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On the well-posed coupling between free fluid and porous viscous flows

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ABSTRACT

We present a well-posed model for the Stokes/Brinkman problem with a family of *jump embedded boundary conditions (J.E.B.C.)* on an immersed interface with weak regularity assumptions. It arises from a general framework recently proposed for fictitious domain problems. Our model is based on algebraic transmission conditions combining the stress and velocity jumps on the interface Σ separating the fluid and porous domains. These conditions are well chosen to get the coercivity of the operator. Then, the general framework allows us to prove new results on the global solvability of some models with physically relevant stress or velocity jump boundary conditions for the momentum transport at a fluid–porous interface. The Stokes/Brinkman problem with Ochoa-Tapia and Whitaker (1995) [9,10] interface conditions and the Stokes/Darcy problem with Beavers and Joseph (1967) [13] conditions are both proved to be well-posed, by an asymptotic analysis. Up to now, only the Stokes/Darcy problem with Saffman (1971) [15] approximate interface conditions with negligible tangential porous velocity was known to be well-posed.

1. Introduction

Notation. Let the domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3 in practice) be an open bounded and Lipschitz continuous domain. Let an interface $\Sigma \subset \mathbb{R}^{d-1}$, Lipschitz continuous, separate Ω into two disjoint connected subdomains: the fluid domain Ω_f and the porous one Ω_p such that $\Omega = \Omega_f \cup \Sigma \cup \Omega_p$. The boundaries of the subdomains are respectively defined by $\partial\Omega_f = \Gamma_f \cup \Sigma$ for $\Omega_f \partial\Omega_p = \Gamma_p \cup \Sigma$ for Ω_p, and $\partial\Omega = \Gamma_f \cup \Gamma_p$ for Ω (see Fig. 1), assuming no cusp singularity at $\Sigma \cap \partial\Omega$. Let $\mathbf{n}$ be the unit normal vector on Σ oriented from Ω_p to Ω_f and $\boldsymbol{\tau}$ any unit tangential vector of a local tangential basis $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{d-1})$ on Σ. For any quantity ψ defined all over Ω, the restrictions on Ω_f and Ω_p are denoted by ψ^f and ψ^p respectively. For a function ψ in $H^1(\Omega_f \cup \Omega_p)$, let ψ^- and ψ^+ be the traces of $\psi_{|\Omega_p}$ and $\psi_{|\Omega_f}$ on each side of Σ respectively, $\overline{\psi}_{|\Sigma} = (\psi^+ + \psi^-)/2$ the arithmetic mean of traces of ψ, and $[\![\psi]\!]_\Sigma = (\psi^+ - \psi^-)$ the jump of traces of ψ on Σ oriented by $\mathbf{n}$.

There exist in the literature different models with physically relevant stress or velocity jump boundary conditions for the tangential momentum transport at the fluid–porous interface Σ; see e.g. [1,2]. For when the homogeneous porous flow is to be governed by the Brinkman equation (cf. [3–8]), the interface condition below linking the jump of shear stress with a continuous velocity was derived with volume averaging techniques by Ochoa-Tapia and Whitaker [9] instead of the usual stress and velocity continuity boundary conditions at the interface [7]:

$$\left(\mu \nabla \mathbf{v}^f \cdot \mathbf{n} - \frac{\mu}{\phi} \nabla \mathbf{v}^p \cdot \mathbf{n}\right)_\Sigma \cdot \boldsymbol{\tau} = \frac{\mu\, \beta_{otw}}{\sqrt{K}} \mathbf{v}_\Sigma \cdot \boldsymbol{\tau} \quad \text{and} \quad \mathbf{v}^f = \mathbf{v}^p = \mathbf{v}_\Sigma \quad \text{on } \Sigma, \tag{1}$$

where the dimensionless parameter β_{otw} is of the order of 1; see [10,2,11,12] for its characterization. We prove in Section 3, as a by-product of our general framework recalled in Section 2, that stress jump boundary conditions of this type yield a well-

