gabriel proved (see for example [3, Corollaries I.6.10 and II.3.7]) that for each finite dimensional algebra A the category of A-modules is equivalent to the category of representations for an appropriate admissible bound quiver. This implies in particular, that we may work with bound quivers instead of algebras.

Now we describe the indecomposable projective representations of $\mathbf{\Gamma}$. For each $x \in \Gamma_0$ we define $P_x \in \operatorname{rep} \mathbf{\Gamma}$ as follows: $P_x(x') := k\mathbf{\Gamma}(x, x')$ for $x' \in \Gamma_0$ and $P_x(\varrho)(\varrho') := \varrho\varrho'$ for morphisms ϱ and ϱ' in $k\mathbf{\Gamma}$ such that $s\varrho' = x$ and $s\varrho = t\varrho'$. Moreover, if ϱ is a morphism in $k\mathbf{\Gamma}$, then we define $p_\varrho : P_{t\varrho} \to P_{s\varrho}$ by $p_\varrho(x)(\varrho') := \varrho'\varrho$ for $x \in \Gamma_0$ and a morphism ϱ' in $k\mathbf{\Gamma}$ with $s\varrho' = t\varrho$ and $t\varrho' = x$. It is an easy exercise to check that the map $\operatorname{Hom}_{\mathbf{\Gamma}}(P_x, M) \to M(x), f \mapsto f(x)(\mathbf{1}_x)$, is an isomorphism of vector spaces for each $x \in \Gamma_0$ and $M \in \operatorname{rep} \mathbf{\Gamma}$. This implies that the above formulas describe the fully faithful contravariant functor $\mathbf{\Gamma} \to \operatorname{rep} \mathbf{\Gamma}$ whose essential image coincides with the full subcategory of the indecomposable projective representations of $\mathbf{\Gamma}$.

Similarly, we describe the indecomposable injective representations of $\mathbf{\Gamma}$. For $x \in \Gamma_0$ we define $Q_x \in \operatorname{rep} \mathbf{\Gamma}$ by $Q_x(x') := (k\mathbf{\Gamma}(x', x))^*$ for $x' \in \Gamma_0$, where $(-)^* : \operatorname{mod} k \to \operatorname{mod} k$ denotes the k-linear dual, and $Q_x(\varrho)(\varphi)(\varrho') := \varphi(\varrho'\varrho)$ for morphisms ϱ and ϱ' in $k\mathbf{\Gamma}$ such that $s\varrho' = t\varrho$ and $t\varrho' = x$, and $\varphi \in (k\mathbf{\Gamma}(s\varrho, x))^*$. Moreover, if ϱ is a morphism in $k\mathbf{\Gamma}$, then we define $q_\varrho : Q_{t\varrho} \to Q_{s\varrho}$ by $q_\varrho(x)(\varphi)(\varrho') := \varphi(\varrho\varrho')$ for $x \in \Gamma_0$, morphisms ϱ and ϱ' in $k\mathbf{\Gamma}$ such that $s\varrho' = x$ and $t\varrho' = s\varrho$, and $\varphi \in (k\mathbf{\Gamma}(x, t\varrho))^*$. Again, the map $(M(x))^* \to \operatorname{Hom}_{\mathbf{\Gamma}}(M, Q_x), \varphi \mapsto (m \mapsto (\varrho \to \varphi(M(\varrho)(m))))$, is an isomorphism for each $x \in \Gamma_0$ and $M \in \operatorname{rep} \mathbf{\Gamma}$, and, consequently, we obtain the fully faithful contravariant functor $\mathbf{\Gamma} \to \operatorname{rep} \mathbf{\Gamma}$ whose essential image coincides with the full subcategory of the indecomposable injective representations of $\mathbf{\Gamma}$.

2. Almost gentle quivers

An admissible monomial bound quiver $\mathbf{\Gamma} = (\Gamma, R)$ is called almost gentle if the following conditions are satisfied:

(1) for each $x \in \Gamma_0$ there are at most two $\alpha \in \Gamma_1$ such that $s\alpha = x$ and at most two $\alpha \in \Gamma_1$ such that $t\alpha = x$,
(2) for each $\alpha \in \Gamma_1$ there is at most one $\alpha' \in \Gamma_1$ such that $s\alpha' = t\alpha$ and $\alpha'\alpha \notin R$, and at most one $\alpha' \in \Gamma_1$ such that $t\alpha' = s\alpha$ and $\alpha\alpha' \notin R$,
(3) for each $\alpha \in \Gamma_1$ there is at most one $\alpha' \in \Gamma_1$ such that $s\alpha' = t\alpha$ and $\alpha'\alpha \in R$, and at most one $\alpha' \in \Gamma_1$ such $t\alpha' = s\alpha$ and $\alpha\alpha' \in R$.

Equivalently, $\mathbf{\Gamma}$ is an almost gentle quiver if and only if there exist functions $S, T : \Gamma_1 \to \{\pm 1\}$, which we call string functions for $\mathbf{\Gamma}$, such that the following conditions are satisfied:

(1) if $s\alpha' = s\alpha''$ and $\alpha' \neq \alpha''$ for $\alpha', \alpha'' \in \Gamma_1$, then $S\alpha' = -S\alpha''$,
(2) if $t\alpha' = t\alpha''$ and $\alpha' \neq \alpha''$ for $\alpha', \alpha'' \in \Gamma_1$, then $T\alpha' = -T\alpha''$,
(3) if $s\alpha' = t\alpha''$ and $\alpha'\alpha'' \notin R$ for $\alpha', \alpha'' \in \Gamma_1$, then $S\alpha' = -T\alpha''$.
(4) if $s\alpha' = t\alpha''$ and $\alpha'\alpha'' \in R$ for $\alpha', \alpha'' \in \Gamma_1$, then $S\alpha' = T\alpha''$.

Note that the string functions for $\mathbf{\Gamma}$ are not uniquely determined by $\mathbf{\Gamma}$. For the rest of the section we fix an almost gentle bound quiver $\mathbf{\Gamma} = (\Gamma, R)$ together with string functions S and T.

Let $R' := R \cup R^{-1} \cup \{\alpha\alpha^{-1}, \alpha^{-1}\alpha \mid \alpha \in \Gamma_1\}$. Then $(\overline{\Gamma}, R')$ is a monomial bound quiver. If $l \in \mathbb{N}_+$, then by a string in $\mathbf{\Gamma}$ of length l we mean a path in $(\overline{\Gamma}, R')$ of length l. Moreover, for each $x \in \Gamma_0$ we introduce two strings $\mathbf{1}_{x,1}$ and $\mathbf{1}_{x,-1}$ such that $\ell(\mathbf{1}_{x,\varepsilon}) := 0$ and $s\mathbf{1}_{x,\varepsilon} := x =: t\mathbf{1}_{x,\varepsilon}$ for $\varepsilon \in \{\pm 1\}$. We put $(\mathbf{1}_{x,\varepsilon})^{-1} := \mathbf{1}_{x,-\varepsilon}$ for $x \in \Gamma_0$ and $\varepsilon \in \{\pm 1\}$. Observe that every path in $\mathbf{\Gamma}$ of positive length is a string in $\mathbf{\Gamma}$. A string ω in $\mathbf{\Gamma}$ is called simple if either $\ell(\omega) = 0$ or ω is a path in $\mathbf{\Gamma}$ (of positive length). Moreover, a string ω in $\mathbf{\Gamma}$ is called directed if either ω or ω^{-1} is a simple string. A string ω in $\mathbf{\Gamma}$ is called a band if $\ell(\omega) > 0$, either $\alpha_1(\omega) \in \Gamma_1$ and $\alpha_{\ell(\omega)}^{-1}(\omega) \in \Gamma_1$ or $\alpha_1^{-1}(\omega) \in \Gamma_1$ and $\alpha_{\ell(\omega)}(\omega) \in \Gamma_1$, $s\omega = t\omega$, ω^n is a string in $\mathbf{\Gamma}$ for each $n \in \mathbb{N}_+$, and there is no string ω' in $\mathbf{\Gamma}$ such that $\ell(\omega') < \ell(\omega)$, $s\omega' = t\omega'$, and $\omega = \omega'^n$ for some $n \in \mathbb{N}_+$.

We extend the functions S and T to the strings in $\mathbf{\Gamma}$ as follows. First, we put $S\alpha^{-1} := T\alpha$ and $T\alpha^{-1} := S\alpha$ for $\alpha \in \Gamma_1$. Next, we put $S\omega := S\alpha_{\ell(\omega)}(\omega)$ and $T\omega := T\alpha_1(\omega)$ for a string ω in $\mathbf{\Gamma}$ of positive length. Finally, we put $S\mathbf{1}_{x,\varepsilon} := \varepsilon$ and $T\mathbf{1}_{x,\varepsilon} := -\varepsilon$ for $x \in \Gamma_0$ and $\varepsilon \in \{\pm 1\}$. Observe that if ω' and ω'' are strings in $\mathbf{\Gamma}$ of positive length such that $s\omega' = t\omega''$ and $\omega'\omega''$ is a string in $\mathbf{\Gamma}$, then $S\omega' = -T\omega''$. Consequently, if ω is a string in $\mathbf{\Gamma}$, $x \in \Gamma_0$, and $\varepsilon \in \{\pm 1\}$, then we say that the composition $\omega\mathbf{1}_{x,\varepsilon}$ ($\mathbf{1}_{x,\varepsilon}\omega$) is defined (and equals ω) if and only if $x = s\omega$ and $\varepsilon = S\omega$ ($x = t\omega$ and $\varepsilon = -T\omega$, respectively).

Let ω be a string in $\mathbf{\Gamma}$ of length l. If $i \in [0, l]$, then we denote by $\omega_{[i]}$ and ${}_{[i]}\omega$ the strings in $\mathbf{\Gamma}$ of length i and $l-i$, respectively, such that $\omega = \omega_{[i]} \cdot {}_{[i]}\omega$. In particular, $\omega_{[0]} = \mathbf{1}_{t\omega,-T\omega}$ and ${}_{[\ell(\omega)]}\omega = \mathbf{1}_{s\omega,S\omega}$.

Fix $x \in \Gamma_0$ and $\varepsilon \in \{\pm 1\}$. By $\Sigma_{x,\varepsilon}$ we denote the set of simple strings σ in $\mathbf{\Gamma}$ such that $s\sigma = x$ and $S\sigma = \varepsilon$. Similarly, $\Sigma'_{x,\varepsilon}$ denotes the set of simple strings σ in $\mathbf{\Gamma}$ such that $t\sigma = x$ and $T\sigma = \varepsilon$. Next,

$$\alpha_{x,\varepsilon} := \begin{cases} \alpha & \alpha \in \Sigma_{x,\varepsilon} \cap \Gamma_1, \\ \varnothing & \text{otherwise}, \end{cases} \quad \text{and} \quad \alpha'_{x,\varepsilon} := \begin{cases} \alpha & \alpha \in \Sigma'_{x,\varepsilon} \cap \Gamma_1, \\ \varnothing & \text{otherwise}. \end{cases}$$

Finally, we denote by $\sigma_{x,\varepsilon}$ and $\sigma'_{x,\varepsilon}$ the strings of maximal length in $\Sigma_{x,\varepsilon}$ and $\Sigma'_{x,\varepsilon}$, respectively.

3. The homotopy category of a gentle quiver

An almost gentle bound quiver $\mathbf{\Gamma} = (\Gamma, R)$ is called gentle if Γ is finite and R consists of paths of length 2. For the rest of the section we assume that $\mathbf{\Gamma} = (\Gamma, R)$ is a fixed gentle bound quiver together with string functions S and T.

