Coleman's power series and Wiles' reciprocity for rank 1 Drinfeld modules ☆

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We introduce the formalism of Coleman's power series for rank 1 Drinfeld modules and apply it to formulate and prove the analogue of Wiles' explicit reciprocity law in this setting.

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1. Introduction

Andrew Wiles discovered an explicit reciprocity law for local fields [16], generalizing earlier work of Artin and Hasse [2] and Iwasawa [8]. Since the publication of [16], different proofs of this reciprocity law were found: see the expositions in [10, Chapter 9], [5, I, §4] (where the main tool are Coleman's power series) and [13, §3.3] (a cohomological approach, inspired by Kato's formulation in [9, §1]).

Wiles' explicit reciprocity law has its foundation in the theory of Lubin–Tate formal groups: torsion points of the formal group generate a "cyclotomic" tower of local fields and the action of local norm symbols on the torsion is expressed by an analytic formula.

A rank 1 Drinfeld module of generic characteristic can be seen as originating a special instance of Lubin–Tate formal group: therefore there should be an analogue of Wiles' reciprocity law in this setting. For the Carlitz module, this was proven by Anglès: his paper [1] follows the approach of [10, Chapter 9].

In the present paper we introduce the formalism of Coleman's power series in the additive setting of (formal) rank 1 Drinfeld modules. As a first application we obtain our main result, the explicit reciprocity law for Drinfeld modules over any global field F (Theorem 24): i.e., we work with any

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sign-normalized rank 1 Drinfeld module, with no restriction on the class number. Following [5, I, §4] we formulate the law directly in its limit form, as the equality of two pairings on systems of groups.

In the classical situation over $\mathbb{Q}_p$, one can exploit the Coleman isomorphism (corresponding to Theorem 11 in our setting) to construct p-adic L-functions: e.g., the Kubota–Leopoldt zeta function comes from the system of cyclotomic units. We hope to be able to recover this aspect of the theory in a future paper.

Let us finally give a rough sketch of the contents of this paper. In Section 2 we introduce the basic notation and properties for formal Drinfeld modules and the "cyclotomic" tower we are working with. In Section 3 we rewrite Coleman's formalism in the setting of Drinfeld modules: our construction is quite detailed, in the hope it can be a good introduction to the subject (the differences with characteristic 0 are irrelevant). In Section 4 we introduce the two pairings and prove their equality.

2. Setting

Let F be a global function field, with field of constants $\mathbb{F}_q$, q a power of p. Given a place v of F, $q_v := q^{\deg(v)}$ will indicate the cardinality of its residue field $\mathbb{F}_v$. Besides, we denote by τ the operator $x \mapsto x^q$ and, for an $\mathbb{F}_p$-algebra R, by $R\{\tau\}$ the ring of skew polynomials with coefficients in R: multiplication in $R\{\tau\}$ is given by composition.

2.1. Review of rank 1 Drinfeld modules

We briefly recall Hayes' theory of explicit construction of class fields by means of rank 1 Drinfeld modules; our main references will be [7] and [6, Chapter 7]. We fix a place ∞ of F and let $A \subset F$ be the ring of functions regular away from ∞.

As in [6,7] we fix a sign-function $sgn : F_\infty \to \mathbb{F}_\infty$: then the basic extensions of F are H and H^+, the Hilbert class field and the normalizing field (see e.g. [7, §14–15] or [6, §7.1 and §7.4]), with B and B^+ the integral closure of A in H and H^+ respectively. (It might be worth to recall that H depends only on the choice of ∞, while H^+ is determined by both ∞ and sgn.) Let Φ be a sgn-normalized rank 1 Drinfeld A-module: i.e., Φ is a ring homomorphism $A \to B^+\{\tau\}$, $a \mapsto \Phi_a$, such that the constant coefficient of Φ_a is a and the leading coefficient map is a $\mathrm{Gal}(\mathbb{F}_\infty/\mathbb{F}_q)$-twist of sgn. We fix such a Φ.

As usual, if I is an ideal of A, $\Phi[I]$ denotes the I-torsion of Φ (i.e., the common zeroes of all Φ_a, $a \in I$) and Φ_I is the unique monic generator of the left ideal of $H^+\{\tau\}$ generated by Φ_a, $a \in I$. One sees immediately that

$$\Phi_I(x) = \prod_{u \in \Phi[I]} (x - u)$$

and since elements in $\Phi[I]$ are all integral above B^+, it follows that $\Phi_I \in B^+\{\tau\}$. By [7, Proposition 11.4], in case $I = \mathfrak{p}^n$, $\mathfrak{p}$ a prime, all irreducible factors (over B^+) of the polynomial $\Phi_I(x)$ are Eisenstein at $\mathfrak{p}$.

To conclude, we recall that the extension $H^+(\Phi[\mathfrak{p}^n])/F$ is abelian and

$$\mathrm{Gal}\big(H^+(\Phi[\mathfrak{p}^n])/H^+\big) \simeq \big(A/\mathfrak{p}^n\big)^*$$

[6, §7.5]: the isomorphism is given by the A-action on $\Phi[\mathfrak{p}^n]$.

Caveat. The notation Φ_a (or Φ_I) will be used to denote both the operator $\Phi_a \in B^+\{\tau\}$ and the polynomial $\Phi_a(x) \in B^+[x]$; the context should make clear which one we mean.

2.2. Our local setting

From now on, we fix a prime ideal $\mathfrak{p}$ in A: since Φ is *sgn*-normalized, it has good reduction in $\mathfrak{p}$. Let $F_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ be the completions at $\mathfrak{p}$. We also fix $\mathbf{C}_{\mathfrak{p}}$, completion of an algebraic closure of $F_{\mathfrak{p}}$, and choose an embedding $H^+ \hookrightarrow \mathbf{C}_{\mathfrak{p}}$; let K be the topological closure of H^+ in $\mathbf{C}_{\mathfrak{p}}$ and $\mathcal{O} = \mathcal{O}_K$ its ring of integers, with maximal ideal $\mathfrak{m} = \mathfrak{m}_K$. Let $\mathfrak{M} := B(0,1) = \{z \in \mathbf{C}_{\mathfrak{p}} : |z| < 1\}$. In what follows, all extensions of F are assumed to be contained in $\mathbf{C}_{\mathfrak{p}}$. The valuation v on $\mathbf{C}_{\mathfrak{p}}$ is normalized so that $v(F_{\mathfrak{p}}^*) = \mathbb{Z}$.

The extension H^+/F is unramified outside of ∞: in particular, it follows that $K/F_{\mathfrak{p}}$ is unramified.

Let $\mathcal{I}$ be the group of fractional ideals of A and $\mathcal{P}^+$ the subgroup of the positively generated principal ones. By Hayes' theory we know that $\mathcal{I}/\mathcal{P}^+ \simeq \mathrm{Gal}(H^+/F)$ [7, Theorem 14.7]; this isomorphism is given by the action of ideals on the set of *sgn*-normalized rank 1 Drinfeld modules. In particular the image of $\mathfrak{p}$ in $\mathcal{I}/\mathcal{P}^+$ corresponds to the Frobenius at $\mathfrak{p}$: its order is $f := [K : F_{\mathfrak{p}}]$, hence we have $\mathfrak{p}^f = (\eta)$, with $\eta \in A$ a positive element (i.e., $sgn(\eta) = 1$). Then, by definition of *sgn*-normalized, Φ_η is monic and therefore $\Phi_\eta = \Phi_{\mathfrak{p}^f}$. Notice that η is uniquely defined once we have fixed *sgn*.

Finally, we remark that, since all factors of $\Phi_{\mathfrak{p}^n}$ are Eisenstein, $\Phi[\mathfrak{p}^n] \subset \mathfrak{M}$ for all n. In particular, all coefficients (but the leading one) of Φ_η are in $\mathfrak{m}$.

2.3. The formal module

Consider the ring of skew power series $\mathcal{O}\{\{\tau\}\}$: it is a local ring, complete in the topology induced by its maximal ideal $\mathfrak{m} + \mathcal{O}\{\{\tau\}\}\tau$. As observed by Rosen [11, p. 247], the homomorphism $\Phi : A \to B^+\{\tau\} \subset \mathcal{O}\{\{\tau\}\}$ can be extended to the localization of A at $\mathfrak{p}$ and then to its completion: we get a formal Drinfeld module $\Phi : A_{\mathfrak{p}} \to \mathcal{O}\{\{\tau\}\}$.