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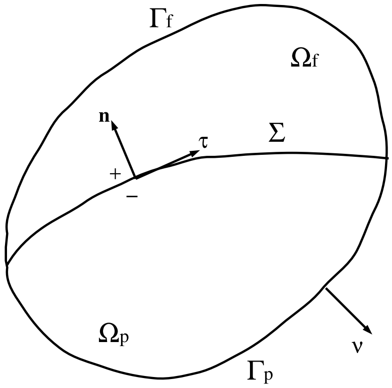

Fig. 1. Configuration for fluid–porous flows inside the domain $\Omega = \Omega_f \cup \Sigma \cup \Omega_p$.

posed fluid–porous Stokes/Brinkman problem whatever the dimensionless parameter $\beta_{otw} \geq 0$. This was not previously stated, to our knowledge.

When the porous flow is governed by the Darcy equation (see e.g. [6]), the well-known Beavers and Joseph interface condition [13] must be used. It links the shear stress at the interface with the jump of tangential velocity:

$$(\mu \nabla \mathbf{v}^f \cdot \mathbf{n})_{|\Sigma} \cdot \boldsymbol{\tau} = \frac{\mu\, \alpha_{bj}}{\sqrt{K}} \left(\mathbf{v}^f - \mathbf{v}^p\right)_\Sigma \cdot \boldsymbol{\tau} \quad \text{and} \quad \mathbf{v}^f \cdot \mathbf{n} = \mathbf{v}^p \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}_\Sigma \quad \text{on } \Sigma, \tag{2}$$

where the dimensionless parameter $\alpha_{bj} = \mathcal{O}(\frac{1}{\sqrt{\phi}})$ depends on the porosity ϕ and may vary between 0.1 and 4 [13,14]. The approximate Saffman interface condition [15], derived by homogenization techniques in [16], is also written when the porous filtration tangential velocity can be neglected with respect to the fluid velocity at the interface: $|\mathbf{v}^p_\Sigma \cdot \boldsymbol{\tau}| \ll |\mathbf{v}^f_\Sigma \cdot \boldsymbol{\tau}|$, i.e. for a permeability value K or Darcy number $\mathrm{Da} = K/H^2$ sufficiently small. The global solvability of the Stokes/Darcy problem with the Saffman condition for $\mathbf{v}^p_\Sigma \cdot \boldsymbol{\tau} \approx 0$ is proved with a mixed hybrid formulation in [17], whatever the dimensionless parameter $\alpha_{bj} \geq 0$, and then by many others with various formulations; see e.g. the recent review [18]. The only result of well-posedness for the full form of Beavers and Joseph condition was recently established in [19] for α^2_{bj} sufficiently small. We prove in Section 4 by a singular perturbation in our general framework with a vanishing viscosity that the above Beavers and Joseph interface conditions yield a well-posed Stokes/Darcy problem whatever the parameter $\alpha_{bj} \geq 0$. Here, the main difficulty lies in how to give a sense to the tangential trace of the porous velocity on the interface with minimal regularity assumptions. This is particularly relevant for thin fluid layers, such as for conducting fractures in porous media flows [20,21,19].

We first begin in Section 2 by describing the general framework with jump embedded boundary conditions studied in [22]. It is derived by a generalization to vector elliptic problems of a previous model stated for scalar problems [23,24]. A short version of the following results can be found in [25].

2. A well-posed Stokes/Brinkman problem with jump embedded boundary conditions

Let $\boldsymbol{\sigma}(\mathbf{v}, p) \equiv -p\,\mathbf{I} + 2\tilde{\mu}\,\mathbf{d}(\mathbf{v})$ denote the Newtonian stress tensor defined with the effective viscosity $\tilde{\mu}$ in the porous domain Ω_p, with $\tilde{\mu} = \mu$ in the fluid domain Ω_f and $\mathbf{d}(\mathbf{v}) \equiv \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$ being the strain rate tensor. We consider the following Stokes/Brinkman problem including *jump embedded boundary conditions (J.E.B.C.)* on the interface Σ which link the trace jumps of both the stress vector $\boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n}$ and the velocity vector $\mathbf{v}$ through the interface Σ:

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{v}, p) = \mathbf{f} \quad \text{in } \Omega_f, \tag{3}$$