Let $\mathcal{K}^b(\mathbf{\Gamma})$ denote the bounded homotopy category of complexes of projective representations of $\mathbf{\Gamma}$. Recall that $\mathcal{K}^b(\mathbf{\Gamma})$ has a structure of a triangulated category [19, Theorem 2.3.1] with the suspension functor given by the degree shift $X \mapsto X[1]$. Moreover, since $\mathbf{\Gamma}$ is Gorenstein, i.e. $\operatorname{pdim}_{\mathbf{\Gamma}} Q < \infty$ for each injective representation Q of $\mathbf{\Gamma}$ and $\operatorname{idim}_{\mathbf{\Gamma}} P < \infty$ for each projective representation P of $\mathbf{\Gamma}$ [14], $\mathcal{K}^b(\mathbf{\Gamma})$ possesses almost split triangles [18, Section 5], thus also the Auslander–Reiten translation, which we denote by $\tau_{\mathcal{K}^b(\mathbf{\Gamma})}$.

Let $R' := \{\alpha\alpha^{-1}, \alpha^{-1}\alpha \mid \underline{\alpha} \in \Gamma_1\}$. Then $(\overline{\Gamma}, R')$ is a monomial bound quiver. If $l \in \mathbb{N}_+$, then by a homotopy string in $\mathbf{\Gamma}$ of length l we mean a path in $(\overline{\Gamma}, R')$ of length l. Moreover, $\mathbf{1}_{x,1}$ and $\mathbf{1}_{x,-1}$ are homotopy strings in $\mathbf{\Gamma}$ of length 0 for each $x \in \Gamma_0$. Observe that every string in $\mathbf{\Gamma}$ is a homotopy string in $\mathbf{\Gamma}$. We extend S and T to the homotopy strings in $\mathbf{\Gamma}$ in the usual way. If ω' and ω'' are homotopy strings in $\mathbf{\Gamma}$ of positive length, then we say that the composition $\omega'\omega''$ is defined (in the obvious way) if $s\omega' = t\omega''$ and one of the following conditions is satisfied, where $\alpha' := \alpha_{\ell(\omega')}(\omega')$ and $\alpha'' := \alpha_1(\omega'')$:

(1) $S\omega' = T\omega''$ and either $\alpha', \alpha'' \in \Gamma_1$ or $\alpha'^{-1}, \alpha''^{-1} \in \Gamma_1$,
(2) $S\omega' = -T\omega''$ and either $\alpha', \alpha''^{-1} \in \Gamma_1$ or $\alpha'^{-1}, \alpha'' \in \Gamma_1$.

Similarly, if ω is a homotopy string in $\mathbf{\Gamma}$ of positive length, $x \in \Gamma_0$, and $\varepsilon \in \{\pm 1\}$, then the composition $\omega\mathbf{1}_{x,\varepsilon}$ ($\mathbf{1}_{x,\varepsilon}\omega$) is defined (and equals ω) if and only if $x = s\omega$ and either $\varepsilon = S\omega$ and $\alpha_{\ell(\omega)}(\omega) \in \Gamma_1$ or $\varepsilon = -S\omega$ and $\alpha_{\ell(\omega)}^{-1}(\omega) \in \Gamma_1$ ($x = t\omega$ and either $\varepsilon = T\omega$ and $\alpha_1(\omega) \in \Gamma_1$ or $\varepsilon = -T\omega$ and $\alpha_1^{-1}(\omega) \in \Gamma_1$, respectively). Finally, if $x', x'' \in \Gamma_0$ and $\varepsilon', \varepsilon'' \in \{\pm 1\}$, then the composition $\mathbf{1}_{x',\varepsilon'}\mathbf{1}_{x'',\varepsilon''}$ is defined (and equals $\mathbf{1}_{x',\varepsilon'}$) if and only if $x' = x''$ and $\varepsilon' = \varepsilon''$. Observe that the above definitions for homotopy strings differ from the ones we have for strings. If ω is a homotopy string in $\mathbf{\Gamma}$, then by σ_ω we denote the string of maximal length among the simple strings σ in $\mathbf{\Gamma}$ such that the composition $\sigma\omega$ (as homotopy strings in $\mathbf{\Gamma}$) is defined.

A simple homotopy string θ in $\mathbf{\Gamma}$ is called an antipath in $\mathbf{\Gamma}$ provided $\alpha_i(\theta)\alpha_{i+1}(\theta) \in R$ (equivalently, $S\alpha_i(\theta) = T\alpha_{i+1}(\theta)$) for each $i \in [1, \ell(\theta) - 1]$. For $x \in \Gamma_0$ and $\varepsilon \in \{\pm 1\}$, let $\Theta_{x,\varepsilon}$ denote the set of all antipaths θ in $\mathbf{\Gamma}$ such that $t\theta = x$ and $T\theta = \varepsilon$. If there is an antipath in $\Theta_{x,\varepsilon}$ of maximal length, then we denote it by $\theta_{x,\varepsilon}$. Otherwise, we put $\theta_{x,\varepsilon} := \varnothing$.

Let ω be a homotopy string in $\mathbf{\Gamma}$. If $\ell(\omega) > 0$, then ω has a unique presentation in the form $\omega = \sigma_1 \cdots \sigma_L$, $L \in \mathbb{N}_+$, such that σ_i is a directed string in $\mathbf{\Gamma}$ of positive length for each $i \in [1, L]$, and the composition of $\sigma_i\sigma_{i+1}$ (as homotopy strings in $\mathbf{\Gamma}$) is defined for each $i \in [1, L-1]$. In the above situation we put $L(\omega) := L$, $\sigma_i(\omega) := \sigma_i$ and $\sigma_i^{-1}(\omega) := \sigma_i^{-1}$ for $i \in [1, L]$, and

$$\deg \omega := |\{i \in [1, L] \mid \sigma_i \text{ is a path in } \mathbf{\Gamma}\}| - |\{i \in [1, L] \mid \sigma_i^{-1} \text{ is a path in } \mathbf{\Gamma}\}|.$$

Moreover, we put $L(\omega) := 0$ and $\deg \omega := 0$ if $\ell(\omega) = 0$. If $i \in [0, L(\omega)]$, then we denote by $\omega^{[i]}$ and ${}^{[i]}\omega$ the homotopy strings in $\mathbf{\Gamma}$ of length $\sum_{j \in [1,i]} \ell(\sigma_j(\omega))$ and $\sum_{j \in [i+1, L(\omega)]} \ell(\sigma_j(\omega))$, respectively, such that $\omega = \omega^{[i]} \cdot {}^{[i]}\omega$. In particular,

$$\omega^{[0]} = \begin{cases} \mathbf{1}_{t\omega, T\omega} & \ell(\omega) > 0 \text{ and } \alpha_1(\omega) \in \Gamma_1, \\ \mathbf{1}_{t\omega, -T\omega} & \text{otherwise}, \end{cases}$$

and

$${}^{[L(\omega)]}\omega = \begin{cases} \mathbf{1}_{s\omega, -S\omega} & \ell(\omega) > 0 \text{ and } \alpha_{\ell(\omega)}^{-1}(\omega) \in \Gamma_1, \\ \mathbf{1}_{t\omega, S\omega} & \text{otherwise}. \end{cases}$$

Moreover, ${}^{[i]}\omega^{[j]} := {}^{[i]}(\omega^{[j]})$ for $i, j \in [0, L(\omega)]$, $i \le j$.

Let ω be a homotopy string in $\mathbf{\Gamma}$ and $m \in \mathbb{Z}$. We define $X = X_{m,\omega} \in \mathcal{K}^b(\mathbf{\Gamma})$ in the following way. First, for $m' \in \mathbb{Z}$ we put $\mathcal{I}_{m'} = \mathcal{I}_{m'}(m, \omega) := \{i \in [0, L(\omega)] \mid m + \deg \omega^{[i]} = m'\}$. Then $X^{m'} := \bigoplus_{i \in \mathcal{I}_{m'}} P_{s\omega^{[i]}}$ for $m' \in \mathbb{Z}$ and

$$(d_X^{m'})_{i,j} := \begin{cases} p_{\sigma_{j+1}(\omega)} & i = j+1 \text{ and } \sigma_{j+1}(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ p_{\sigma_j^{-1}(\omega)} & i = j-1 \text{ and } \sigma_j^{-1}(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ 0 & \text{otherwise}, \end{cases}$$

for $m' \in \mathbb{Z}$, $j \in \mathcal{I}_{m'}$, and $i \in \mathcal{I}_{m'+1}$. The objects of $\mathcal{K}^b(\mathbf{\Gamma})$ of the above form are called the string complexes.

For a homotopy string ω in $\mathbf{\Gamma}$ and $m \in \mathbb{Z}$ we denote by $\Upsilon_{m,\omega}$ the map $\Upsilon : X_{m,\omega} \to X_{m+\deg\omega, \omega^{-1}}$ defined by

$$\Upsilon_{i,j}^{m'} := \begin{cases} p_{s\omega^{[j]}} & i = L(\omega) - j, \\ 0 & \text{otherwise}, \end{cases}$$

for $m' \in \mathbb{Z}$, $j \in \mathcal{I}_{m'}(m, \omega)$, and $i \in \mathcal{I}_{m'}(m + \deg \omega, \omega^{-1})$. Observe that $\Upsilon_{m,\omega}$ is an isomorphism for each homotopy string ω in $\mathbf{\Gamma}$ and $m \in \mathbb{Z}$ — the inverse map is given by $\Upsilon_{m+\deg\omega, \omega^{-1}}$.

Let ω be a homotopy string in $\mathbf{\Gamma}$. Let σ be a path in $\mathbf{\Gamma}$ of positive length such that the composition $\sigma\omega$ (as homotopy strings in $\mathbf{\Gamma}$) is defined. For $m \in \mathbb{Z}$ we denote by $F'_{m,\sigma,\omega}$ the map $F' : X_{m, \mathbf{1}_{t\sigma, -T\sigma}} \to X_{m,\omega}$ defined by

$$F'^{m}_{i,0} := \begin{cases} p_\sigma & i = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$i \in \mathcal{I}_m(m, \omega)$. Similarly, let σ be a path in $\mathbf{\Gamma}$ of positive length such that the composition $\sigma^{-1}\omega$ is defined. We denote by $F''_{m,\sigma,\omega}$ the map $F'' : X_{m,\omega} \to X_{m,\mathbf{1}_{s\sigma,S\sigma}}$ defined by

$$F'^{m}_{0,j} := \begin{cases} p_\sigma & j = 0, \\ 0 & \text{otherwise}, \end{cases}$$

for $j \in \mathcal{I}_m(m, \omega)$. Next, if $m \in \mathbb{Z}$ and σ is a path in $\mathbf{\Gamma}$ of positive length such that the composition $\omega\sigma^{-1}$ is defined, then we put $G'_{m,\sigma,\omega} := \Upsilon_{m+\deg\omega,\omega^{-1}} \circ F'_{m+\deg\omega,\sigma,\omega^{-1}} \circ \Upsilon_{m+\deg\omega,\mathbf{1}_{t\sigma,T\sigma}} : X_{m+\deg\omega,\mathbf{1}_{t\sigma,T\sigma}} \to X_{m,\omega}$. Finally, if $m \in \mathbb{Z}$ and σ is a path in $\mathbf{\Gamma}$ of positive length such that the composition $\omega\sigma$ is defined, then we put $G''_{m,\sigma,\omega} := \Upsilon_{m+\deg\omega,\mathbf{1}_{s\sigma,S\sigma}} \circ F''_{m+\deg\omega,\sigma,\omega^{-1}} \circ \Upsilon_{m,\omega} : X_{m,\omega} \to X_{m+\deg\omega,\mathbf{1}_{s\sigma,-S\sigma}}$.