Proposition 1 *(Rosen). There exists a unique $\lambda \in K\{\{\tau\}\}$ of the form $\lambda = 1 + \cdots$ and such that $a\lambda = \lambda\Phi_a$ for all $a \in A_{\mathfrak{p}}$. Besides, λ converges on $\mathfrak{M}$ and it enjoys the following properties*:

(1) $\lambda = \sum c_i \tau^i$ with $v(c_i) \geqslant -i$;
(2) if $v(x) > (q-1)^{-1}$, then $v(\lambda(x)) = v(x)$.

For the proof the reader is referred to [11, Propositions 2.1 and 2.3].

2.4. The "cyclotomic" tower

Define the tower of field extensions K_n/K by $K_n := K(\Phi[\mathfrak{p}^{fn}])$, $n \geqslant 1$, with ring of integers $\mathcal{O}_n$ and maximal ideal $\mathfrak{m}_n$. (This choice of indexing seemed to us more convenient even if slightly unusual – commonly, one calls the tamely ramified extension K_0.) Let $Tr_m^n : K_n \to K_m$ and $N_m^n : K_n^* \to K_m^*$ denote respectively trace and norm.

We also put $K_\infty := \bigcup_{n=1}^{\infty} K_n$.

Consider the Tate module $T_{\mathfrak{p}}\Phi := \varprojlim \Phi[\mathfrak{p}^{fn}]$ (the limit is taken with respect to $x \mapsto \Phi_\eta(x)$).[1] The ring $A_{\mathfrak{p}}$ acts on $T_{\mathfrak{p}}\Phi$ via Φ: i.e., $a \cdot u := \Phi_a(u)$. Since Φ has rank 1, $T_{\mathfrak{p}}\Phi$ is a free $A_{\mathfrak{p}}$-module of rank 1.

Let $\omega = \{\omega_n\}_{n \geqslant 1}$ be a generator: $T_{\mathfrak{p}}\Phi = A_{\mathfrak{p}} \cdot \omega$. This means that the sequence $\{\omega_n\}$ satisfies

$$\Phi_\eta^n(\omega_n) = 0 \neq \Phi_\eta^{n-1}(\omega_n) \quad \text{and} \quad \Phi_\eta(\omega_{n+1}) = \omega_n.$$

(Here and in the following, when we write Φ_a^n the power is always taken in $\mathcal{O}\{\tau\}$ – that is, with respect to composition.)

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[1] More canonically, $T_{\mathfrak{p}}\Phi$ is usually defined as $\mathrm{Hom}_{A_{\mathfrak{p}}}(F_{\mathfrak{p}}/A_{\mathfrak{p}}, \varprojlim \Phi[\mathfrak{p}^n])$ ([6, Definition 4.10.9] and following remarks), but the two are isomorphic and our definition suits better our purpose.

By definition $K_n = K(\omega_n)$. Being a root of an Eisenstein polynomial, ω_n is a uniformizer for the field K_n: it follows that the extensions K_n/K are totally ramified, $\mathrm{Gal}(K_n/K) \simeq (A/\mathfrak{p}^{fn})^*$ and $\mathcal{O}_n = \mathcal{O}[[\omega_n]] = \mathcal{O}[\omega_n]$.

The Galois action on $T_\mathfrak{p}\Phi$ is via the "Carlitz–Hayes" character $\chi : G_K \to A_\mathfrak{p}^*$, defined by $\sigma\omega = \chi(\sigma) \cdot \omega$ for $\sigma \in G_K := \mathrm{Gal}(K_{sep}/K)$: that is, $\chi(\sigma)$ is the unique element in $A_\mathfrak{p}^*$ such that $\Phi_{\chi(\sigma)}(\omega_n) = \sigma\omega_n$ for all n.

Lemma 2. *The elements ω_n form a compatible system under the norm maps*:

$$N_n^{n+1}(\omega_{n+1}) = \omega_n.$$

Proof. Either $[K_{n+1} : K_n]$ is odd or the characteristic is 2: in both cases, it follows that $-N_n^{n+1}(\omega_{n+1})$ is the constant term of the minimal polynomial of ω_{n+1} on K_n. It is immediate to see that the latter is $\Phi_\eta(x) - \omega_n$ (it is Eisenstein). $\quad\square$

We also remark that the $\mathrm{Gal}(K_{n+i}/K_n)$-orbit of ω_{n+i} is exactly $\omega_{n+i} + \Phi[\mathfrak{p}^{if}]$.

Lemma 3. *Let $\mathfrak{D}_n := \mathfrak{D}_{K_n/F_\mathfrak{p}}$ be the different of $K_n/F_\mathfrak{p}$: then $\mathfrak{D}_n$ is generated by an element of valuation*

$$nf - \frac{1}{q_\mathfrak{p} - 1}.$$

Moreover, ω_n has valuation

$$v(\omega_n) = [K_n : K]^{-1} = q_\mathfrak{p}^{1-nf}(q_\mathfrak{p} - 1)^{-1}.$$

Proof. The last assertion is obvious from the already remarked fact that ω_n is a uniformizer in a totally ramified extension. As for the first, let ψ_n be the irreducible polynomial of ω_n over H^+, $n \geqslant 1$: then $\mathfrak{D}_n = \mathfrak{D}_{K_n/K}$ because $K/F_\mathfrak{p}$ is unramified and since $\mathfrak{D}_{K_n/K} = (\psi_n'(\omega_n))$ we just need to compute this derivative.

From [7, Proposition 11.4] we get the equality in $B^+[x]$

$$\psi_n(x)\Phi_{\mathfrak{p}^{fn-1}}(x) = \Phi_{\mathfrak{p}^{fn}}(x) = \Phi_{\eta^n}(x).$$

Differentiating and evaluating in ω_n we get

$$v\big(\psi_n'(\omega_n)\big) = nf - v\big(\Phi_{\mathfrak{p}^{fn-1}}(\omega_n)\big) = nf - (q_\mathfrak{p} - 1)^{-1}.$$

(Observe that $\Phi_{\mathfrak{p}^{fn-1}}(\omega_n)$ has valuation $(q_\mathfrak{p} - 1)^{-1}$ because $\Phi_{\mathfrak{p}^{fn-1}}(x)$ is a monic polynomial of degree $q_\mathfrak{p}^{nf-1}$ all whose coefficients but the leading one are in $\mathfrak{m}_K$.) $\quad\square$

Corollary 4. *Let $Tr_n : K_n \to F_\mathfrak{p}$ be the trace map. Then*

$$v\big(Tr_n(x)\big) \geqslant \big\lfloor v(x) + nf - (q_\mathfrak{p} - 1)^{-1} \big\rfloor$$

where $\lfloor r \rfloor$ denotes the largest integer $\leqslant r$. Similarly, for $m \geqslant 1$,

$$v\big(Tr_m^n(x)\big) > v(x) + (n - m)f - v(\omega_m).$$

Proof. Let $k = \lfloor v(x) + nf - (q_{\mathfrak{p}} - 1)^{-1} \rfloor$: then $x\mathcal{O}_n \subseteq \mathfrak{p}^k \mathfrak{D}_n^{-1}$ by Lemma 3 and this means that $Tr_n(x\mathcal{O}_n) \subseteq \mathfrak{p}^k A_{\mathfrak{p}}$, by a basic property of the different (see e.g. [14, III, §3, Proposition 7]).