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{v}, p) + \mu\, \mathbf{K}^{-1}\, \mathbf{v} = \mathbf{f} \quad \text{in } \Omega_p, \tag{4}$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_f \cup \Omega_p, \tag{5}$$

$$\mathbf{v} = 0 \quad \text{on } \Gamma_f \cup \Gamma_p, \tag{6}$$

$$[\![\boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n}\,]\!]_\Sigma = \mathbf{M}\, \overline{\mathbf{v}}_{|\Sigma} \quad \text{on } \Sigma, \tag{7}$$

$$\overline{\boldsymbol{\sigma}(\mathbf{v}, p) \cdot \mathbf{n}}_{|\Sigma} = \mathbf{S}\, [\![\mathbf{v}\,]\!]_\Sigma \quad \text{on } \Sigma. \tag{8}$$

Here, the viscosity coefficient μ and effective viscosity $\tilde{\mu}$ in the porous medium are bounded positive functions such that $\mu_0 = \min(\mu, \tilde{\mu}) > 0$, the symmetric permeability tensor $\mathbf{K} \equiv (K_{ij})_{1 \leq i,j \leq d}$ is uniformly positive definite, and the *transfer matrices* $\mathbf{S}$, $\mathbf{M}$ on Σ are measurable, bounded and uniformly semi-positive matrices verifying ellipticity assumptions:

$$\mathbf{K} \in \left(L^\infty(\Omega)\right)^{d\times d}; \quad \exists K_0 > 0,\ \forall \boldsymbol{\xi} \in \mathbb{R}^d, \qquad \mathbf{K}(x)^{-1}\cdot\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq K_0\,|\boldsymbol{\xi}|^2 \quad \text{a.e. in } \Omega_p. \tag{A1}$$

$$\mathbf{M}, \mathbf{S} \in \left(L^\infty(\Sigma)\right)^{d\times d}; \quad \exists M_0, S_0 \geq 0,\ \forall \boldsymbol{\xi} \in \mathbb{R}^d, \qquad \mathbf{M}(x)\cdot\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq M_0\,|\boldsymbol{\xi}|^2, \quad \mathbf{S}(x)\cdot\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq S_0\,|\boldsymbol{\xi}|^2 \quad \text{a.e. on } \Sigma. \tag{A2}$$

With usual notation for Sobolev spaces (see e.g. [26,27]), we now define the Hilbert spaces:

$$H^1_{0\Gamma_f}(\Omega_f)^d \equiv \left\{\mathbf{w} \in H^1(\Omega_f)^d;\ \mathbf{w}_{|\Gamma_f} = 0 \text{ on } \Gamma_f\right\}, \qquad H^1_{0\Gamma_p}(\Omega_p)^d \equiv \left\{\mathbf{w} \in H^1(\Omega_p)^d;\ \mathbf{w}_{|\Gamma_p} = 0 \text{ on } \Gamma_p\right\},$$

$$\mathbf{W} \equiv \left\{\mathbf{w} \in L^2(\Omega)^d,\ \mathbf{w}_{|\Omega_f} \in H^1_{0\Gamma_f}(\Omega_f)^d \text{ and } \mathbf{w}_{|\Omega_p} \in H^1_{0\Gamma_p}(\Omega_p)^d;\ \nabla\cdot\mathbf{w} = 0 \text{ in } \Omega_f \cup \Omega_p\right\}$$

equipped with the natural inner product and associated norm in $H^1(\Omega_f \cup \Omega_p)^d$.

Let us note that for $\mathbf{v} \in \mathbf{W}$ satisfying (3) or (4) with $\mathbf{f} \in L^2(\Omega)^d$ such that $\nabla\cdot\boldsymbol{\sigma}(\mathbf{v}, p) \in L^2(\Omega)^d$, we can define $\boldsymbol{\sigma}(\mathbf{v}, p)\cdot\mathbf{n}^{\pm}_{|\Sigma}$ in $H^{-\frac{1}{2}}(\Sigma)^d$; see [28,29]. The model with the J.E.B.C. (7)–(8) also allows a possible pressure jump $[\![p]\!]_\Sigma \neq 0$ in $H^{-\frac{1}{2}}(\Sigma)$ with additional regularity assumptions.