A homotopy string ω in $\mathbf{\Gamma}$ is called a homotopy band if $\deg\omega = 0$, $L(\omega) > 0$, either $\sigma_1(\omega)$ and $\sigma^{-1}_{L(\omega)}(\omega)$ are paths in $\mathbf{\Gamma}$ or $\sigma_1^{-1}(\omega)$ and $\sigma_{L(\omega)}(\omega)$ are paths in $\mathbf{\Gamma}$, $s\omega = t\omega$, $S\omega = -T\omega$, and there is no homotopy string ω' in $\mathbf{\Gamma}$ such that $\ell(\omega') < \ell(\omega)$, $s\omega' = t\omega'$, and $\omega = \omega'^n$ for some $n \in \mathbb{N}_+$. If ω is a homotopy band in $\mathbf{\Gamma}$ and $i \in [0, L(\omega) - 1]$, then we put $\omega^{(i)} := {}^{[i]}\omega \cdot \omega^{[i]}$. Observe that it may happen that $\omega^{(i)}$ is not a homotopy band in the above situation.

Let ω be a homotopy band in $\mathbf{\Gamma}$, μ an indecomposable automorphism of a finite dimensional vector space K, and $m \in \mathbb{Z}$. We define $Y = Y_{m,\omega,\mu} \in \mathcal{K}^b(\mathbf{\Gamma})$ in the following way. First, for $m' \in \mathbb{Z}$ we put $\mathcal{J}_{m'} := \{i \in [1, L(\omega)] \mid m + \deg\omega^{[i]} = m'\}$. Then $Y^{m'} := \bigoplus_{i\in\mathcal{J}_{m'}} P_{s\omega^{[i]}} \otimes_k K$ for $m' \in \mathbb{Z}$ and

$$(d_Y^{m'})_{i,j} := \begin{cases} p_{\sigma_i(\omega)} \otimes \mathrm{Id} & i = j+1,\ (i,j) \neq (L(\omega),1),\ \text{and } \sigma_{j+1}(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ p_{\sigma_j^{-1}(\omega)} \otimes \mathrm{Id} & i = j-1,\ (i,j) \neq (1, L(\omega)),\ \text{and } \sigma_j^{-1}(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ p_{\sigma_1(\omega)} \otimes \mu & j = L(\omega), i = 1, L(\omega) > 2,\ \text{and } \sigma_1(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ p_{\sigma_1^{-1}(\omega)} \otimes \mu & j = 1, i = L(\omega), L(\omega) > 2,\ \text{and } \sigma_1^{-1}(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ p_{\sigma_1(\omega)} \otimes \mu + p_{\sigma_2^{-1}(\omega)} \otimes \mathrm{Id} & j = L(\omega), i = 1, L(\omega) = 2,\ \text{and } \sigma_1(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ p_{\sigma_1^{-1}(\omega)} \otimes \mu + p_{\sigma_2(\omega)} \otimes \mathrm{Id} & j = 1, i = L(\omega), L(\omega) = 2,\ \text{and } \sigma_1^{-1}(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ 0 & \text{otherwise}, \end{cases}$$

for $m' \in \mathbb{Z}$, $j \in \mathcal{J}_{m'}$, and $i \in \mathcal{J}_{m'+1}$. The objects of $\mathcal{K}^b(\mathbf{\Gamma})$ of the above form are called the band complexes.

The following description of the indecomposable objects in $\mathcal{K}^b(\mathbf{\Gamma})$ was obtained in [7].

Proposition 3.1. *Let $\mathbf{\Gamma}$ be a gentle bound quiver. If X is an indecomposable object in $\mathcal{K}^b(\mathbf{\Gamma})$, then either $X \simeq X_{m,\omega}$ for some $m \in \mathbb{Z}$ and a homotopy string ω in $\mathbf{\Gamma}$ or $X \simeq Y_{m,\omega,\mu}$ for some $m \in \mathbb{Z}$, a homotopy band ω in $\mathbf{\Gamma}$, and an automorphism μ of a finite dimensional vector space.*

4. The repetitive quiver of a gentle quiver

Throughout this section we fix a gentle bound quiver $\mathbf{\Gamma} = (\Gamma, R)$ together with string functions S and T. We also denote by Σ the set of maximal paths in $\mathbf{\Gamma}$.

Our first aim in this section is to define the repetitive quiver $\hat{\mathbf{\Gamma}} = (\hat{\Gamma}, \hat{R})$ of $\mathbf{\Gamma}$. We put $\hat{\Gamma}_0 := \Gamma_0 \times \mathbb{Z}$ and $\hat{\Gamma}_1 := (\Gamma_1 \times \mathbb{Z}) \cup (\Sigma^* \times \mathbb{Z})$, where $\Sigma^* := \{\sigma^* \mid \sigma \in \Sigma\}$. Moreover, $s(\alpha[m]) := (s\alpha)[m]$ and $t(\alpha[m]) := (t\alpha)[m]$ for $\alpha \in \Gamma_1$ and $m \in \mathbb{Z}$, and $s(\sigma^*[m]) := (t\sigma)[m+1]$ and $t(\sigma^*[m]) := (s\sigma)[m]$ for $\sigma \in \Sigma$ and $m \in \mathbb{Z}$. For a path σ in Γ and $m \in \mathbb{Z}$ we define the path $\sigma[m]$ in $\hat{\Gamma}$ in the obvious way. Let

$$\begin{aligned} \mathbb{Z}R :=& \{\sigma[m] \mid \sigma \in R,\ m \in \mathbb{Z}\} \cup \\ & \{\sigma'^*[m-1]\sigma''^*[m] \mid \sigma', \sigma'' \in \Sigma,\ t\sigma' = s\sigma'',\ m \in \mathbb{Z}\} \cup \\ & \{\alpha[m]\sigma^*[m] \mid \alpha \in \Gamma_1,\ \sigma \in \Sigma,\ s\alpha = s\sigma,\ S\alpha = -S\sigma,\ m \in \mathbb{Z}\} \cup \\ & \{\sigma^*[m]\alpha[m+1] \mid \alpha \in \Gamma_1,\ \sigma \in \Sigma,\ t\alpha = t\sigma,\ T\alpha = -T\sigma,\ m \in \mathbb{Z}\}. \end{aligned}$$

Every path in $\hat{\Gamma}$ of the form $\sigma''[m]\sigma^*[m]\sigma'[m+1]$, where $\sigma \in \Sigma$, $\sigma = \sigma'\sigma''$ for paths σ' and σ'' in Γ, and $m \in \mathbb{Z}$, is called a full path. Let Λ be the set of full paths in $\hat{\Gamma}$. Then

$$\begin{aligned} \hat{R} :=& \mathbb{Z}R \cup \{\lambda' - \lambda'' \mid \lambda', \lambda'' \in \Lambda,\ s\lambda' = s\lambda'',\ \lambda' \neq \lambda''\} \cup \\ & \{\beta\lambda \mid \beta \in \hat{\Gamma}_1,\ \lambda \in \Lambda,\ s\beta = t\lambda\} \cup \{\lambda\beta \mid \beta \in \hat{\Gamma}_1,\ \lambda \in \Lambda,\ s\lambda = t\beta\}. \end{aligned}$$

We have the Nakayama automorphism ν of $\hat{\mathbf{\Gamma}}$ given by $\nu(x[m]) := x[m+1]$ and $\nu(\alpha[m]) := \alpha[m+1]$ for $x \in \Gamma_0$, $\alpha \in \Gamma_1 \cup \Sigma^*$, and $m \in \mathbb{Z}$.

Let $\underline{\mathrm{rep}}\,\hat{\mathbf{\Gamma}}$ denote the stable category of $\mathbf{\Gamma}$. Since $\mathrm{rep}\,\hat{\mathbf{\Gamma}}$ is a Frobenius category, $\underline{\mathrm{rep}}\,\hat{\mathbf{\Gamma}}$ is a triangulated category with the suspension functor given by the inverse Ω^{-1} of the Heller syzygy functor Ω. Every exact sequence in $\mathrm{rep}\,\hat{\mathbf{\Gamma}}$ induces a triangle in $\underline{\mathrm{rep}}\,\hat{\mathbf{\Gamma}}$. Moreover, $\underline{\mathrm{rep}}\,\hat{\mathbf{\Gamma}}$ possesses almost split triangles, which come from the almost split sequences in $\mathrm{rep}\,\hat{\mathbf{\Gamma}}$. In particular, the Auslander–Reiten translation in $\underline{\mathrm{rep}}\,\hat{\mathbf{\Gamma}}$ is given by the Auslander–Reiten translation $\tau_{\hat{\mathbf{\Gamma}}}$ in $\mathrm{rep}\,\hat{\mathbf{\Gamma}}$.

We define the functions $S, T : \hat{\Gamma}_1 \to \{\pm 1\}$ by

$$S\beta := \begin{cases} S\alpha & \beta = \alpha[m] \text{ for } \alpha \in \Gamma_1 \text{ and } m \in \mathbb{Z}, \\ -T\sigma & \beta = \sigma^*[m] \text{ for } \sigma \in \Sigma \text{ and } m \in \mathbb{Z}, \end{cases}$$

and

$$T\beta := \begin{cases} T\alpha & \beta = \alpha[m] \text{ for } \alpha \in \Gamma_1 \text{ and } m \in \mathbb{Z}, \\ -S\sigma & \beta = \sigma^*[m] \text{ for } \sigma \in \Sigma \text{ and } m \in \mathbb{Z}, \end{cases}$$

for $\beta \in \hat{\Gamma}_1$. One easily checks that $(\hat{\Gamma}, \hat{\underline{R}})$, where $\hat{\underline{R}} := \mathbb{Z}R \cup \{\lambda \mid \lambda \in \Lambda\}$, is an almost gentle quiver with string functions S and T. By a path in $\hat{\mathbf{\Gamma}}$ we mean a path in $(\hat{\Gamma}, \hat{\underline{R}})$. Similarly, by a string (band) in $\hat{\mathbf{\Gamma}}$ we mean a string (band, respectively) in $(\hat{\Gamma}, \hat{\underline{R}})$. For a string ω in $\mathbf{\Gamma}$ and $m \in \mathbb{Z}$ we define the string $\omega[m]$ in $\hat{\mathbf{\Gamma}}$ in the obvious way.

With a string ζ in $\hat{\mathbf{\Gamma}}$ we associate the representation V_ζ of $\hat{\mathbf{\Gamma}}$ in the following way. First, we put $I_x := \{i \in [0, \ell(\zeta)] \mid s\zeta_{[i]} = x\}$ for $x \in \Gamma_0$. Then we define V_ζ by $V_\zeta(x) := k^{I_x}$ for $x \in \Gamma_0$ and

$$(V_\zeta(\alpha))_{i,j} := \begin{cases} \mathrm{Id} & i = j-1 \text{ and } \alpha = \alpha_j(\zeta), \text{ or} \\ & i = j+1 \text{ and } \alpha = \alpha_{j+1}^{-1}(\zeta), \\ 0 & \text{otherwise}, \end{cases}$$

for $\alpha \in \Gamma_1$, $j \in I_{s\alpha}$, and $i \in I_{t\alpha}$. If $i \in I_x$ for $x \in \Gamma_0$, then we denote by $e_i(\zeta)$ the corresponding basis element of $V_\zeta(x)$. Similarly, if $i \in I_x$ for $x \in \Gamma_0$, then $e_i^*(\zeta)$ denotes the corresponding element of the basis of $(V_\zeta(x))^*$ dual to $(e_j(\zeta))_{j \in I_x}$. The representations of $\hat{\mathbf{\Gamma}}$ of the above form are called the string representations of $\hat{\mathbf{\Gamma}}$.