In the same way, using the fact (obvious from Lemma 3) that the generator of $\mathfrak{D}_{K_n/K_m}$ has valuation $(n - m)f$, one gets

$$v\left(Tr_m^n(x)\right) \geqslant \left\lfloor \frac{v(x) + (n - m)f}{v(\omega_m)} \right\rfloor v(\omega_m);$$

the second statement follows. $\square$

2.5. Local class field theory

The Carlitz–Hayes character induces an isomorphism of topological groups

$$\chi^{-1} : A_{\mathfrak{p}}^* \to \mathrm{Gal}(K_\infty/K)$$

(recall that we put $K_\infty = \bigcup_{n=1}^\infty K_n$). This should be compared with the local norm symbol map

$$(\cdot, K_\infty/F_{\mathfrak{p}}) : F_{\mathfrak{p}}^* \to \mathrm{Gal}(K_\infty/F_{\mathfrak{p}}).$$

By class field theory, the image of $A_{\mathfrak{p}}^*$ in $\mathrm{Gal}(K_\infty/F_{\mathfrak{p}})$ is exactly $\mathrm{Gal}(K_\infty/K)$. Let $a \in A$ be a generator of a prime ideal $\mathfrak{q} \neq \mathfrak{p}$ and assume $sgn(a) = 1$, so that $\Phi_a = \Phi_{\mathfrak{q}}$; moreover, let $Frob_{\mathfrak{q}} \in \mathrm{Gal}(H^+(\Phi[\mathfrak{p}^\infty])/F)$ denote the Frobenius at $\mathfrak{q}$. Passing from local to global class field theory one finds $Frob_{\mathfrak{q}} = (a^{-1}, K_\infty/F_{\mathfrak{p}})$. By [6, Proposition 7.5.4] Φ_a acts on ω as $Frob_{\mathfrak{q}}$ and hence $\chi^{-1}(a) = (a^{-1}, K_\infty/F_{\mathfrak{p}})$.

By Tchebotarev for any n all elements in $\mathrm{Gal}(H^+(\Phi[\mathfrak{p}^n])/H^+)$ can be represented as $Frob_{\mathfrak{q}}$ for some $\mathfrak{q}$ as above. Therefore the corresponding a's are dense in $A_{\mathfrak{p}}^*$ and we get

$$(u, K_\infty/K)(\omega) = \Phi_{u^{-1}}(\omega) \tag{1}$$

for all $u \in A_{\mathfrak{p}}^*$.

3. Coleman's formalism

Let $\mathcal{K}$ be a local field and $\{\mathcal{K}_n\}$ the tower of extensions of $\mathcal{K}$ generated by the torsion of a Lubin–Tate formal module: in [4], Coleman discovered an isomorphism between $\varprojlim \mathcal{K}_n^*$ and the subgroup of $\mathcal{O}_{\mathcal{K}}((x))^*$ fixed under a certain operator $\mathcal{N}$. The same formalism can be used in the context of Drinfeld modules, as follows.

3.1. A bit of functional analysis

Let R be a subring of $\mathbf{C}_{\mathfrak{p}}$: then, as usual, $R((x)) := R[[x]](x^{-1})$ is the ring of formal Laurent series with coefficients in R. Moreover, following [4] we define $R[[x]]_1$ and $R((x))_1$ as the subrings consisting of those (Laurent) power series which converge on the punctured open ball

$$B' := B(0, 1) - \{0\} \subset \mathbf{C}_{\mathfrak{p}}.$$

These latter rings are endowed with a structure of topological R-algebras, induced by the family of seminorms $\{\|\cdot\|_r\}$, where r varies in $|\mathbf{C}_{\mathfrak{p}}| \cap (0, 1)$ and $\|f\|_r := \sup\{|f(z)| : |z| = r\}$. One easily checks that this is the same as the "compact-open" topology of [4, p. 93]: in particular a sequence $\{f_n\}$ in $\mathcal{O}((x))_1$ converges to $f \in \mathcal{O}((x))_1$ if and only if for each closed annulus C around zero in B' and for

each $\epsilon > 0$ there exists a positive integer $N(C, \epsilon)$ such that $|f_n(a) - f(a)| < \epsilon$ for all $a \in C$ and all $n \geqslant N(C, \epsilon)$.

Let $\phi \in \mathcal{O}\{\tau\}$ be an additive polynomial having all its zeroes in $\mathfrak{M}$. One checks easily that, since $|\phi(z)| \leqslant |z|$ for all $z \in B'$, the map $\circ \phi : g \mapsto g \circ \phi$ defines a continuous endomorphism of $K[[x]]_1$.

Lemma 5. *Let ϕ be as above; furthermore assume that $\phi(x)$ is separable. Then the image of $\circ \phi : K[[x]]_1 \to K[[x]]_1$ consists exactly of those g such that $g(x + u) = g(x)$ for all u zeroes of ϕ.*

This is essentially Lemma 3 of [4].

Proof. For $u \in \mathfrak{M}$ let T_u be the automorphism of $\mathbf{C_p}[[x]]_1$ given by $g \mapsto g(x + u)$. The inclusion

$$\circ \phi \big(K[[x]]_1 \big) \subset \bigcap_{\phi(u)=0} \ker(T_u - \mathrm{id})$$

is clear.

Vice versa, assume that $f \in K[[x]]_1$ is T_u-invariant for all u's. Let $K[[x]]_{\bar{r}}$ consist of those power series converging on the closed ball $\bar{B}(0, r)$: then $K[[x]]_1$ is the inverse limit of $K[[x]]_{\bar{r}}$ for $r < 1$. The Weierstrass division theorem holds in each $K[[x]]_{\bar{r}}$ and reasoning as in [4, Lemma 3] one can find f_i's such that

$$f(x) = \sum_{i=0}^{n-1} f_i(0)\phi(x)^i + f_n(x)\phi(x)^n$$

for all $n \geqslant 0$. It remains to show that $f_n \phi^n$ tends to zero: one proves inductively that $\|f_n\|_r \leqslant \|f\|_r \|\phi\|_r^{-n}$, which implies, for $s > r$,

$$\big\| f_n \phi^n \big\|_r \leqslant \|f_n\|_s \|\phi^n\|_r \leqslant \|f\|_s \left(\frac{\|\phi\|_r}{\|\phi\|_s} \right)^n$$

because $\|g\|_r \leqslant \|g\|_s$. To conclude notice that $\|\phi\|_r < \|\phi\|_s$. $\quad \square$

Remark. For the goals of this paper, it would have been enough to prove the weaker statement that $(\circ \phi)(K[[x]]_1)$ is the closure of the subspace of T_u-invariant polynomials. This can be done without Weierstrass theory, as follows.

Suppose $f \in K[x]$ is T_u-invariant for all the zeroes of ϕ: we can assume inductively (taking as first step the constants) that if $g \in K[x]$ enjoys this property and $\deg(g) < \deg(f)$ then g belongs to the image of $\circ \phi$. By the euclidean algorithm for $K[x]$, $f = f_1 \phi + r$; evaluation in the zeroes of ϕ shows that $r = f(0)$ is a constant and then it is immediate to check that f_1 is T_u-invariant. Finally observe that $K[x]$ is dense in $K[[x]]_1$ and the maps are all continuous.

Corollary 6. *The map $\circ \phi$ induces an isomorphism of topological algebras between $K[[x]]_1$ and its image.*

Proof. Observe that $K[[x]]_1$ is a Fréchet space over K (for definitions and basic properties, see e.g. [12, I, §8]). Lemma 5 implies that $\circ \phi(K[[x]]_1)$, being closed, is Fréchet as well. Since $\circ \phi$ is injective, the corollary follows from the open mapping theorem, as in [12, Corollary 8.7]. $\quad \square$

3.1.1. Some topological rings

To enhance clarity, we add a brief digression on topological structures for rings of power series. As above R is a subring of $\mathbf{C_p}$.

We let $R[[x]] \simeq R^{\mathbb{N}}$ be a topological R-algebra with the product topology: that is, a fundamental system of neighbourhoods of 0 is given by

$$U_{\varepsilon,n} := \left\{ \sum a_i x^i \in R[[x]]: |a_i| < \varepsilon \ \forall i < n \right\}.$$

When we just write $R((x))$, we think of it as the additive group with the topology induced by the one on $R[[x]]$ (observe however that with this topology $R((x))$ is not exactly a topological ring, since multiplication by x^{-1} is not continuous).