Then, as a consequence of the general framework stated in [22], the problem (3)–(8) satisfies in Ω the nice weak formulation below:

Find $\mathbf{v} \in \mathbf{W}$ *such that* $\forall \mathbf{w} \in \mathbf{W}$, $a(\mathbf{v}, \mathbf{w}) = l(\mathbf{w})$ with

$$a(\mathbf{v},\mathbf{w}) \;=\; 2\int_{\Omega_f} \mu\,\mathbf{d}(\mathbf{v}):\mathbf{d}(\mathbf{w})\,\mathrm{d}x + 2\int_{\Omega_p} \tilde{\mu}\,\mathbf{d}(\mathbf{v}):\mathbf{d}(\mathbf{w})\,\mathrm{d}x + \int_{\Omega_p} \mu\,\mathbf{K}^{-1}\,\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x + \int_\Sigma \mathbf{M}\,\overline{\mathbf{v}}_{|\Sigma}\cdot\overline{\mathbf{w}}_{|\Sigma}\,\mathrm{d}s + \int_\Sigma \mathbf{S}\,[\![\mathbf{v}]\!]_\Sigma\cdot[\![\mathbf{w}]\!]_\Sigma\,\mathrm{d}s$$

$$l(\mathbf{w}) = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x. \tag{9}$$

Also, the following well-posedness result is ensured by [22, Theorem 1.1].

Theorem 2.1 (*Global Solvability of the Stokes/Brinkman Model with J.E.B.C.*)**.** *If the ellipticity assumptions* (A1), (A2) *hold, the problem* (3)–(8) *with* $\mathbf{f} \in L^2(\Omega)^d$ *has a unique solution* $(\mathbf{v}, p) \in \mathbf{W} \times L^2(\Omega)$ *satisfying the weak form* (9) *for all* $\mathbf{w} \in \mathbf{W}$ *and such that* $p^f = p_0^f + C^0 + C^1/2$ *and* $p^p = p_0^p + C^0 - C^1/2$ *where* $p_0 \in L^2_0(\Omega) = \{q \in L^2(\Omega),\ \int_\Omega q\,\mathrm{d}x = 0\}$ *and* $C^0,\ C^1$ *are constants defined by*

$$C^0 = \frac{1}{|\Sigma|}\left\langle \overline{\boldsymbol{\sigma}(\mathbf{v}, p_0)\cdot\mathbf{n}}_{|\Sigma} - \mathbf{S}\,[\![\mathbf{v}]\!]_\Sigma, \mathbf{n}\right\rangle_{-\frac{1}{2},\Sigma} \quad \text{and} \quad C^1 = \frac{1}{|\Sigma|}\left\langle [\![\boldsymbol{\sigma}(\mathbf{v}, p_0)\cdot\mathbf{n}]\!]_\Sigma - \mathbf{M}\,\overline{\mathbf{v}}_{|\Sigma}, \mathbf{n}\right\rangle_{-\frac{1}{2},\Sigma}.$$

Hence, to satisfy (7), (8) *in the sense of* $H^{-\frac{1}{2}}(\Sigma)^d$, *the pressure field* $p \in L^2(\Omega)$ *must be adjusted from the zero-average pressure* $p_0 \in L^2_0(\Omega)$ *such that* $\overline{(p - p_0)}_{|\Sigma} = C^0$ *and* $[\![p - p_0]\!]_\Sigma = C^1$.

Moreover, there exists a constant $\alpha_0(\Omega_f, \Omega_p, K_0, \mu_0) > 0$ *such that*

$$\|\mathbf{v}\|_{\mathbf{W}} + \|p_0\|_{0,\Omega} \leq \frac{c(\Omega_f, \Omega_p, \mu, \tilde{\mu}, \|\mathbf{K}^{-1}\|_\infty)}{\alpha_0}\,\|\mathbf{f}\|_{0,\Omega}.$$