For a string ζ in $\hat{\mathbf{\Gamma}}$ of length l we denote by υ_ζ the map $\upsilon : X_\zeta \to X_{\zeta^{-1}}$ given by $\upsilon(e_i(\zeta)) := e_{l-i}(\zeta^{-1})$ for $i \in [0, l]$. Observe that υ_ζ is an isomorphism for each string ζ in $\hat{\mathbf{\Gamma}}$ (and the inverse map is given by $\upsilon_{\zeta^{-1}}$). Moreover, if ζ' and ζ'' are strings in $\hat{\mathbf{\Gamma}}$ such that $V_{\zeta'} \simeq V_{\zeta''}$, then either $\zeta' = \zeta''$ or $\zeta'^{-1} = \zeta''$.

Let ζ' and ζ'' be strings in $\hat{\mathbf{\Gamma}}$ such that $t\zeta' = t\zeta''$ and $T\zeta' = T\zeta''$. Put $l' := \ell(\zeta')$, $l'' := \ell(\zeta'')$, and

$$l := \max\{i \in [0, \min(l', l'')] \mid \alpha_j(\zeta') = \alpha_j(\zeta'') \text{ for each } j \in [1, i]\}.$$

If $\zeta' \neq \zeta''$, then we write $\zeta' <_{\mathrm{t}} \zeta''$ if either $l'' > l$ and $\alpha_{l+1}(\zeta'') \in \Gamma_1$ or $l' > l$ and $\alpha_{l+1}^{-1}(\zeta') \in \Gamma_1$. Moreover, we write $\zeta' \leq_{\mathrm{t}} \zeta''$ if either $\zeta' = \zeta''$ or $\zeta' <_{\mathrm{t}} \zeta''$. If $\zeta' \leq_{\mathrm{t}} \zeta''$, then by $f_{\zeta',\zeta''}$ we denote the map $f : V_{\zeta'} \to V_{\zeta''}$ given by $f(e_i(\zeta')) := e_i(\zeta'')$ for $i \in [0, l]$ and $f(e_i(\zeta')) := 0$ for $i \in [l+1, l']$.

Dually, let ζ' and ζ'' be strings in $\hat{\mathbf{\Gamma}}$ such that $s\zeta' = s\zeta''$ and $S\zeta' = S\zeta''$. We write $\zeta' \leq_{\mathrm{s}} \zeta''$ if $\zeta'^{-1} \leq_{\mathrm{t}} \zeta''^{-1}$. If $\zeta' \leq_{\mathrm{s}} \zeta''$, then we put $g_{\zeta',\zeta''} := \upsilon_{\zeta''^{-1}} \circ f_{\zeta'^{-1},\zeta''^{-1}} \circ \upsilon_{\zeta'}$.

Let ω be a band in $\hat{\mathbf{\Gamma}}$ and μ an indecomposable automorphism of a finite dimensional vector space K. We define the representation $W_{\omega,\mu}$ of $\hat{\mathbf{\Gamma}}$ as follows. First, for $x \in \Gamma_0$ we put $J_x := \{i \in [1, \ell(\omega)] \mid s\alpha_i(\omega) = x\}$. Then $W_{\omega,\mu}(x) := K^{J_x}$ for $x \in Q_0$ and

$$(W_{\omega,\mu}(\alpha))_{i,j} := \begin{cases} \mathrm{Id} & j \in [2, \ell(\omega)], i = j-1, \text{ and } \alpha = \alpha_j(\omega), \text{ or} \\ & j \in [1, \ell(\omega)-1], i = j+1, \text{ and } \alpha = \alpha_{j+1}^{-1}(\omega), \\ \mu & j = 1, i = \ell(\omega), \text{ and } \alpha = \alpha_1(\omega), \text{ or} \\ & j = \ell(\omega), i = 1, \text{ and } \alpha = \alpha_1^{-1}(\omega), \\ 0 & \text{otherwise}, \end{cases}$$

for $\alpha \in \Gamma_1$, $j \in J_{s\alpha}$, and $i \in J_{t\alpha}$. The representations of $\hat{\mathbf{\Gamma}}$ of the above form are called the band representations of $\hat{\mathbf{\Gamma}}$.

Let Ξ be the set consisting of all paths in $\hat{\mathbf{\Gamma}}$ and a chosen full path starting at y for each $y \in \hat{\Gamma}_0$. For $\xi \in \Xi$ we denote by ξ^* unique $\xi' \in \Xi$ such that $t\xi' = s\xi$ and $\xi\xi'$ is a full path. Observe that $(\sigma[m])^* = \sigma^*[m]$ for all $\sigma \in \Sigma$ and $m \in \mathbb{Z}$. Moreover, $(\xi^*)^* = \nu\xi$ for each $\xi \in \Xi$.

Fix $y \in \hat{\Gamma}_0$. If $y' \in \hat{\Gamma}_0$ and $\Xi(y', y) := \{\xi \in \Xi \mid s\xi = y',\ t\xi = y\}$, then (the residue classes of) $(\xi)_{\xi \in \Xi(y',y)}$ form a basis of $k\hat{\mathbf{\Gamma}}(y', y)$. For each $y' \in \hat{\Gamma}_0$ we identify $(\xi^*)_{\xi \in \Xi(y',y)}$ with the basis of $(k\hat{\mathbf{\Gamma}}(y', y))^*$ dual to $(\xi)_{\xi \in \Xi(y',y)}$. This identification induces isomorphisms $(k\hat{\mathbf{\Gamma}}(y', y))^* \simeq k\hat{\mathbf{\Gamma}}(\nu y, y')$, $y' \in \hat{\Gamma}_0$, which extend to an isomorphism $Q_y \simeq P_{\nu y}$, which we also treat as identification.

Let ζ be a string in $\hat{\mathbf{\Gamma}}$ of positive length. Then we have a presentation $\zeta = \xi_1\xi_2 \cdots \xi_L$, $L \in \mathbb{N}_+$, where ξ_i is a directed string in $\hat{\mathbf{\Gamma}}$ of positive length for each $i \in [1, L]$, and $\xi_i\xi_{i+1}$ is not a directed string in $\hat{\mathbf{\Gamma}}$ for each $i \in [1, L-1]$. In the above situation we put $L(\zeta) := L$, and $\xi_i(\zeta) := \xi_i$ and $\xi_i^{-1}(\zeta) := \xi_i^{-1}$ for $i \in [1, L]$. Moreover, if ζ is a band in $\hat{\mathbf{\Gamma}}$, then we put $\zeta^{(0)} := \zeta$ and $\zeta^{(i)} := \xi_{i+1} \cdots \xi_L\xi_1 \cdots \xi_i$ for $i \in [1, L-1]$. We also put $L(\mathbf{1}_{x,\varepsilon}) := 0$ for $x \in \hat{\Gamma}_0$ and $\varepsilon \in \{\pm 1\}$.

Now we define the operation $(-)^\times$ on the strings in $\hat{\mathbf{\Gamma}}$. First, we put $(\mathbf{1}_{y,\varepsilon})^\times := \mathbf{1}_{\nu y,-\varepsilon}$ for $y \in \hat{\Gamma}_0$ and $\varepsilon \in \{\pm 1\}$. Next, if ξ is a directed string of positive length, then we put $\xi^\times := (\xi^*)^{-1}$ if ξ is a path in $\hat{\mathbf{\Gamma}}$, and $\xi^\times := (\xi^{-1})^*$ if ξ^{-1} is a path in

$\hat{\mathbf{\Gamma}}$. Finally, if ζ is an arbitrary string in $\hat{\mathbf{\Gamma}}$ of positive length, then $\zeta^\times := \xi_1(\zeta)^\times \cdots \xi_{L(\zeta)}(\zeta)^\times$. The operation $(-)^\times$ is clearly invertible. We denote the inverse operation by $(-)^+$.

We also need some additional operations on the strings in $\hat{\mathbf{\Gamma}}$. Let ζ be a string in $\hat{\mathbf{\Gamma}}$, $l := \ell(\zeta)$, and $L := L(\zeta)$. If $L > 0$, then we denote by $\partial\zeta$ the unique string ζ' in $\hat{\mathbf{\Gamma}}$ such that $\zeta = \xi_1(\zeta)\cdot\zeta'$. We put

$$\partial'\zeta := \begin{cases} \partial\zeta & L > 0 \text{ and } \xi_1(\zeta) \text{ is a path in } \mathbf{\Gamma}, \\ \zeta & \text{otherwise,} \end{cases}$$

and

$$\partial''\zeta := \begin{cases} \partial\zeta & L > 0 \text{ and } \xi_1^{-1}(\zeta) \text{ is a path in } \mathbf{\Gamma}, \\ \zeta & \text{otherwise.} \end{cases}$$

Next, we put

$$\delta'_{\mathrm{t}}\zeta := \begin{cases} {}_{[1]}\zeta & l > 0 \text{ and } \alpha_1(\zeta) \in \hat{\Gamma}_1, \\ \sigma_{t\zeta,-T\zeta}\zeta & \text{otherwise.} \end{cases}$$

$$\delta'_{\mathrm{s}}\zeta := \begin{cases} \zeta_{[l-1]} & l > 0 \text{ and } \alpha^{-1}_{\ell(\zeta)}(\zeta) \in \hat{\Gamma}_1, \\ \zeta\sigma^{-1}_{s\zeta,-S\zeta} & \text{otherwise,} \end{cases}$$

$$\delta''_{\mathrm{t}}\zeta := \begin{cases} {}_{[1]}\zeta & l > 0 \text{ and } \alpha^{-1}_1(\zeta) \in \hat{\Gamma}_1, \\ \sigma'^{-1}_{t\zeta,-T\zeta}\zeta & \text{otherwise,} \end{cases}$$

and

$$\delta''_{\mathrm{s}}\zeta := \begin{cases} \zeta_{[l-1]} & l > 0 \text{ and } \alpha_{\ell(\zeta)}(\zeta) \in \hat{\Gamma}_1, \\ \zeta\sigma'_{s\zeta,-S\zeta} & \text{otherwise.} \end{cases}$$

Moreover, we put $\delta'\zeta := \delta'_{\mathrm{s}}\delta'_{\mathrm{t}}\zeta = \delta'_{\mathrm{t}}\delta'_{\mathrm{s}}\zeta$ and $\delta''\zeta := \delta''_{\mathrm{s}}\delta''_{\mathrm{t}}\zeta = \delta''_{\mathrm{t}}\delta''_{\mathrm{s}}\zeta$. Finally, we put $\Delta\zeta := \delta'(\zeta^\times)$. Observe that Δ is invertible and $\Delta^{-1}\zeta = \delta''(\zeta^+)$.