To compare structures remember that if $\{f_n\}$ converges to f in $R[[x]]_1$ then the individual coefficients of the power series f_n converge to those of f: it follows that the inclusion $R[[x]]_1 \hookrightarrow R[[x]]$ is continuous. Observe, however, that $R((x))_1$ is not continuously injected in $R((x))$: e.g., if $|a| < 1$ then $a^{n!}x^{-n}$ converges to 0 in $\mathcal{O}((x))_1$, but not in $\mathcal{O}((x))$. This example shows as well that $R((x))_1$ is not complete.

When R is a subring of the closed ball $\overline{B}(0,1) \subset \mathbf{C_p}$, $R[[x]]_1 \simeq R[[x]]$ as topological spaces and $R((x))_1 = R((x))$ as sets. Moreover in this case we can furnish $R((x))$ with a third topology, defined (if the restriction of v to R is discrete) by the valuation $v(\sum a_i x^i) := \min_i\{v(a_i)\}$. Once again, continuity of the inclusion $(R[[x]], v) \hookrightarrow R[[x]]_1$ fails to extend to Laurent series.

Lastly we remark that, for ϕ as above, $\circ\phi$ is a continuous endomorphism of $K((x))$ (but, of course, not of $K((x))_1$, unless ϕ has no zeroes in B') and even an automorphism if $\phi(x)$ is separable. In fact $\phi(x)$ separable means that its degree 1 coefficient is not zero, hence one can find $\psi \in K[[x]]$ such that $\psi(\phi(x)) = x$.

3.2. Coleman's theorems

Theorem 7 *(Coleman). There exist unique continuous operators*

$$\mathcal{T}, \mathcal{N} : K((x))_1 \to K((x))_1$$

such that respectively

$$\sum_{u \in \Phi[\mathfrak{p}^f]} g(x+u) = (\mathcal{T}g) \circ \Phi_\eta,$$

$$\prod_{u \in \Phi[\mathfrak{p}^f]} g(x+u) = (\mathcal{N}g) \circ \Phi_\eta.$$

Moreover, $\mathcal{T}$ is a homomorphism of the additive group $K((x))_1$ and $\mathcal{N}$ of $K((x))_1^*$.

Proof. On $K[[x]]_1$ the theorem is an immediate consequence of Corollary 6: $\mathcal{T}$ (respectively $\mathcal{N}$) is just the composition of $(\circ\Phi_\eta)^{-1}$ with $\sum T_u$ (respectively $\prod T_u$).

In order to extend $\mathcal{T}$ and $\mathcal{N}$ to $K((x))_1$, remember that Φ_η belongs to $x\mathcal{O}[x]$: then, if $g \in K((x))_1$, for some $i \geqslant 0$ one has $\Phi_\eta(x)^i g \in K[[x]]_1$ and we put $\mathcal{T}(g) := x^{-i}\mathcal{T}(\Phi_\eta(x)^i g)$, $\mathcal{N}(g) := x^{-q_\mathfrak{p}^{if}}\mathcal{N}(\Phi_\eta(x)^i g)$. These are well defined: e.g., if $g \in K[[x]]_1$,

$$\mathcal{N}(\Phi_\eta(x)g) \circ \Phi_\eta = \prod T_u(\Phi_\eta(x)g) = \prod T_u(\Phi_\eta)T_u(g) = \Phi_\eta^{q_\mathfrak{p}^f}\prod T_u(g) = (x^{q_\mathfrak{p}^f}\mathcal{N}g) \circ \Phi_\eta.$$

Additivity of $\mathcal{T}$ and multiplicativity of $\mathcal{N}$ are immediate. $\quad\square$

As usual, we call $\mathcal{T}$ and $\mathcal{N}$ respectively the Coleman trace and norm.

Lemma 8. *The equality* $\mathcal{N}^k g \circ \Phi_\eta^k = \displaystyle\prod_{u \in \Phi[\mathfrak{p}^{fk}]} g(x+u)$ *holds for any* $g \in K((x))_1^*$.

Proof. Assume by induction that the statement is true up to $k-1$. Let $W \subset \Phi[\mathfrak{p}^{fk}]$ be a set such that $\Phi_\eta : W \to \Phi[\mathfrak{p}^{fk-f}]$ is a bijection. We have the following equalities:

$$\mathcal{N}^k g\big(\Phi_\eta^k(x)\big) = \mathcal{N}^{k-1}\mathcal{N}g\big(\Phi_\eta^{k-1}\big(\Phi_\eta(x)\big)\big) = \prod_{v \in \Phi[\mathfrak{p}^{fk-f}]} \mathcal{N}g\big(\Phi_\eta(x)+v\big) = \prod_{w \in W} \mathcal{N}g\big(\Phi_\eta(x+w)\big)$$

$$= \prod_{w \in W}(\mathcal{N}g \circ \Phi_\eta)(x+w) = \prod_W \prod_{u \in \Phi[\mathfrak{p}^f]} g(x+u+w) = \prod_{t \in \Phi[\mathfrak{p}^{fk}]} g(x+t). \quad \square$$

Lemma 9. *One computes*

$$\big(\mathcal{N}^k g\big)(\omega_n) = N_n^{n+k}\big(g(\omega_{n+k})\big).$$

Similarly, $\mathcal{T}^k g(\omega_n) = Tr_n^{n+k}(g(\omega_{n+k}))$.

Proof. Replace $\omega_n = \Phi_\eta^k(\omega_{n+k})$ and apply Lemma 8. $\quad \square$

Lemma 10. *The restriction to* $\mathcal{O}((x))_1^*$ *of the sequence of operators* $\mathcal{N}^k$ *converges to a continuous endomorphism* $\mathcal{N}^\infty$.

Proof. Observe that $\mathcal{O}((x))_1^* = x^{\mathbb{Z}} \times \mathcal{O}[[x]]^*$ and that $\mathcal{N}x = x$. Therefore the lemma is proven if we show that, for any $g \in \mathcal{O}[[x]]^*$, $\mathcal{N}^k g$ is a Cauchy sequence with respect to the valuation topology on $\mathcal{O}[[x]]$ (uniformly on g). More precisely, we are going to prove that $\mathcal{N}^{k+1}g \equiv \mathcal{N}^k g \bmod \mathfrak{m}_K^{k+1}$ by induction on k.

First notice that

$$\Phi_\eta(x) = \prod_{u \in \Phi[\mathfrak{p}^f]}(x-u) \equiv x^{q_{\mathfrak{p}}^f} \quad \bmod \mathfrak{m}_K$$

since $v(u) > 0$ for $u \in \Phi[\mathfrak{p}^f]$. Therefore

$$g^{q_{\mathfrak{p}}^f} \equiv \prod g(x+u) = \mathcal{N}g \circ \Phi_\eta \equiv \mathcal{N}g\big(x^{q_{\mathfrak{p}}^f}\big) \equiv \big(\mathcal{N}g(x)\big)^{q_{\mathfrak{p}}^f} \quad \bmod \mathfrak{m}_K$$

where the last congruence is true because $q_{\mathfrak{p}}^f = |\mathcal{O}_K/\mathfrak{m}_K|$. It follows that $\mathcal{N}g \equiv g \bmod \mathfrak{m}_K$.

Now put $h := \dfrac{\mathcal{N}^k g}{\mathcal{N}^{k-1}g}$, so that our claim becomes $\mathcal{N}h \equiv 1 \bmod \mathfrak{m}_K^{k+1}$. By the induction hypothesis $h = 1 + \pi^k g_1$ and this implies

$$(\mathcal{N}h \circ \Phi_\eta)(x) = \prod_{u \in \Phi[\mathfrak{p}^f]}\big(1 + \pi^k g_1(x+u)\big) \equiv \big(1 + \pi^k g_1(x)\big)^{q_{\mathfrak{p}}^f} \equiv 1 \quad \bmod \mathfrak{m}_K^{k+1}.$$

To conclude, observe that since Φ_η is monic $v(h \circ \Phi_\eta) = v(h)$ for any h (where v is the valuation on $\mathcal{O}[[x]]$ defined in Section 3.1.1). $\quad \square$

Of course $\mathcal{N} \circ \mathcal{N}^\infty = \mathcal{N}^\infty$ and $\mathcal{N}^\infty$ is a projection.