Remark 1 (*Generalizations*)**.** For practical problems, the case of a nonhomogeneous Dirichlet boundary condition: $\mathbf{v} = \mathbf{v}_D$ on $\Gamma_f \cup \Gamma_p$ with $\mathbf{v}_D \in H^{\frac{1}{2}}(\Gamma_f \cup \Gamma_p)^d$ and the compatibility condition $\int_{\Gamma_f\cup\Gamma_p} \mathbf{v}_D\cdot\mathbf{n}\,\mathrm{d}s = 0$, can be treated also by defining an ad hoc divergence-free extension of $\mathbf{v}_D$ (see e.g. [30]), and adding its contribution in the source term $\mathbf{f}$ of the present problem (9). The generalization to unsteady Stokes/Brinkman problems is also straightforward.

3. The Stokes/Brinkman problem with Ochoa-Tapia and Whitaker interface conditions

We now consider that $\tilde{\mu} = \mu/\phi$, where $\phi\ \in]0, 1]$ is the porosity of the porous medium, and stress jump interface conditions of Ochoa-Tapia and Whitaker type [9] like in (1), the original ones reading, with $\beta_\tau = \beta_{otw}$ and $\beta_n = 0$,

$$[\![\boldsymbol{\sigma}(\mathbf{v}, p)\cdot\mathbf{n}]\!]_\Sigma = \mathbf{M}\,\mathbf{v} \quad \text{with } M_{jj} = \frac{\mu\,\beta_\tau}{\sqrt{K_\tau}},\ j = 1, \ldots, d-1,\ M_{dd} = \frac{\mu\,\beta_n}{\sqrt{K_n}} \quad \text{and} \quad [\![\mathbf{v}]\!]_\Sigma = 0 \quad \text{on } \Sigma, \tag{10}$$

where $\mathbf{M}$ is a positive diagonal matrix with $\beta_\tau, \beta_n \geq 0$ a.e. on Σ and K_τ, K_n permeability coefficients. Then, as a consequence of the general framework stated in [22], the problem (3)–(6) and (10) satisfies in Ω the weak formulation below:

Find $\mathbf{v} \in \mathbf{V} = \{\mathbf{u} \in H^1_0(\Omega)^d;\ \nabla\cdot\mathbf{u} = 0\}$ *such that*

$$2\int_{\Omega_f} \mu\,\mathbf{d}(\mathbf{v}):\mathbf{d}(\mathbf{w})\,\mathrm{d}x + 2\int_{\Omega_p} \frac{\mu}{\phi}\,\mathbf{d}(\mathbf{v}):\mathbf{d}(\mathbf{w})\,\mathrm{d}x + \int_{\Omega_p} \mu\,\mathbf{K}^{-1}\,\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x + \int_\Sigma \mathbf{M}\,\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}s = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x, \quad \forall \mathbf{w} \in \mathbf{V}. \tag{11}$$

Also, the following well-posedness result is ensured as a corollary of Theorem 2.1.

Corollary 3.1 (*Global Solvability of Stokes/Brinkman Problem with OT–W*)**.** *If the ellipticity assumptions (A1), (A2) hold, the problem (3)–(6) and (10) with* $\mathbf{f} \in L^2(\Omega)^d$ *has a unique solution* $(\mathbf{v}, p) \in \mathbf{V} \times L^2(\Omega)$ *satisfying the weak form (11) for all* $\mathbf{w} \in \mathbf{V}$ *and such that* $p^f = p_0^f + C^1/2$ *and* $p^p = p_0^p - C^1/2$ *with* $p_0 \in L_0^2(\Omega)$ *and the constant* C^1 *defined by*

$$C^1 = \frac{1}{|\Sigma|} \langle [\![\boldsymbol{\sigma}(\mathbf{v}, p_0) \cdot \mathbf{n}]\!]_\Sigma - \mathbf{M}\,\mathbf{v}, \mathbf{n} \rangle_{-\frac{1}{2},\Sigma} .$$