Let ζ be a string in $\hat{\mathbf{\Gamma}}$. We describe a projective cover $\pi_\zeta : P_\zeta \to V_\zeta$ of V_ζ. Write $\zeta = \xi_1\xi_2^{-1}\cdots\xi_{2L-1}\xi_{2L}^{-1}$, where $L \in \mathbb{N}_+$ and $\xi_1, \ldots, \xi_{2L}$ are simple strings in $\mathbf{\Gamma}$ such that $\ell(\xi_i) > 0$ for each $i \in [2, 2L-1]$. Let $l_i := \sum_{j\in[1,2i-1]}\ell(\xi_j)$ for $i \in [1, L]$. Then $\pi_\zeta : P_\zeta \to V_\zeta$, where $P_\zeta := \bigoplus_{i\in[1,L]} P_{s\xi_{2i-1}}$ and $(\pi_\zeta)_i$ corresponds to $e_{l_i}(\zeta)$ under the canonical isomorphism $\operatorname{Hom}_{\mathbf{\Gamma}}(P_{s\xi_{2i-1}}, V_\zeta) \simeq V_\zeta(s\xi_{2i-1})$ for each $i \in [1, L]$, is the minimal projective cover of V_ζ. Observe that $\Omega V_\zeta := \operatorname{Ker}\pi_\zeta \simeq V_{\Delta^{-1}\zeta}$. We identify ΩV_ζ with $V_{\Delta^{-1}\zeta}$. Further, P_ζ is an injective envelope of $V_{\Delta^{-1}\zeta}$. More precisely, if $\Delta^{-1}\zeta = \xi_1'^{-1}\xi_2'\cdots\xi'^{-1}_{2L-1}\xi'_{2L}$ for $L \in \mathbb{N}_+$ and simple strings $\xi'_1, \ldots, \xi'_{2L}$ in $\hat{\mathbf{\Gamma}}$ such that $\ell(\xi'_i) > 0$ for each $i \in [2, 2L-1]$, then $P_\zeta = \bigoplus_{i\in[1,L]} Q_{t\xi'_{2i-1}}$. Moreover, if $l'_i := \sum_{j\in[1,2i-1]}\ell(\xi'_j)$ for $i \in [1, L]$ and $\iota_\zeta : V_{\Delta^{-1}\zeta} \to P_\zeta$ is such that $(\iota_\zeta)_i$ corresponds to $(-1)^i e^*_{l'_i}(\Delta^{-1}\zeta)$ under the canonical isomorphism $(V_{\Delta^{-1}\zeta}(t\xi_{2i-1}))^* \simeq \operatorname{Hom}_{\hat{\mathbf{\Gamma}}}(V_{\Delta^{-1}\zeta}, Q_{t\xi_{2i-1}})$ for each $i \in [1, L]$, then the sequence $0 \to V_{\Delta^{-1}\zeta} \xrightarrow{\iota_\zeta} P_\zeta \xrightarrow{\zeta} V_\zeta \to 0$ is exact. We will use sequences of the above form to calculate the action of Ω on morphisms in $\underline{\operatorname{rep}}\,\hat{\mathbf{\Gamma}}$. In particular, it follows that $\Omega f_{\zeta',\zeta''}$ and $f_{\Delta^{-1}\zeta',\Delta^{-1}\zeta''}$ ($\Omega g_{\zeta',\zeta''}$ and $g_{\Delta^{-1}\zeta',\Delta^{-1}\zeta''}$) coincide up to sign for strings ζ' and ζ'' in $\hat{\mathbf{\Gamma}}$ such that $t\zeta' = t\zeta''$, $T\zeta' = T\zeta''$, and $\zeta' \leq_{\mathrm{t}} \zeta''$ ($s\zeta' = s\zeta''$, $S\zeta' = S\zeta''$, and $\zeta' \leq_{\mathrm{s}} \zeta''$).

Similarly as above we show that $\Omega W_{\zeta,\mu} \simeq W_{\zeta^+,(-1)^{L(\zeta)/2}\mu^{-1}}$.

5. The Happel functor

Throughout this section we fix a gentle bound quiver $\mathbf{\Gamma} = (\Gamma, R)$ together with string functions S and T. We also denote by Σ the set of maximal paths in $\mathbf{\Gamma}$.

Let $\mathcal{D}^b(\mathbf{\Gamma})$ denote the derived category of $\operatorname{rep}\mathbf{\Gamma}$. It is known that $\mathcal{D}^b(\mathbf{\Gamma})$ is a triangulated category and $\mathcal{K}^b(\mathbf{\Gamma})$ can be viewed as a full triangulated subcategory of $\mathcal{D}^b(\mathbf{\Gamma})$. We identify $M \in \operatorname{rep}\mathbf{\Gamma}$ with the complex in $\mathcal{D}^b(\mathbf{\Gamma})$ concentrated in degree 0. In [16] Happel constructed a fully faithful triangle functor $\mathcal{D}^b(\mathbf{\Gamma}) \to \underline{\operatorname{rep}}\,\hat{\mathbf{\Gamma}}$ which extends the inclusion functor $\operatorname{rep}\mathbf{\Gamma} \to \underline{\operatorname{rep}}\,\hat{\mathbf{\Gamma}}$. By Ψ we denote the restriction of this functor to $\mathcal{K}^b(\mathbf{\Gamma})$.

Let ω be a homotopy string $\mathbf{\Gamma}$. We define the string $\psi\omega$ in $\hat{\mathbf{\Gamma}}$ by induction on $L(\omega)$ as follows. First, $\psi\omega := \sigma_\omega[0]\cdot(\sigma_{\omega^{-1}})^{-1}[0]$ if $L(\omega) = 0$. Next, if $L(\omega) > 0$, then

$$\psi\omega := \begin{cases} \sigma_\omega[0]\cdot(\sigma[0])^+\cdot\delta''_{\mathrm{s}}(\zeta^+) & \sigma \text{ is a path in } \mathbf{\Gamma}, \\ \sigma_\omega[0]\cdot\sigma[0]\cdot\delta'_{\mathrm{s}}(\zeta^\times) & \sigma^{-1} \text{ is a path in } \mathbf{\Gamma} \text{ and } \ell(\zeta) > 0, \\ \sigma_\omega[0]\cdot\sigma_{[\ell(\sigma)-1]}[0] & \sigma^{-1} \text{ is a path in } \mathbf{\Gamma} \text{ and } \ell(\zeta) = 0, \end{cases}$$

where $\sigma := \sigma_1(\omega)$ and $\zeta := \partial'(\psi({}^{[1]}\omega))$.

The meaning of the above assignment is explained in the following.

Proposition 5.1. *Let ω be a homotopy string in $\mathbf{\Gamma}$ and $m \in \mathbb{Z}$. Then $\Psi X_{m,\omega} \simeq V_{\Delta^{-m}(\psi\omega)}$. Moreover, under the isomorphisms of the above form, we have the following:*

(1) *If σ is a path in $\mathbf{\Gamma}$ of positive length such that the composition $\sigma\omega$ is defined, then $\Psi F'_{m,\sigma,\omega} = \varepsilon f_{\Delta^{-m}(\psi \mathbf{1}_{t\sigma,-T\sigma}),\Delta^{-m}(\psi\omega)}$ for some $\varepsilon \in \{\pm 1\}$.*

(2) *If σ is a path in $\mathbf{\Gamma}$ of positive length such that $\sigma^{-1}\omega$ is defined, then $\Psi F''_{m,\sigma,\omega} = \varepsilon f_{\Delta^{-m}(\psi\omega),\Delta^{-m}(\psi \mathbf{1}_{s\sigma,S\sigma})}$ for some $\varepsilon \in \{\pm 1\}$.*

(3) *If σ is a path in $\mathbf{\Gamma}$ of positive length such that $\omega\sigma^{-1}$ is defined, then $\Psi G'_{m,\sigma,\omega} = \varepsilon g_{\Delta^{-\deg\omega - m}(\psi \mathbf{1}_{t\sigma,T\sigma}),\Delta^{-m}(\psi\omega)}$ for some $\varepsilon \in \{\pm 1\}$.*

(4) *If σ is a path in $\mathbf{\Gamma}$ of positive length such that $\omega\sigma$ is defined, then $\Psi G''_{m,\sigma,\omega} = \varepsilon g_{\Delta^{-m}(\psi\omega),\Delta^{-\deg\omega - m}(\psi \mathbf{1}_{s\sigma,-S\sigma})}$ for some $\varepsilon \in \{\pm 1\}$.*

Proof. We prove the above claims by induction on $L(\omega)$. If $L(\omega) = 0$, then the proofs are immediate (observe that $X_{m,\omega} \simeq X_{0,\omega}[-m]$ for each $m \in \mathbb{Z}$). It also follows easily that $\Psi \Upsilon_{0,\omega}$ and $\upsilon_{\psi\omega}$ coincide in this case.

Now assume that $L(\omega) > 0$ and let $\sigma' := \sigma_1(\omega)$ and $\omega' := {}^{[1]}\omega$. We also assume that σ' is a path in $\mathbf{\Gamma}$, the case when σ'^{-1} is a path in $\mathbf{\Gamma}$ is similar.

By induction hypothesis $\Psi X_{0,\mathbf{1}_{t\sigma',-T\sigma'}} \simeq V_{\psi \mathbf{1}_{t\sigma',-T\sigma'}}$ and $\Psi X_{0,\omega'} \simeq V_{\psi\omega'}$. Moreover, if $f := \Psi F'_{0,\sigma',\omega'}$, then f equals up to sign $f_{\psi \mathbf{1}_{t\sigma',-T\sigma'},\psi\omega'}$. Let $\zeta := \Delta(\psi \mathbf{1}_{t\sigma',-T\sigma'})$ and

$$\begin{tikzcd} V_{\psi \mathbf{1}_{t\sigma',-T\sigma'}} \arrow[r, "\iota_\zeta"] \arrow[d, "f"] & P_\zeta \arrow[r, "\pi_\zeta"] \arrow[d, "g"] & V_\zeta \arrow[d, "\mathrm{Id}"] \\ V_{\psi\omega'} \arrow[r, "f'"] & V \arrow[r, "f''"] & V_\zeta \end{tikzcd}$$

be the push-out diagram (recall that P_ζ, ι_ζ and π_ζ are defined at the end of the previous section). Then

$$V_{\psi \mathbf{1}_{t\sigma',-T\sigma'}} \xrightarrow{f} V_{\psi\omega'} \xrightarrow{f'} V \xrightarrow{f''} V_\zeta$$

is a triangle in $\underline{\mathrm{rep}}\, \hat{\mathbf{\Gamma}}$. Direct calculations show that $V \simeq V_{\Delta(\psi\omega)}$ in $\underline{\mathrm{rep}}\, \hat{\mathbf{\Gamma}}$ (note that V may be a decomposable module, so this isomorphism may not hold in $\mathrm{rep}\, \hat{\mathbf{\Gamma}}$). Moreover, by choosing g in an appropriate way we get that f' and f'' equal up to sign $g_{\psi\omega',\Delta(\psi\omega)}$ and $\upsilon_{\zeta^{-1}} \circ f_{\Delta(\psi\omega),\zeta^{-1}}$, respectively. Since we have an isomorphism $X_{m,\omega} \simeq X_{-1,\omega}[-m-1]$ for each $m \in \mathbb{Z}$ and a triangle

$$X_{0,\mathbf{1}_{t\sigma',-T\sigma'}} \xrightarrow{F'_{0,\sigma',\omega'}} X_{0,\omega'} \xrightarrow{F'} X_{-1,\omega} \xrightarrow{F''} X_{-1,\mathbf{1}_{t\sigma',-T\sigma'}}$$

in $\mathcal{K}^b(\mathbf{\Gamma})$, we get the first claim. Moreover, under the appropriate isomorphisms, $\Psi F' = f'$ and $\Psi F'' = f''$.