Theorem 11 *(Coleman). The evaluation map* $ev : f \mapsto \{f(\omega_n)\}$ *gives an isomorphism*

$$\left(\mathcal{O}((x))^*\right)^{\mathcal{N}=\mathrm{id}} \simeq \varprojlim K_n^*$$

where the inverse limit is taken with respect to the norm maps.

Proof. The map is injective, because a function is uniquely determined by its values at the ω_n's (e.g., observe that $|\omega_n| < 1$ and use [6, Proposition 2.11]).

Notice that $\mathcal{O}((x))^* = x^{\mathbb{Z}} \times \mathcal{O}[[x]]^*$ and $\varprojlim K_n^* = \omega^{\mathbb{Z}} \times \varprojlim \mathcal{O}_n^*$: since $ev(x) = \omega$, it suffices to show $(\mathcal{O}[[x]]^*)^{\mathcal{N}=\mathrm{id}} \simeq \varprojlim \mathcal{O}_n^*$.

Consider the diagram

$$
\begin{array}{ccc}
\mathcal{O}[[x]]^* & \xrightarrow{\ ev\ } & \prod \mathcal{O}_n^* \\
\downarrow{\scriptstyle \mathcal{N}/\mathrm{id}} & & \downarrow{\scriptstyle N/\mathrm{id}} \\
\mathcal{O}[[x]]^* & \xrightarrow{\ ev\ } & \prod \mathcal{O}_n^*
\end{array}
$$

where N is the norm map $(x_n) \mapsto (N_n^{n+1} x_{n+1})$. It commutes by Lemma 9; $\varprojlim \mathcal{O}_n^*$ is the kernel of the right-hand side: hence $ev(g) \in \varprojlim \mathcal{O}_n^*$ iff $\mathcal{N}g = g$. Since $(\mathcal{O}[[x]]^*)^{\mathcal{N}=\mathrm{id}} = \mathcal{N}^\infty(\mathcal{O}[[x]]^*)$ is compact (because so is $\mathcal{O}[[x]]^*$), the theorem is proven if we show that the image of ev is dense in a set containing $\varprojlim \mathcal{O}_n^*$. For this we use the argument given in [15, p. 307]. For any $u = (u_n) \in \varprojlim \mathcal{O}_n^*$ and any k there exists $g \in \mathcal{O}[[x]]^*$ such that $g(\omega_{2k}) = u_{2k}$. Let $h := \mathcal{N}^k g$. Remembering (from the proof of Lemma 10) that $\mathcal{N}^k g \equiv \mathcal{N}^{k+r} g \mod \mathfrak{m}^k$ for any $r \geqslant 0$, we get

$$h(\omega_i) \equiv \mathcal{N}^{2k-i} g(\omega_i) = N_i^{2k}\big(g(\omega_{2k})\big) = u_i \mod \mathfrak{m}^k$$

for all $i = 1, \dots, k$: density follows. $\square$

In particular, $\omega \in \varprojlim \mathcal{O}_n^*$ corresponds to $x \in \mathcal{O}((x))^*$ and, more generally, for $a \in A_{\mathfrak{p}}^*$ the element $a \cdot \omega$ corresponds to $\Phi_a(x) \in \mathcal{O}((x))^*$ (this is equivalent to changing generator of $T_{\mathfrak{p}}\Phi$).

We conclude with a lemma we are going to use in the next section.

Lemma 12. *Let* $\mathrm{dlog} : K((x))_1^* \to K((x))_1$ *be the logarithmic derivative operator,* $\mathrm{dlog}(g) := \frac{g'}{g}$. *Then*

$$\mathcal{T} \, \mathrm{dlog}\, g = \eta \, \mathrm{dlog}\, \mathcal{N}g.$$

Proof. One just computes

$$(\mathcal{T} \, \mathrm{dlog}\, g) \circ \Phi_\eta = \sum T_u(\mathrm{dlog}\, g) = \mathrm{dlog} \prod T_u(g) = \mathrm{dlog}(\mathcal{N}g \circ \Phi_\eta) = \eta\big(\mathrm{dlog}(\mathcal{N}g) \circ \Phi_\eta\big)$$

using the fact that dlog is a homomorphism and $\frac{d}{dx}\Phi_\eta(x) = \eta$. $\square$

3.3. Higher rank Drinfeld modules

As the reader may have noticed, the rank of the Drinfeld module Φ plays essentially no part in Theorem 7. This suggests the possibility of extending Coleman's results to Drinfeld modules of any rank: we sketch an approach.

In this subsection (and only here) our notations are slightly modified. For simplicity we take $F := \mathbb{F}_q(T)$ and $A := \mathbb{F}_q[T]$: then, fixing a prime ideal $\mathfrak{p} = (\pi)$ of A, we have $K = F_{\mathfrak{p}}$ and $\mathcal{O} = A_{\mathfrak{p}}$. Let

$\Phi : A \to \mathcal{O}\{\tau\}$ be a rank r Drinfeld module, $r > 1$; we need the hypothesis that Φ has good reduction mod $\mathfrak{m}_K$ and that the height is maximal: $h = r$. In particular, it follows that all zeroes of Φ_π are in the open unit ball $B(0, 1) \subset \mathbf{C_p}$. We also assume that $\Phi_\pi(x)$ is a monic polynomial. Then, reasoning exactly as in Theorem 7 and Lemma 10, one proves the following.

Theorem 13. *There exists a continuous homomorphism* $\mathcal{N} : K((x))_1^* \to K((x))_1^*$ *such that*

$$\prod_{u \in \Phi[\mathfrak{p}]} g(x + u) = (\mathcal{N}g) \circ \Phi_\pi.$$

The restriction to $\mathcal{O}((x))_1^*$ *of the sequence of operators* $\mathcal{N}^k$ *converges to a continuous endomorphism* $\mathcal{N}^\infty$.

As in Section 2.4, we choose a sequence $\{\omega_n\}_{n \geqslant 1}$ so that

$$\Phi_\pi^n(\omega_n) = 0 \neq \Phi_\pi^{n-1}(\omega_n) \quad \text{and} \quad \Phi_\pi(\omega_{n+1}) = \omega_n$$

and construct a tower $\{K_n\}$ by $K_n := K_{n-1}(\omega_n)$, with $K_0 := K$. Because of the rank, these extensions are much smaller than $K(\Phi[\mathfrak{p}^n])$ and they are not Galois; however, since the polynomials $\Phi_\pi(x)x^{-1}$ and $\Phi_\pi(x) - \omega_n$ are Eisenstein, it still holds that each K_n/K is totally ramified, with uniformizer ω_n.

For any $n \geqslant 1$ there is a norm map

$$N_n^{n+1} : K_{n+1}^* \to K_n^*, \quad a \mapsto \prod_{\sigma \in S_{n+1}} \sigma(a)$$

where the product is taken on the set of embeddings

$$S_{n+1} := \{\sigma : K_{n+1} \hookrightarrow \mathbf{C_p} : \sigma|_{K_n} = \mathrm{id}_{K_n}\}.$$

It follows from the additivity of Φ_π that the assignment $\sigma \mapsto \sigma(\omega_{n+1}) - \omega_{n+1}$ is a bijection $S_{n+1} \to \Phi[\mathfrak{p}]$: therefore $N_n^{n+1}(\omega_{n+1}) = \omega_n$ and, more generally,

$$(\mathcal{N}g)(\omega_n) = \mathcal{N}g(\Phi_\pi(\omega_{n+1})) = \prod_{u \in \Phi[\mathfrak{p}]} g(\omega_{n+1} + u) = N_n^{n+1}(g(\omega_{n+1})).$$

Theorem 14. *The evaluation map* $ev : f \mapsto \{f(\omega_n)\}$ *gives an isomorphism*

$$(\mathcal{O}((x))^*)^{\mathcal{N}=\mathrm{id}} \simeq \varprojlim K_n^*.$$

The proof is the same as for Theorem 11.

4. The explicit reciprocity law

The reader is reminded that Tr_n, N_n denote respectively trace and norm from K_n to $F_\mathfrak{p}$. Also, we let

$$(\cdot, L^{ab}/L) : L^* \to G_L^{ab}$$

be the local norm symbol map and write Col_u for the power series in $\mathcal{O}((x))^*$ associated to $u \in \varprojlim K_n^*$ by Coleman's isomorphism of Theorem 11. To lighten notation, in this section the action of $A_\mathfrak{p}$ via Φ will be often denoted by $a \cdot x := \Phi_a(x)$.