Sketch of proof. The existence and uniqueness of $\mathbf{v} \in \mathbf{V}$ satisfying (11) are ensured by the Lax–Milgram theorem. The pressure field $p_0 \in L_0^2(\Omega)$ can be also recovered by the De Rham theorem [30,29] which involves the inf-sup condition between the velocity and pressure spaces [31]. Then, on constructing an ad hoc divergence-free extension as for [22, Theorem 1.1] (see also [29]), this allows us to verify the stress jump condition (10) in $H^{-\frac{1}{2}}(\Sigma)^d$ with the pressure field $p \in L^2(\Omega)$ fitted such that we have formally $[\![p - p_0]\!]_\Sigma = C^1$ and $\overline{(p - p_0)}_{|\Sigma} = 0$. □

We can also interpret this solution as the limit solution of the problem (3)–(8) with penalized velocity jumps on Σ when the penalty parameter $\varepsilon > 0$ tends to zero and we have the following convergence result.

Theorem 3.2 (*Convergence to the Stokes/Brinkman Problem with OT–W*)**.** *For any* $\varepsilon > 0$*, the solution* $(\mathbf{v}_\varepsilon, p_\varepsilon)$ *of the problem (3)–(8) from Theorem 2.1 with* $\mathbf{M}$ *defined in (10) and* $\mathbf{S} = \frac{1}{\varepsilon}\mathbf{I}$ *strongly converges to the solution* $(\mathbf{v}, p)$ *of Corollary 3.1 in* $\mathbf{W} \times L^2(\Omega)$ *when* $\varepsilon \to 0$*. Moreover, there exists a constant* $C(\Omega_f, \Omega_p, \mu, \phi, K_0, \|\mathbf{K}^{-1}\|) > 0$ *such that the following error estimate holds,* $\boldsymbol{\psi}$ *being the weak limit of* $\frac{1}{\varepsilon}[\![\mathbf{v}_\varepsilon]\!]_\Sigma$ *in* $L^2(\Sigma)^d$*:*

$$\|\mathbf{v}_\varepsilon - \mathbf{v}\|_{\mathbf{W}} + \|p_{0\varepsilon} - p_0\|_{0,\Omega} \le C\, \|\boldsymbol{\psi}\|_{0,\Sigma} \sqrt{\varepsilon} \quad \textit{and} \quad \|[\![\mathbf{v}_\varepsilon]\!]_\Sigma\|_{0,\Sigma} \le \|\boldsymbol{\psi}\|_{0,\Sigma}\, \varepsilon.$$

With additional regularity assumptions such that $\boldsymbol{\psi} \in H^{\frac{1}{2}}(\Sigma)^d$*, then the previous estimate becomes optimal in* $\mathcal{O}(\varepsilon)$*.*

Sketch of proof. The solution $\mathbf{v}_\varepsilon \in \mathbf{W}$ satisfies with (9) the weak form below:

$$2\int_{\Omega_f} \mu\, \mathbf{d}(\mathbf{v}_\varepsilon) : \mathbf{d}(\mathbf{w})\, dx + 2\int_{\Omega_p} \frac{\mu}{\phi} \mathbf{d}(\mathbf{v}_\varepsilon) : \mathbf{d}(\mathbf{w})\, dx + \int_{\Omega_p} \mu\, \mathbf{K}^{-1} \mathbf{v}_\varepsilon \cdot \mathbf{w}\, dx + \int_\Sigma \mathbf{M}\, \overline{\mathbf{v}}_{\varepsilon|\Sigma} \cdot \overline{\mathbf{w}}_{|\Sigma}\, ds$$
$$+ \frac{1}{\varepsilon} \int_\Sigma [\![\mathbf{v}_\varepsilon]\!]_\Sigma \cdot [\![\mathbf{w}]\!]_\Sigma\, ds = \int_\Omega \mathbf{f} \cdot \mathbf{w}\, dx, \quad \forall \mathbf{w} \in \mathbf{W}. \tag{12}$$