Let σ be a path in $\mathbf{\Gamma}$ of positive length such that $s\sigma = t\omega$ and $S\sigma = T\omega$. Then we have the commutative diagram

$$\begin{tikzcd} X_{0,\mathbf{1}_{t\sigma,-T\sigma}} \arrow[r] \arrow[d, "F'_{0,\sigma,\mathbf{1}_{t\sigma',-T\sigma'}}"] & 0 \arrow[r] \arrow[d] & X_{-1,\mathbf{1}_{t\sigma,-T\sigma}} \arrow[r, "\mathrm{Id}"] \arrow[d, "F'_{-1,\sigma,\omega}"] & X_{-1,\mathbf{1}_{t\sigma,-T\sigma}} \arrow[d, "F'_{-1,\sigma,\mathbf{1}_{t\sigma',-T\sigma'}}"] \\ X_{0,\mathbf{1}_{t\sigma',-T\sigma'}} \arrow[r, "F'_{0,\sigma',\omega'}"] & X_{0,\omega'} \arrow[r, "F'"] & X_{-1,\omega} \arrow[r, "F''"] & X_{-1,\mathbf{1}_{t\sigma',-T\sigma'}} \end{tikzcd}$$

in $\mathcal{K}^b(\mathbf{\Gamma})$, which gives rise to the commutative diagram

$$\begin{tikzcd} V_{\psi \mathbf{1}_{t\sigma,-T\sigma}} \arrow[r] \arrow[d, "\Psi F'_{0,\sigma,\mathbf{1}_{t\sigma',-T\sigma'}}"] & 0 \arrow[r] \arrow[d] & V_{\Delta(\psi \mathbf{1}_{t\sigma,-T\sigma})} \arrow[r, "\mathrm{Id}"] \arrow[d, "\Psi F'_{-1,\sigma,\omega}"] & V_{\Delta(\psi \mathbf{1}_{t\sigma,-T\sigma})} \arrow[d, "\Psi F'_{-1,\sigma,\mathbf{1}_{t\sigma',-T\sigma'}}"] \\ V_{\psi \mathbf{1}_{t\sigma',-T\sigma'}} \arrow[r, "f"] & V_{\psi\omega'} \arrow[r, "f'"] & V_{\Delta(\psi\omega)} \arrow[r, "f''"] & V_\zeta \end{tikzcd}$$

in $\underline{\mathrm{rep}}\, \hat{\mathbf{\Gamma}}$. By induction hypothesis $\Psi F'_{0,\sigma,\mathbf{1}_{t\sigma',-T\sigma'}}$ and $\Psi F'_{-1,\sigma,\mathbf{1}_{t\sigma',-T\sigma'}}$ equal up to sign $f_{\psi \mathbf{1}_{t\sigma,-T\sigma},\psi \mathbf{1}_{t\sigma',-T\sigma'}}$ and $f_{\Delta(\psi \mathbf{1}_{t\sigma,-T\sigma}),\zeta}$, respectively. Moreover, $\Psi F'_{m,\sigma,\mathbf{1}_{t\sigma',-T\sigma'}} = \Omega^{m+1} \Psi F'_{-1,\sigma,\mathbf{1}_{t\sigma',-T\sigma'}}$ and this follows from the above diagram that $\Psi F'_{-1,\sigma,\omega}$ equals up to sing $f_{\Delta(\psi \mathbf{1}_{t\sigma,-T\sigma}),\Delta(\psi\omega)}$, hence (1) follows. The remaining claims are proved similarly. □

Observe that $\omega' = \omega''$ if $\psi\omega' = \psi\omega''$ for homotopy strings ω' and ω'' in $\mathbf{\Gamma}$. In fact we have more.

Lemma 5.2. *Let ω' and ω'' be homotopy strings in $\mathbf{\Gamma}$ and $m \in \mathbb{Z}$. If $\psi\omega' = \Delta^m(\psi\omega'')$, then $m = 0$ and $\omega' = \omega''$.*

Proof. In light of the preceding remark it is sufficient to prove that $m = 0$. Without loss of generality we may assume that $m \leq 0$. Observe that $t(\psi\omega') \in \Gamma_0[0]$. Moreover, easy induction shows that $t(\Delta^m(\psi\omega'')) \in \Gamma_0[-m'']$ for some $m'' \in \mathbb{N}_+$ provided $m < 0$, hence the claim follows. □

Corollary 5.3. *Let ω be a homotopy string in $\mathbf{\Gamma}$. Then $(\psi\omega)^{-1} = \Delta^{-\deg\omega}(\psi\omega^{-1})$.*

Proof. Recall that $X_{0,\omega} \simeq X_{\deg\omega,\omega^{-1}}$, thus Proposition 5.1 implies that $V_{\psi\omega} \simeq V_{\Delta^{-\deg\omega}(\psi\omega^{-1})}$. Hence, either $\psi\omega = \Delta^{-\deg\omega}(\psi\omega^{-1})$ or $(\psi\omega)^{-1} = \Delta^{-\deg\omega}(\psi\omega^{-1})$. In the former case the previous lemma implies that $\omega^{-1} = \omega$, which is impossible. □

Now we calculate the images of the band complexes. For this we need an additional function ψ' between the homotopy strings in $\mathbf{\Gamma}$ of positive length and the strings in $\hat{\mathbf{\Gamma}}$. Let ω be a homotopy string in $\mathbf{\Gamma}$ of positive length. Put $L := L(\omega)$ and $\sigma := \sigma_1(\omega)$. If $L = 1$, then $\psi'\omega := (\sigma[0])^+$ provided σ is a path in $\mathbf{\Gamma}$, and $\psi'\omega := \sigma[0]$ provided σ^{-1} is a path in $\mathbf{\Gamma}$. If $L > 1$, then

$$\psi'\omega := \begin{cases} \psi'\sigma \cdot (\psi'({}^{[1]}\omega))^+ & \sigma \text{ is a path in } \mathbf{\Gamma}, \\ \psi'\sigma \cdot (\psi'({}^{[1]}\omega))^\times & \sigma^{-1} \text{ is a path in } \mathbf{\Gamma}. \end{cases}$$

One can easily deduce the formula for $\psi'\omega$ as follows:

$$\psi'\omega = (\sigma_1(\omega))^{\times n_1} \cdots (\sigma_L(\omega))^{\times n_L},$$

where

$$n_i := \begin{cases} -\deg\omega^{[i]} & \sigma_i(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ -\deg\omega^{[i]} - 1 & \sigma_i^{-1}(\omega) \text{ is a path in } \mathbf{\Gamma}, \end{cases}$$

for $i \in [1, L]$, and $(-)^{\times n}$ denotes the n-th power of $(-)^\times$ for $n \in \mathbb{Z}$. The above formula implies immediately that $\psi'\omega^{-1} = (\psi'\omega)^{-1}$ if $\deg\omega = 0$. Moreover, if $\deg\omega = 0$ and ω' is a homotopy string in $\mathbf{\Gamma}$ of positive length such that the composition $\omega\omega'$ is defined, then $\psi'(\omega\omega') = \psi'\omega \cdot \psi'\omega'$. Finally, it follows that $\psi'\omega$ is a band in $\hat{\mathbf{\Gamma}}$ if ω is a homotopy band in $\mathbf{\Gamma}$.

We also need the following property of ψ'.

Lemma 5.4. *Let ω be a homotopy string in $\mathbf{\Gamma}$ such that $L(\omega) > 0$ and $\deg\omega = 0$. If ω' is a homotopy string in $\mathbf{\Gamma}$ such that the composition $\omega\omega'$ is defined, then $\psi(\omega\omega') = \sigma_\omega[0] \cdot \psi'\omega \cdot \partial'(\psi\omega')$. In particular, $\psi\omega = \sigma_\omega[0] \cdot \psi'\omega \cdot (\sigma_{\omega^{-1}})^{-1}[0]$.*

Proof. If $L(\omega) = 2$, then the claim follows by direct calculations. Observe that in this case either $\omega = \sigma'\sigma''^{-1}$ or $\omega = \sigma'^{-1}\sigma''$ for paths σ' and σ'' in $\mathbf{\Gamma}$ of positive length.

Now assume that $L := L(\omega) > 2$. There are some cases to consider.

First assume that there exists $i \in [2, L-2]$ such that $\deg\omega^{[i]} = 0$. Observe that $\deg{}^{[i]}\omega = 0$. Then we have the following sequence of equalities

$$\begin{aligned} \psi(\omega\omega') &\overset{\text{ind}}{=} \sigma_\omega[0] \cdot \psi'\omega^{[i]} \cdot \partial'(\psi'({}^{[i]}\omega\omega')) \overset{\text{ind}}{=} \\ &\overset{\text{ind}}{=} \sigma_\omega[0] \cdot \psi'\omega^{[i]} \cdot \psi'{}^{[i]}\omega \cdot \partial'(\psi\omega') = \sigma_\omega[0] \cdot \psi'\omega \cdot \partial'(\psi'\omega') \end{aligned}$$

hence the claim follows in this case.

If the above condition is not satisfied, then $\deg{}^{[1]}\omega^{[L-1]} = 0$. Put $\sigma' := \sigma_1(\omega)$ and $\sigma'' := \sigma_L(\omega)$. Assume in addition that σ' is a path in $\mathbf{\Gamma}$, the other case is similar. Then σ''^{-1} is a path in $\mathbf{\Gamma}$. Moreover, if $\ell(\partial'(\psi\omega')) > 0$, then we have the following sequence of equalities

$$\begin{aligned} \psi(\omega\omega') &\overset{\text{def}}{=} \sigma_\omega[0] \cdot (\sigma'[0])^+ \cdot \delta''_s((\partial'(\psi({}^{[1]}\omega \cdot \omega')))^+) \\ &\overset{\text{ind}}{=} \sigma_\omega[0] \cdot (\sigma'[0])^+ \cdot \delta''_s((\psi'({}^{[1]}\omega^{[L-1]}))^+ \cdot (\partial'(\psi(\sigma''\omega')))^+) \\ &\overset{\text{def}}{=} \sigma_\omega[0] \cdot \psi'\sigma' \cdot \delta''_s((\psi'({}^{[1]}\omega^{[L-1]}))^+ \cdot (\sigma''[0])^+ \cdot (\delta'_s((\partial'(\psi\omega'))^\times))^+) \\ &= \sigma_\omega[0] \cdot \psi'\sigma' \cdot (\psi'({}^{[1]}\omega^{[L-1]}))^+ \cdot (\psi'\sigma'')^+ \cdot \partial'(\psi\omega') \\ &= \sigma_\omega[0] \cdot \psi'\sigma' \cdot (\psi'({}^{[1]}\omega))^+ \cdot \partial'(\psi\omega') \overset{\text{def}}{=} \sigma_\omega[0] \cdot \psi'\omega \cdot \partial'(\psi'\omega'). \end{aligned}$$

On the other hand, if $\ell(\partial'(\psi'\omega')) = 0$, then by repeating some of the calculations above we get

$$\begin{aligned} \psi(\omega\omega') &= \sigma_\omega[0] \cdot \psi'\sigma' \cdot \delta''_s((\psi'({}^{[1]}\omega^{[L-1]}))^+ \cdot (\sigma''_{[\ell(\sigma'')-1]}[0])^+) \\ &= \sigma_\omega[0] \cdot \psi'\sigma' \cdot (\psi'({}^{[1]}\omega^{[L-1]}))^+ \cdot (\sigma''[0])^+ \\ &= \sigma_\omega[0] \cdot \psi'\omega \cdot \partial'(\psi'\omega'), \end{aligned}$$

what finishes the proof. □

The above lemma will be used in the proof of the following.