4.1. The Kummer pairing

Next to $\varprojlim K_n^*$ we consider $\varinjlim K_n$, defined as the direct limit of the maps $\Phi_\eta : K_n \to K_{n+1}$: that is, $\varinjlim K_n$ consists of sequences $a = (a_n)_{n \geqslant N}$ (for some $N \in \mathbb{N}$) such that $a_N \in K_N$ and $a_{n+1} = \Phi_\eta(a_n)$, modulo the relation $(a_n)_{n \geqslant N} = (b_n)_{n \geqslant M}$ if $a_n = b_n$ for $n \gg 0$.

Kummer theory yields a pairing $(,)_n : K_n \times K_n^* \to \Phi[\mathfrak{p}^{fn}]$ defined by

$$(a, u)_n := \left((u, K_n^{ab}/K_n) - 1\right)({}_\eta\sqrt[n]{a}).$$

Here ${}_\eta\sqrt[n]{a}$ is a solution of $\Phi_\eta^n(x) = a$: since any two roots differ by an element in $\Phi[\mathfrak{p}^{fn}] \subset K_n$, the value of $(a, u)_n$ is independent of the choice of ${}_\eta\sqrt[n]{a}$.

Observe that, since by definition ${}_\eta\sqrt[n]{a} = {}_\eta\sqrt[n+1]{\Phi_\eta(a)}$,

$$\left(\Phi_\eta(a), u\right)_{n+1} = \left((u, K_{n+1}^{ab}/K_{n+1}) - 1\right)\left({}_\eta\sqrt[n+1]{\Phi_\eta(a)}\right) = \left((N_n^{n+1}(u), K_n^{ab}/K_n) - 1\right)({}_\eta\sqrt[n]{a}) = \left(a, N_n^{n+1}(u)\right)_n.$$

This means that, given $a = (a_n) \in \varinjlim K_n$ and $u = (u_n) \in \varprojlim K_n^*$, one has (for any n large enough that a_n exists)

$$(a_{n+1}, u_{n+1})_{n+1} = (a_n, u_n)_n.$$

Therefore we can define a limit form of the Kummer pairing

$$(,) : \varinjlim K_n \times \varprojlim K_n^* \to \Phi[\mathfrak{p}^\infty]$$

by $(a, u) := (a_n, u_n)_n$ for $n \gg 0$.

One checks immediately that $(,)$ is bilinear, additive in the first variable and multiplicative in the second. In particular, since values are in a group of exponent p, it follows that $(\cdot, \zeta)_n = 0$ for any root of unity $\zeta \in K_n^*$.

Lemma 15. *All the pairings $(,)_n$ are continuous. Furthermore, $(a, \cdot)_n \equiv 0$ for any $a \in K_n$ such that $v(a) > nf + (q_\mathfrak{p} - 1)^{-1}$.*

Proof. For any $a \in K_n$, $(a, \cdot)_n$ is continuous: therefore the first assertion follows from the second, which in turn is an easy application of Krasner's Lemma. In fact, if one can choose $\alpha = {}_\eta\sqrt[n]{a}$ so that $v(\alpha)$ is big enough, then it follows $(a, u) = 0$ because $|(a, u)| \leqslant |\alpha| = |(u, K_n^{ab}/K_n)(\alpha)|$ and $\Phi[\mathfrak{p}^{nf}]$ is a discrete subset of $\mathbf{C}_\infty$.

We are left with a valuation computation. First of all, observe that for any $j \geqslant 1$ if $u \in \Phi[\mathfrak{p}^j] - \Phi[\mathfrak{p}^{j-1}]$ then

$$v_j := v(u) = \left[K\left(\Phi[\mathfrak{p}^j]\right) : K\right]^{-1} = \left|(A/\mathfrak{p}^j)^*\right|^{-1} = \frac{1}{q_\mathfrak{p}^{j-1}(q_\mathfrak{p} - 1)}$$

(because $\dfrac{\Phi_{\mathfrak{p}^j}(x)}{\Phi_{\mathfrak{p}^{j-1}}(x)}$ is Eisenstein). In particular the smallest non-zero elements in $\Phi[\mathfrak{p}^{nf}]$ have valuation $v_1 = (q_\mathfrak{p} - 1)^{-1}$. We also put $v_0 := \infty$.

Now choose α a root of $\Phi_\eta^n(x) = a$ such that $v(\alpha)$ is maximal: we get

$$a = \prod_{u \in \Phi[\mathfrak{p}^{fn}]} (\alpha + u) \quad \text{and} \quad v(a) = \sum_{u \in \Phi[\mathfrak{p}^{fn}]} v(\alpha + u),$$

with $v(\alpha + u) = \min\{v(\alpha), v(u)\}$ for all u's because of the maximality hypothesis. Hence if $v_j \geqslant v(\alpha) > v_{j+1}$ we obtain

$$v(a) = \sum_{u \in \Phi[\mathfrak{p}^j]} v(\alpha) + \sum_{u \in \Phi[\mathfrak{p}^{nf}] - \Phi[\mathfrak{p}^j]} v(u) = q_{\mathfrak{p}}^j v(\alpha) + nf - j$$

and $1 + nf - j + (q_{\mathfrak{p}} - 1)^{-1} = q_{\mathfrak{p}}^j v_j + nf - j \geqslant v(a) > nf - j + (q_{\mathfrak{p}} - 1)^{-1}$. $\square$

For a more detailed analysis of how $v(a)$ determines the extension $K_1(\sqrt[n]{a})/K_1$ see [1, Proposition 2.1].

We remark that the computation in the proof of Lemma 15 yields immediately the following result.

Lemma 16. *For any $a = (a_n)_{n \geqslant N} \in \varinjlim \mathfrak{m}_n$ there exists a constant $c(a)$ such that $v(a_n) \geqslant nf - c(a)$ for all $n \geqslant N$.*

Lemma 17. *Let $a \in K_n^*$: then $(a, a)_n = 0$.*

Proof. Let α be a representative of $\sqrt[n]{a}$ and put $L := K_n(\alpha)$. Kummer theory identifies $\mathrm{Gal}(L/K_n)$ with a subgroup V of $\Phi[\mathfrak{p}^{nf}]$: then one sees that

$$a = \prod_{u \in \Phi[\mathfrak{p}^{nf}]/V} \prod_{v \in V} (\alpha + u + v) = \prod_{\Phi[\mathfrak{p}^{nf}]/V} N_{L/K_n}(\alpha + u)$$

and consequently $(a, K_n^{ab}/K_n)$ acts trivially on L. $\square$

Lemma 18. *Let $b \in \mathfrak{m}_n - \{0\}$: then $(c, 1 - b)_n = (\frac{bc}{1-b}, b^{-1})_n$ for all $c \in K_n$.*

Proof. If $c = 0$ both sides are 0. If not, by applying Lemma 17 to $a = c(1 - b)$ we get, by bilinearity,

$$(c, 1 - b)_n = (cb, c)_n + (cb, 1 - b)_n$$

and, by recurrence,

$$(c, 1 - b)_n = \sum_{j=1}^{\infty} (cb^j, cb^{j-1})_n$$

(the sum converges because only a finite number of terms are not 0, by Lemma 15).

By Lemma 17 we have $(cb^j, cb^{j-1})_n = (cb^j, cb^j)_n + (cb^j, b^{-1})_n = (cb^j, b^{-1})_n$ and therefore

$$(c, 1 - b)_n = \sum_{j=1}^{\infty} (cb^j, b^{-1})_n = \left(c \sum_{j=1}^{\infty} b^j, b^{-1} \right)_n = \left(\frac{cb}{1-b}, b^{-1} \right)_n.   \square$$

Proposition 19. *Let $a \in \varinjlim \mathfrak{m}_n$, $u \in \varprojlim K_n^*$: then*

$$(a, u) = \left(a_n \omega_n \, \mathrm{dlog} \, Col_u(\omega_n), \omega_n \right)_n$$

for all n sufficiently large.

Proof. The statement is clearly true for $u = \omega$. As for $u \in \varprojlim \mathcal{O}_n^*$, we are going to prove that, more generally,

$$(c, w)_n = \left(c\omega_n \frac{\mathrm{dlog}\, w}{d\omega_n}, \omega_n \right)_n$$

for all $(c, w) \in \mathfrak{m}_n^{s_n} \times \mathcal{O}_n^*$ for some s_n. Here $\mathrm{dlog} : \mathcal{O}_n^* \to \Omega_{\mathcal{O}_n/\mathcal{O}}$ is the map $x \mapsto \frac{dx}{x}$. The module of differentials is free over $\mathcal{O}_n/\mathfrak{D}_n$ with generator $d\omega_n$: by Lemma 3 if $c \in \mathfrak{m}_n$ with $v(c) > \frac{2}{q_{\mathfrak{p}} - 1} - v(\omega_n)$ and $\delta \in \mathfrak{D}_n$ the inequality

$$v(c\omega_n\delta) \geqslant v(c) + v(\omega_n) + nf - \frac{1}{q_{\mathfrak{p}} - 1} > nf + \frac{1}{q_{\mathfrak{p}} - 1}$$

holds and so Lemma 15 shows that $(c\omega_n \frac{\mathrm{dlog}\, w}{d\omega_n}, \omega_n)_n$ is well defined. Thanks to Lemma 16 this is enough for our purposes.

Observe that it suffices to prove the claim for $w = 1 - \zeta\omega_n^k$ (ζ varying among roots of unity in K_n), because one can choose a topological basis of $1 + \mathfrak{m}_n$ consisting of elements of this form and the pairing is continuous and linear.

Applying Lemma 18 with $x = \zeta\omega_n^k$ we get

$$(c, 1 - \zeta\omega_n^k)_n = \left(\frac{c\zeta\omega_n^k}{1 - \zeta\omega_n^k}, \omega_n^{-k} \right)_n = \left(c\omega_n \frac{\mathrm{dlog}(1 - \zeta\omega_n^k)}{d\omega_n}, \omega_n \right)_n$$

because

$$\mathrm{dlog}\big(1 - \zeta\omega_n^k\big) = \frac{-k\zeta\omega_n^{k-1}}{1 - \zeta\omega_n^k} d\omega_n.$$

To conclude just notice that, for $u \in \varprojlim K_n^*$, $\mathrm{dlog}\, Col_u(\omega_n) = \frac{\mathrm{dlog}\, u_n}{d\omega_n}$. (Caveat! In this last formula the symbol dlog appears with two different meanings: on the left-hand side $\mathrm{dlog}\, Col_u$ is the power series $\frac{1}{Col_u(x)} \frac{d}{dx} Col_u(x)$, evaluated in ω_n, while on the right-hand side $\mathrm{dlog} : \mathcal{O}_n^* \to \Omega_{\mathcal{O}_n/\mathcal{O}}$ is the map we defined above.) $\quad\square$

4.2. The analytic pairing

As above, let $u \in \varprojlim K_n^*$. Lemmata 9 and 12 together with the $\mathcal{N}$-invariance of Col_u yield

$$Tr_n^{n+k} \mathrm{dlog}\, Col_u(\omega_{n+k}) = \eta^k \mathrm{dlog}\, Col_u(\omega_n).$$

Besides, for $a = (a_n) \in \varinjlim \mathfrak{m}_n$, $\lambda(a_{n+k}) = \eta^k \lambda(a_n)$ by definition of λ (Proposition 1). It follows that

$$Tr_{n+k}\big(\eta^{-n-k}\lambda(a_{n+k}) \mathrm{dlog}\, Col_u(\omega_{n+k})\big) = \eta^k Tr_n\big(\eta^{-n}\lambda(a_n) \mathrm{dlog}\, Col_u(\omega_n)\big).$$

Lemma 20. $Tr_n(\eta^{-n}\lambda(a_n) \mathrm{dlog}\, Col_u(\omega_n)) \in A_{\mathfrak{p}}$ for $n \gg 0$.

Proof. By Lemma 16, $v(a_n) \geqslant nf - c$ for some constant c (depending on a): Proposition 1 implies that the same is true for $\lambda(a_n)$. Since $Col_u \in x^{\mathbb{Z}} \times \mathcal{O}[[x]]^*$ one has $v(\mathrm{dlog}\, Col_u(\omega_n)) \geqslant -v(\omega_n)$ and the last value, in turn, is controlled by Lemma 3. Now apply Corollary 4. $\quad\square$

Therefore one can define a second pairing

$$[\,,\,] : \varinjlim \mathfrak{m}_n \times \varprojlim K_n^* \to \Phi\big[\mathfrak{p}^\infty\big]$$

putting

$$[a, u] := Tr_n\big(\eta^{-n}\lambda(a_n)\,\mathrm{dlog}\,Col_u(\omega_n)\big) \cdot \omega_n$$

for $n \gg 0$. (Recall that, by definition, $a \cdot \omega_n = \Phi_a(\omega_n)$.)

4.2.1.   It is convenient to define also a level n pairing $[\,,\,]_n : \mathfrak{m}_n^{t_n} \times K_n^* \to \Phi[\mathfrak{p}^{nf}]$ for some $t_n \geqslant 1$.
    The logarithmic differential we used in the proof of Proposition 19 can be extended to a homomorphism

$$\frac{\mathrm{dlog}}{d\omega_n} : K_n^* = \omega_n^{\mathbb{Z}} \times \mathcal{O}_n^* \to \mathfrak{m}_n^{-1}/\mathfrak{D}_n$$

by putting $\dfrac{\mathrm{dlog}\,\omega_n^i}{d\omega_n} := i\omega_n^{-1}$.

Lemma 21. *The pairing*

$$[a, u]_n := Tr_n\left(\eta^{-n}\lambda(a)\frac{\mathrm{dlog}\,u}{d\omega_n}\right) \cdot \omega_n$$

is well defined for $v(a) \geqslant 2(q-1)^{-1}$.

Proof. We have to show that $Tr_n(\eta^{-n}\lambda(a)b)$ belongs to $A_\mathfrak{p}$ for any $b \in \mathfrak{m}_n^{-1}$ and that

$$v\big(Tr_n\big(\eta^{-n}\lambda(a)\delta\big)\big) \geqslant nf$$

for $\delta \in \mathfrak{D}_n$. Since the hypothesis implies $v(\lambda(a)) = v(a)$, both assertions are easy consequences of Lemma 3 and Corollary 4. □

    It is clear that if $(a, u) \in \varinjlim \mathfrak{m}_n \times \varprojlim K_n^*$, the equality $[a, u] = [a_n, u_n]_n$ holds for $n \gg 0$.

Proposition 22. *For n large enough, $[a, u] = [a_n\omega_n\,\mathrm{dlog}\,Col_u(\omega_n), \omega_n]_n$.*

Proof. To lighten notation, put $D = \omega_n\,\mathrm{dlog}\,Col_u(\omega_n)$; observe that $D \in \mathcal{O}_n$. One has to check that

$$Tr_n\big(\eta^{-n}\lambda(a_n)D\omega_n^{-1}\big) \cdot \omega_n = Tr_n\big(\eta^{-n}\lambda(a_nD)\omega_n^{-1}\big) \cdot \omega_n,$$

i.e. that

$$v\left(Tr_n\left(\frac{\lambda(a_nD) - \lambda(a_n)D}{\omega_n}\right)\right) \geqslant 2nf.$$

By Proposition 1,

$$v\big(\lambda(a_nD) - \lambda(a_n)D\big) = v\left(\sum_{i=1}^{\infty} c_i a_n^{q^i}\big(D^{q^i} - D\big)\right) \geqslant \min_{i \geqslant 1}\{q^i v(a_n) - i\}$$

which for $n \gg 0$ is at least $2nf - c$ for some c independent of n, by Lemma 16. Now apply Corollary 4 to get rid of c. $\quad\square$

4.3. The reciprocity law

In this paragraph we prove Wiles' explicit reciprocity law for rank 1 Drinfeld modules (Theorem 24).