Proposition 5.5. *Let ω be a homotopy band in $\mathbf{\Gamma}$ and μ an indecomposable automorphism of a finite dimensional vector space. Let*

$$\varepsilon := \begin{cases} 1 & \sigma_1(\omega) \text{ is a path in } \mathbf{\Gamma}, \\ -1 & \sigma_1^{-1}(\omega) \text{ is a path in } \mathbf{\Gamma}. \end{cases}$$

Then $\Psi Y_{0,\omega,\mu} \simeq W_{\psi'\omega,\varepsilon'\mu^{\varepsilon}}$ for some $\varepsilon' \in \{\pm 1\}$ depending only on ω.

Proof. Put $L := L(\omega)$. Let $\sigma' := \sigma_1(\omega)$, $\sigma'' := \sigma_{L(\omega)}^{-1}(\omega)$, and $\omega' := {}^{[1]}\omega^{[L-1]}$. Observe that $\deg \omega' = 0$. Consequently, $\psi'\omega' = \sigma_{\omega'}[0] \cdot \psi'\omega' \cdot (\sigma_{\omega'^{-1}})^{-1}[0]$ if $L > 2$ (by the previous lemma) and $\psi'\omega' = \sigma_{\omega'}[0] \cdot (\sigma_{\omega'^{-1}})^{-1}[0]$ otherwise (by definition). Let K be the domain of μ.

We assume that σ' is a path in $\mathbf{\Gamma}$ — the other case is similar. Then σ'' is also a path in $\mathbf{\Gamma}$ and we have a triangle

$$X_{0,\mathbf{1}_{t\sigma',-T\sigma'}} \otimes_k K \xrightarrow{F} X_{0,\omega'} \otimes_k K \to Y_{-1,\omega,\mu} \to X_{-1,\mathbf{1}_{t\sigma',-T\sigma'}} \otimes K$$

in $\mathcal{K}^b(\mathbf{\Gamma})$, where $F := F'_{0,\sigma',\omega'} \otimes \mu + G_{0,\sigma'',\omega'} \otimes \mathrm{Id}$. Observe that Proposition 5.1 implies that $\Psi X_{0,\mathbf{1}_{t\sigma',-T\sigma'}} \simeq V_{\psi \mathbf{1}_{t\sigma',-T\sigma'}}$ and $\Psi X_{0,\omega'} \simeq V_{\psi\omega'}$. If $f := \Psi F$, then under the above isomorphisms $f = \varepsilon_1 f_{\psi \mathbf{1}_{t\sigma',-T\sigma'},\psi\omega'} \otimes \mu + \varepsilon_2 g_{\psi \mathbf{1}_{t\sigma',-T\sigma'},\psi\omega'} \otimes \mathrm{Id}$ for some $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ (depending only on ω) according to Proposition 5.1(1) and (3). Since we have the triangle

$$V_{\psi \mathbf{1}_{t\sigma',-T\sigma'}} \otimes K \xrightarrow{f} V_{\psi\omega'} \otimes K \to W_{\sigma'[0]\cdot\psi'\omega'\cdot\sigma''^{-1}[0],\varepsilon_0\mu^{-1}} \to \Omega^{-1}(V_{\psi \mathbf{1}_{t\sigma',-T\sigma'}} \otimes K)$$

in $\underline{\mathrm{rep}}\, \hat{\mathbf{\Gamma}}$, where $\varepsilon_0 := -\varepsilon_1\varepsilon_2$, $\Psi Y_{-1,\omega,\mu} \simeq W_{\sigma'[0]\cdot\psi'\omega'\cdot\omega''^{-1}[0],\varepsilon_0\mu^{-1}}$ (in the calculation of this triangle we use the form of $\psi'\omega'$ calculated at the beginning of the proof). Since $Y_{0,\omega,\mu} \simeq \Psi Y_{-1,\omega,\mu}[-1]$ and

$$\Omega W_{\sigma'[0]\cdot\psi'\omega'\cdot\sigma''^{-1}[0],\varepsilon_0\mu^{-1}} \simeq W_{\psi'\omega,(-1)^{L(\omega)/2}\varepsilon_0\mu},$$

the claim follows. $\square$

Corollary 5.6. *Let $m \in \mathbb{Z}$, ω be a homotopy band in $\mathbf{\Gamma}$, and μ an automorphism of a finite dimensional vector space. Then $\Psi Y_{m,\omega,\mu} \simeq W_{(\psi'\omega)^{\times(-m)},\varepsilon'\mu^{\varepsilon''}}$ for some $\varepsilon', \varepsilon'' \in \{\pm 1\}$ depending only on ω and m.*

Proof. Since $Y_{m,\omega,\mu} \simeq Y_{0,\omega,\mu}[-m]$, the claim follows from the previous proposition. $\square$

6. Almost split triangles

Throughout this section we fix a gentle bound quiver $\mathbf{\Gamma} = (\Gamma, R)$ together with string functions S and T. Moreover, ψ is the map which associates with a homotopy string in $\mathbf{\Gamma}$ a string in $\hat{\mathbf{\Gamma}}$ as defined in Proposition 5.1.

Our aim in this section is to determine the shape of the almost split triangles in $\mathcal{K}^b(\mathbf{\Gamma})$. In order to restrict the number of technical definitions we will not describe the maps appearing in the almost split triangles, however the interested reader can easily check that "natural" candidates are the correct ones.

We first recall that a triangle $X \to Y \to Z \to X[1]$ is $\mathcal{K}^b(\mathbf{\Gamma})$ is an almost split triangle in $\mathcal{K}^b(\mathbf{\Gamma})$ if and only if the corresponding triangle $\Psi X \to \Psi Y \to \Psi Z \to \Omega^{-1}\Psi X$ in $\underline{\mathrm{rep}}\, \hat{\mathbf{\Gamma}}$ is an almost split triangle in $\underline{\mathrm{rep}}\, \hat{\mathbf{\Gamma}}$ [18, Proposition 5.2], where similarly as in the previous section Ψ denotes the restriction of the Happel functor to $\mathcal{K}^b(\mathbf{\Gamma})$.

As a consequence we immediately obtain the following.

Main Theorem (*Part I: Band complexes*). *Let $m \in \mathbb{Z}$ and ω be a homotopy band in $\mathbf{\Gamma}$. If $0 \to \mu' \to \bigoplus_{i\in[1,n]} \mu_i \to \mu'' \to 0$ is an almost split sequence in the category of the automorphisms of finite dimensional vector spaces, where $\mu', \mu_1, \ldots, \mu_n$ and μ'' are indecomposable automorphisms of finite dimensional vector spaces, then*

$$Y_{m,\omega,\mu'} \to \bigoplus_{i\in[1,n]} Y_{m,\omega,\mu_i} \to Y_{m,\omega,\mu''} \to Y_{m,\omega,\mu'}[1]$$

is an almost split triangle in $\mathcal{K}^b(\mathbf{\Gamma})$.

Proof. It follows from the above remark and Corollary 5.6, since for each band ζ in $\hat{\mathbf{\Gamma}}$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$ we have an almost split triangle

$$W_{\zeta,\varepsilon'\mu'^{\varepsilon}} \to \bigoplus_{i\in[1,n]} W_{\zeta,\varepsilon'\mu_i^{\varepsilon}} \to W_{\zeta,\varepsilon'\mu''^{\varepsilon}} \to \Omega^{-1} W_{\zeta,\varepsilon'\mu'^{\varepsilon}}$$

in $\underline{\mathrm{rep}}\, \hat{\mathbf{\Gamma}}$ (compare [10,26, Theorem 4.1(2)]). $\square$

It remains to describe the almost split triangles in $\mathcal{K}^b(\mathbf{\Gamma})$ involving the string complexes. We first describe the almost split triangles $\underline{\mathrm{rep}}\, \hat{\mathbf{\Gamma}}$ involving the string representations.

For a string ζ in $\hat{\mathbf{\Gamma}}$ we put

$$_{+}\zeta := \begin{cases} \sigma_{s\alpha',-S\alpha'}\alpha'^{-1}\zeta & \alpha' \neq \varnothing, \\ _{[1]}(\partial''\zeta) & \alpha' = \varnothing \text{ and } \ell(\partial''\zeta) > 0, \\ \varnothing & \alpha' = \varnothing \text{ and } \ell(\partial''\zeta) = 0, \end{cases}$$

where

$$\alpha' := \begin{cases} \alpha'_{t\zeta,-T\zeta} & \alpha'_{t\zeta,-T\zeta} \neq \varnothing \text{ and } \alpha'^{-1}_{t\zeta,-T\zeta}\zeta \text{ is a string in } \hat{\mathbf{\Gamma}}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Next, we put $\zeta_{+} := (_{+}(\zeta^{-1}))^{-1}$ for a string ζ in $\hat{\mathbf{\Gamma}}$. Finally, if ζ is a string in $\hat{\mathbf{\Gamma}}$, then

$$_{+}\zeta_{+} := \begin{cases} (_{+}\zeta)_{+} & _{+}\zeta \neq \varnothing, \\ _{+}(\zeta_{+}) & \zeta_{+} \neq \varnothing. \end{cases}$$

We leave it to the reader to verify that the above definition is correct and $_{+}\zeta_{+} \neq \varnothing$ for each string ζ in $\hat{\mathbf{\Gamma}}$. Moreover, [10,26, Theorem 4.1(1)] imply that for each string ζ in $\hat{\mathbf{\Gamma}}$ we have an almost split triangle in $\underline{\operatorname{rep}}\,\hat{\mathbf{\Gamma}}$ of the form

$$V_{\zeta} \to V_{_{+}\zeta} \oplus V_{\zeta_{+}} \to V_{_{+}\zeta_{+}} \to \Omega^{-1}V_{\zeta},$$

where $V_{\varnothing} := 0$.

We translate the above construction to $\mathcal{K}^{b}(\mathbf{\Gamma})$.

Let ω be a homotopy band in $\mathbf{\Gamma}$. We denote by $r(\omega)$ the maximal $i \in [0, \ell(\omega)]$ such that $\alpha_j(\omega) \in \Gamma_1$ for each $j \in [1, i]$ and $\alpha_j(\omega)\alpha_{j+1}(\omega) \in R$ for each $j \in [1, i-1]$. We put

$$\omega' := \begin{cases} _{[1]}(\sigma_{r(\omega)}(\omega)) \cdot {}^{[r(\omega)]}\omega & r(\omega) > 0, \\ \omega & r(\omega) = 0, \end{cases}$$

and $\sigma := \sigma_{\omega}$. Then we define

$$_{+}\omega := \begin{cases} \theta^{-1}_{t\sigma,-T\sigma}\sigma\omega & \ell(\sigma) > 0, \\ \theta^{-1}_{t\omega',-T\omega'}\omega' & \ell(\sigma) = 0, \ell(\theta_{t\omega',-T\omega'}) > 0, \text{ and } \ell(\omega') > 0, \\ ({}^{[1]}\theta_{t\omega',-T\omega'})^{-1} & \ell(\sigma) = 0, \ell(\theta_{t\omega',-T\omega'}) > 0, \text{ and } \ell(\omega') = 0, \\ {}^{[1]}\omega' & \ell(\sigma) = 0, \ell(\theta_{t\omega',-T\omega'}) = 0, \ell(\omega') > 0, \text{ and } \alpha_1^{-1}(\omega') \in \Gamma_1, \\ \omega' & \ell(\sigma) = 0, \ell(\theta_{t\omega',-T\omega'}) = 0, \ell(\omega') > 0, \text{ and } \alpha_1(\omega') \in \Gamma_1, \\ \varnothing & \ell(\sigma) = 0, \ell(\theta_{t\omega',-T\omega'}) = 0, \text{ and } \ell(\omega') = 0. \end{cases}$$

and

$$m'(\omega) := \begin{cases} \ell(\theta_{t\sigma,-T\sigma}) - 1 & \ell(\sigma) > 0, \\ \ell(\theta_{t\omega',-T\omega'}) + r(\omega) - 1 & \ell(\sigma) = 0. \end{cases}$$

We first prove that the above definitions are correct.