Proposition 23. *Let* $a \in \mathfrak{m}_n$, $v(a) \geqslant 2(q-1)^{-1}$: *then* $[a, \omega_n]_n = (a, \omega_n)_n$.

Proof. Our proof is divided in many steps, along the lines of [5, I, §4].

1. To start with, we put $a_m := \Phi_\eta^{m-n}(a)$ for all $m \geqslant n$ and let $b_m := a_m \omega_m^{-1}$. Thanks to Lemmata 16 and 3 there is a constant c (depending only on a) such that $v(b_m) \geqslant mf - c$.

As a consequence of Lemma 17 we get

$$0 = \left(a_m + \omega_m, (1 + b_m)\omega_m\right)_m = (a_m, \omega_m)_m + (a_m, 1 + b_m)_m + (\omega_m, 1 + b_m)_m$$

for $m \geqslant n$.

2. Claim. *If* $m \gg 0$, *then* $(a_m, 1 + b_m)_m = 0$.

Proof. By definition of the Kummer pairing, for $m > n$ we have

$$(a_m, 1 + b_m)_m = \left(a_n, N_n^m(1 + b_m)\right)_n.$$

Since $1 + b_m$ tends to 1, so does also $N_n^m(1 + b_m)$: the claim is proven because $(a, \cdot)_n$ is continuous and $\Phi[\mathfrak{p}^{nf}]$ discrete. $\quad\square$

3. Claim. $(a_n, \omega_n)_n = \omega_{2m} - \Phi_{N_m(1+b_m)^{-1}}(\omega_{2m})$ *for* $m \gg 0$.

Proof. From the above, we have

$$(a_n, \omega_n)_n = (a_m, \omega_m)_m = -(\omega_m, 1 + b_m)_m = \left(1 - \left(1 + b_m, K_m^{ab}/K_m\right)\right)\left({}_\eta\sqrt[m]{\omega_m}\right).$$

We can take ${}_\eta\sqrt[m]{\omega_m} = \omega_{2m}$. The extension $K_{2m}/F_\mathfrak{p}$ is abelian: hence

$$\left(1 + b_m, K_m^{ab}/K_m\right)_{|K_{2m}} = \left(N_m(1 + b_m), F_\mathfrak{p}^{ab}/F_\mathfrak{p}\right)_{|K_{2m}}.$$

Now one applies formula (1). $\quad\square$

4. Claim. $N_m(1 + b_m)^{-1} \equiv 1 - Tr_m(b_m) \bmod \mathfrak{p}^{2mf}$ *for* m *big enough.*

Proof. Take $k \in \mathbb{N}$ such that $kf > c + 1$ where c is the constant which appeared in step 1. We can assume $m \gg k$.

Put $\beta := Tr_{m-k}^m(b_m)$: then $v(\beta) \geqslant v(b_m) + kf - v(\omega_{m-k})$ by Corollary 4. We have

$$N_{m-k}^m\left(\frac{1}{1 + b_m}\right) = 1 - \beta + \delta,$$

where δ is an element of K_{m-k} such that $v(\delta) \geqslant 2v(b_m)$. Now apply N_{m-k} to the above equation to obtain

$$N_m(1 + b_m)^{-1} = N_{m-k}(1 - \beta + \delta) = 1 - Tr_{m-k}(\beta - \delta) + \theta \equiv 1 - Tr_m(b_m) \mod \mathfrak{p}^{2mf},$$

because summands in θ have valuation at least

$$2v(\beta - \delta) \geqslant 2 \min\{(m + k)f - c - v(\omega_{m-k}), 2mf - 2c\}$$

and $v(Tr_{m-k}(\delta)) \geqslant \lfloor (3m - k)f - 2c - (q_\mathfrak{p} - 1)^{-1} \rfloor$. $\quad \square$

5. From steps 3 and 4 we get $(a, \omega_n)_n = Tr_m(a_m \omega_m^{-1}) \cdot \omega_{2m}$. On the other hand $[a, \omega_n]_n = [(a_m), \omega] = [a_m, \omega_m]_m$. Since $Col_\omega = x$, $\mathrm{dlog}\, Col_\omega = \frac{1}{x}$ and by definition we obtain

$$[a_m, \omega_m]_m = Tr_m\left(\frac{\lambda(a_m)}{\eta^m \omega_m}\right) \cdot \omega_m = \frac{1}{\eta^m} Tr_m\left(\frac{\lambda(a_m)}{\omega_m}\right) \cdot (\eta^m \cdot \omega_{2m}) = Tr_m\left(\frac{\lambda(a_m)}{\omega_m}\right) \cdot \omega_{2m}.$$

The proof is completed by the same reasoning as in Proposition 22. $\quad \square$

Combining Propositions 19, 22 and 23, we get our reciprocity law:

Theorem 24. *The two pairings* $(\,,)$ *and* $[\,,]$ *on* $\varprojlim \mathfrak{m}_n \times \varprojlim K_n^* \to \Phi[\mathfrak{p}^\infty]$, *defined respectively by*

$$[a, u] := Tr_n\left(\eta^{-n} \lambda(a_n)\, \mathrm{dlog}\, Col_u(\omega_n)\right) \cdot \omega_n$$

and

$$(a, u) := \left((u_n, K_n^{ab}/K_n) - 1\right)\left({}_\eta \sqrt[n]{a_n}\right)$$

for $n \gg 0$, *are equal.*

4.3.1. The Kummer pairing in practice

A weaker form of our explicit reciprocity law can be used to calculate the Kummer pairing also when $u_n \in K_n^*$ is not a coordinate in an inverse limit or, even if it is the case, one does not know how to explicitly find Col_u.

Given $u_n \in K_n^*$ and $a_n \in \mathfrak{m}_n$ we want to compute $(a_n, u_n)_n$. We need to impose that there exists u_m for some convenient $m \geqslant n$ such that $N_n^m(u_m) = u_n$. If so we have

$$(a_n, u_n)_n = \left(\Phi_\eta(a_n), u_{n+1}\right)_{n+1} = \cdots = \left(\Phi_\eta^j(a_n), u_{n+j}\right)_{n+j}$$

for any integer $0 \leqslant j \leqslant m - n$. Put $a_{n+j} := \Phi_\eta^j(a_n)$.

In particular suppose we can take m big enough to have $v(a_m) > 2/(q - 1)$ (and hence $v(a_m) > \frac{2}{q_\mathfrak{p}-1} - v(\omega_n)$). (An estimate on the required size of $m - n$ can be obtained from the computations proving Lemma 16.) Then

$$(a_m, u_m)_m = \left(a_m \omega_m \frac{\mathrm{dlog}\, u_m}{d\omega_m}, \omega_m\right)_m$$

(see the proof of Proposition 19) and by Proposition 23 we have

$$(a_m, u_m)_m = \left(a_m \omega_m \frac{d\log u_m}{d\omega_m}, \omega_m \right)_m = \left[a_m \omega_m \frac{d\log u_m}{d\omega_m}, \omega_m \right]_m.$$

By Proposition 1 we have $v(\lambda(z) - z) \geqslant \min_{i \geqslant 1}\{q^i v(z) - i\}$; in particular this minimum is attained in $i = 1$ when $v(z) > 2(q-1)^{-1}$. Hence a simple computation shows that the further condition $v(a_m) \geqslant \frac{1}{q}(mf + 1 + (q_{\mathfrak{p}} - 1)^{-1} + v(\omega_m))$ implies

$$\left[a_m \omega_m \frac{d\log u_m}{d\omega_m}, \omega_m \right]_m = Tr_m \left(\frac{a_m}{\eta^m} \frac{d\log u_m}{d\omega_m} \right) \cdot \omega_m = [a_m, u_m]_m.$$

The limit form of the Kummer pairing, as in Theorem 24, is useful rather for the purposes of Iwasawa theory (which is not yet well understood in our setting) than for a concrete calculation of the pairing at a level n. Historically, the interest in computing the Kummer pairing for local fields at a finite level was originated by the study of diophantine equations, in particular the Fermat one. For a survey on various explicit reciprocity laws in the local case we refer to [3].

Acknowledgments

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