Lemma 6.1. *Let $x \in \Gamma_0$ and $\varepsilon \in \{\pm 1\}$. If $\alpha_{x,\varepsilon} = \varnothing$, then $\theta_{x,\varepsilon} \neq \varnothing$.*

Proof. We show that $\ell(\theta) \leq |\Gamma_1|$ for each $\theta \in \Theta_{x,\varepsilon}$. Assume this is not the case and fix $\theta \in \Theta_{x,\varepsilon}$ such that $l := \ell(\theta) > |\Gamma_1|$. Then there exist $i, j \in [1, l]$, $i < j$, such that $\alpha_i(\theta) = \alpha_j(\theta)$. An easy induction shows that $\alpha_{j+1-i}(\theta) = \alpha_1(\theta) = \alpha'_{x,\varepsilon}$. Consequently, $\alpha_{j-i}(\theta) \in \Sigma_{x,\varepsilon}$, which is impossible. □

Let ω, σ, and ω' be as in the definition of $_{+}\omega$. Obviously, $\alpha_{t\sigma,-T\sigma} = \varnothing$, hence the above lemma implies that $\theta_{t\sigma,-T\sigma} \neq \varnothing$. Now assume that $\ell(\sigma) = 0$. This assumption means that $\alpha_{t\omega,\varepsilon} = \varnothing$, where

$$\varepsilon = \begin{cases} T\omega & r(\omega) > 0, \\ -T\omega & r(\omega) = 0. \end{cases}$$

Consequently, if $r(\omega) = 0$, i.e. $\omega' = \omega$, then $\theta_{t\omega',-T\omega'} \neq \varnothing$. On the other hand, if $r(\omega) > 0$, then $\theta_{t\omega,T\omega} \neq \varnothing$. Moreover, in this case $\alpha_1(\omega) \cdots \alpha_{r(\omega)}(\omega)\theta \in \Theta_{t\omega,T\omega}$ for each $\theta \in \Theta_{t\omega',-T\omega'}$, hence we also get that $\theta_{t\omega',-T\omega'} \neq \varnothing$.

The following lemma is crucial.

Lemma 6.2. *Let ω be a homotopy band in $\mathbf{\Gamma}$. Then $_{+}\omega = \varnothing$ if and only if $_{+}(\psi\omega) = \varnothing$. Moreover, if $_{+}\omega \neq \varnothing$, then $_{+}(\psi\omega) = \Delta^{-m'(\omega)}(\psi(_{+}\omega))$.*

Proof. We make some remarks about the proof and leave the details to the reader. Let σ and ω' be as in the definition of ${}_+\omega$.

First, if $\ell(\sigma) > 0$, then we prove $\Delta^{-1}({}_+(\psi\omega)) = \Delta^{-\ell(\theta_{t\sigma,-T\sigma})}(\psi({}_+\omega))$. In the proof we use the following fact, which can be easily proved by induction.

Fact. *Let ω_0 be a homotopy string in $\mathbf{\Gamma}$ such that $\sigma_{t\partial'(\psi\omega_0),-T\partial'(\psi\omega_0)} = \sigma_{\omega_0}[0]$. Let $l \in [0, \ell(\omega_0)]$. If $\sigma_i(\omega_0) \in \Gamma_1^{-1}$ for each $i \in [1, l]$, then*

$$\Delta^{-l}(\psi\omega_0) = \sigma_{t\partial'(\psi\omega_0'),-T\partial'(\psi\omega_0')} \cdot \partial'(\psi\omega_0').$$

where $\omega_0' := {}^{[l]}\omega$.

Next, we assume that $\ell(\sigma) = 0$ and either $\ell(\theta_{t\omega',-T\omega'}) > 0$ or $\ell(\omega') > 0$. In this case we show either ${}_+(\Delta^{r(\omega)-1}(\psi\omega)) = \Delta^{-\ell(\theta_{t\omega',-T\omega'})}(\psi({}_+\omega))$ if $\omega' \neq {}^{[r(\omega)]}\omega$, or ${}_+(\Delta^{r(\omega)}(\psi\omega)) = \Delta^{-\ell(\theta_{t\omega',-T\omega'})+1}(\psi({}_+\omega))$, otherwise. In addition to the above fact, we use here also the following.

Fact. *Let ω_0 be a homotopy string in $\mathbf{\Gamma}$ such that $\ell(\sigma_{\omega_0}) = 0$. Let $r \in [0, \ell(\omega_0)]$. If $\sigma_i(\omega_0) \in \Gamma_1$ for each $i \in [1, r]$, then*

$$\Delta^r(\psi\omega_0) = \partial'(\psi\omega_0'),$$

where $\omega_0' := {}^{[r]}\omega_0$.

Finally, we prove that ${}_+(\Delta^{r(\omega)}(\psi\omega)) = \varnothing$ if $\ell(\sigma) = 0$, $\ell(\theta'_{t\omega',-T\omega'}) = 0$, and $\ell(\omega') = 0$. In this proof we also use the latter fact. $\square$

Dually, we put $\omega_+ := ({}_+(\omega^{-1}))^{-1}$ for a homotopy band ω in $\mathbf{\Gamma}$. Lemma 6.2 and Corollary 5.3 imply that $(\psi\omega)_+ = \psi(\omega_+)$ for each homotopy band ω in $\mathbf{\Gamma}$. Finally, we put

$${}_+\omega_+ := \begin{cases} ({}_+\omega)_+ & {}_+\omega \neq \varnothing, \\ {}_+(\omega_+) & \omega_+ \neq \varnothing, \end{cases} \quad \text{and} \quad m''(\omega) := \begin{cases} m'(\omega) & {}_+\omega \neq \varnothing, \\ m'(\omega_+) & \omega_+ \neq \varnothing. \end{cases}$$

We obtain the following description of the almost split triangles in $\mathcal{K}^b(\mathbf{\Gamma})$ involving the string complexes, where $X_{m,\varnothing}$ is the zero complex for $m \in \mathbb{Z}$.

Main Theorem (*Part II: String Complexes*). *Let ω be a homotopy string in $\mathbf{\Gamma}$ and $m \in \mathbb{Z}$. Then we have an almost split triangle in $\mathcal{K}^b(\mathbf{\Gamma})$ of the form*

$$X_{m,\omega} \to X_{m+m'(\omega),{}_+\omega} \oplus X_{m,\omega_+} \to X_{m+m''(\omega),{}_+\omega_+} \to X_{m-1,\omega}.$$

As a consequence we obtain the following description of the almost split sequences with indecomposable middle terms containing the string complexes (we encourage the reader to compare this result with [5]).

Corollary 6.3. *Let ω be a homotopy string in $\mathbf{\Gamma}$. Then $\varnothing \in \{{}_+\omega, \omega_+\}$ if and only if $\omega = \theta_{x,\varepsilon}^{\varepsilon'}$ for $x \in \Gamma_0$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$ such that $\alpha_{x,\varepsilon} = \varnothing$. Moreover, if this is the case, then ${}_+\omega_+ = \theta_{t\sigma,-T\sigma}^{\varepsilon'}$, where $\sigma := \sigma_{s\omega^{\varepsilon'},-S\omega^{\varepsilon'}}$.*

References

[1] H. Asashiba, The derived equivalence classification of representation-finite selfinjective algebras, J. Algebra 214 (1) (1999) 182–221.
[2] I Assem, D. Happel, Generalized tilted algebras of type A_n, Comm. Algebra 9 (20) (1981) 2101–2125.
[3] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras. Vol. 1, in: London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006, p. x+458.
[4] I. Assem, A. Skowroński, Iterated tilted algebras of type $\tilde{\mathbf{A}}_n$, Math. Z. 195 (2) (1987) 269–290.
[5] D. Avella-Alaminos, Ch. Geiß, Combinatorial derived invariants for gentle algebras, J. Pure Appl. Algebra 212 (1) (2008) 228–243.
[6] A.A. Beĭlinson, Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra, Funktsional. Anal. i Prilozhen. 12 (3) (1978) 68–69 (in Russian) English transl., Funct. Anal. Appl. 12 (3) (1978) 214–216.
[7] V. Bekkert, H.A. Merklen, Indecomposables in derived categories of gentle algebras, Algebr. Represent. Theory 6 (3) (2003) 285–302.
[8] R. Bocian, Th. Holm, A. Skowroński, Derived equivalence classification of one-parametric self-injective algebras, J. Pure Appl. Algebra 207 (3) (2006) 491–536.
[9] Th. Brüstle, Derived-tame tree algebras, Compos. Math. 129 (3) (2001) 301–323.
[10] M.C.R. Butler, C.M. Ringel, Auslander–Reiten sequences with few middle terms and applications to string algebras, Comm. Algebra 15 (1–2) (1987) 145–179.
[11] E. Cline, B. Parshall, L. Scott, Derived categories and Morita theory, J. Algebra 104 (2) (1986) 397–409.
[12] W. Geigle, H. Lenzing, A class of weighted projective curves arising in representation theory of finite-dimensional algebras, in: G.-M. Greuel, G. Trautmann (Eds.), Singularities, Representation of Algebras, and Vector Bundles, in: Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297.
[13] Ch. Geiß, Derived tame algebras and Euler-forms, Math. Z. 239 (4) (2002) 829–862. With an appendix by the author and B. Keller.
[14] Ch. Geiß, I. Reiten, Gentle algebras are Gorenstein, in: Representations of Algebras and Related Topics, in: Fields Inst. Commun., vol. 45, Amer. Math. Soc., Providence, RI, 2005, pp. 129–133.
[15] D. Happel, On the derived category of a finite-dimensional algebra, Comment. Math. Helv. 62 (3) (1987) 339–389.
[16] D. Happel, Triangulated Categories in the Representation Theory of Finite-dimensional Algebras, in: London Math. Soc. Lecture Note Ser., vol. 119, Cambridge Univ. Press, Cambridge, 1988, p. x+208.
[17] D. Happel, Auslander-Reiten triangles in derived categories of finite-dimensional algebras, Proc. Amer. Math. Soc. 112 (3) (1991) 641–648.

[18] D. Happel, B. Keller, I. Reiten, Bounded derived categories and repetitive algebras, J. Algebra 319 (4) (2008) 1611–1635.
[19] S. König, A. Zimmermann, Derived Equivalences for Group Rings, in: Lecture Notes in Math., vol. 1685, Springer, Berlin, 1998, p. x+246.
[20] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra 61 (3) (1989) 303–317.
[21] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) 39 (3) (1989) 436–456.
[22] C.M. Ringel, The repetitive algebra of a gentle algebra, Bol. Soc. Mat. Mexicana (3) 3 (2) (1997) 235–253.
[23] J. Schröer, A. Zimmermann, Stable endomorphism algebras of modules over special biserial algebras, Math. Z. 244 (2003) 515–530.
[24] J.-L. Verdier, Categories dérivées, in: P. Deligne (Ed.), Cohomologie Étale, in: Lecture Notes in Mathematics, vol. 569, Springer, Berlin, 1977, pp. 262–311.
[25] D. Vossieck, The algebras with discrete derived category, J. Algebra 243 (1) (2001) 168–176.
[26] B. Wald, J. Waschbüsch, Tame biserial algebras, J. Algebra 95 (2) (1985) 480–500.