On choosing $\mathbf{w} = \mathbf{v}_\varepsilon$, we get, using the Korn and Friedrichs–Poincaré inequalities in Ω_f, Ω_p together with the inequality $a\,b \le (a^2 + b^2)/2,\ \forall a, b \in \mathbb{R}$,

$$\mu_0 \int_{\Omega_f \cup \Omega_p} |\nabla \mathbf{v}_\varepsilon|^2\, dx + \mu_0 K_0 \int_{\Omega_p} |\mathbf{v}_\varepsilon|^2\, dx + \int_\Sigma \mathbf{M}\, \overline{\mathbf{v}}_{\varepsilon|\Sigma} \cdot \overline{\mathbf{v}}_{\varepsilon|\Sigma}\, ds + \frac{1}{\varepsilon} \int_\Sigma |[\![\mathbf{v}_\varepsilon]\!]_\Sigma|^2\, ds \le \frac{c(\Omega_f, \Omega_p)}{\mu_0} \|\mathbf{f}\|_{0,\Omega}^2 .$$

With this bound, there exists $\mathbf{v} \in \mathbf{W}$ such that, up to a subsequence, $\mathbf{v}_\varepsilon$ tends to $\mathbf{v}$ in $\mathbf{W}$ or $H^1(\Omega_f \cup \Omega_p)^d$ weakly when $\varepsilon \to 0$ and strongly in $L^2(\Omega)^d$. Indeed, since the trace application is continuous, we have $\mathbf{v}_{|\Gamma_f \cup \Gamma_p} = 0$. Moreover we have $\|[\![\mathbf{v}_\varepsilon]\!]_\Sigma\|_{0,\Sigma} \le c(\Omega_f, \Omega_p, \mu_0, \mathbf{f})\sqrt{\varepsilon}$ and thus $[\![\mathbf{v}]\!]_\Sigma = 0$, $\overline{\mathbf{v}}_{|\Sigma} = \mathbf{v}_{|\Sigma}$ and $\mathbf{v}$ belongs to the subspace $\mathbf{V}$ of $\mathbf{W}$. Then $p_{0\varepsilon}$ defined by Theorem 2.1 is bounded in $L_0^2(\Omega)$ since we have, using the Nečas theorem [30,31],

$$\|p_{0\varepsilon}\|_{0,\Omega} \le c(\Omega_f, \Omega_p) \left(\|\nabla p_{0\varepsilon}\|_{-1,\Omega_f} + \|\nabla p_{0\varepsilon}\|_{-1,\Omega_p} \right) \le C\, \|\mathbf{v}_\varepsilon\|_{\mathbf{W}} + \|\mathbf{f}\|_{0,\Omega}. \tag{13}$$

Thus, there exists $p_0 \in L_0^2(\Omega)$ such that, up to a subsequence, $p_{0\varepsilon}$ tends to p_0 weakly in $L^2(\Omega)$. Now taking the limit of (12) when $\varepsilon \to 0$, there exists $\boldsymbol{\psi} \in L^2(\Sigma)^d$ such that $\frac{1}{\varepsilon}[\![\mathbf{v}_\varepsilon]\!]_\Sigma$ tends weakly to $\boldsymbol{\psi}$ in $L^2(\Sigma)^d$ and we get that $\mathbf{v}$ is the unique solution in $\mathbf{V}$ (the uniqueness being proved directly with $\mathbf{f} = 0$ and $\mathbf{w} = \mathbf{v} \in \mathbf{V} \subset \mathbf{W}$) satisfying

$$2\int_{\Omega_f} \mu\, \mathbf{d}(\mathbf{v}) : \mathbf{d}(\mathbf{w})\, dx + 2\int_{\Omega_p} \frac{\mu}{\phi} \mathbf{d}(\mathbf{v}) : \mathbf{d}(\mathbf{w})\, dx + \int_{\Omega_p} \mu\, \mathbf{K}^{-1} \mathbf{v} \cdot \mathbf{w}\, dx + \int_\Sigma \mathbf{M}\, \mathbf{v}_{|\Sigma} \cdot \overline{\mathbf{w}}_{|\Sigma}\, ds + \int_\Sigma \boldsymbol{\psi} \cdot [\![\mathbf{w}]\!]_\Sigma\, ds$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{w}\, dx, \quad \forall \mathbf{w} \in \mathbf{W}. \tag{14}$$

Hence, $\mathbf{v} \in \mathbf{V}$ also satisfies (11) for all $\mathbf{w} \in \mathbf{V}$. Also, using test functions $\mathbf{w} = \boldsymbol{\varphi} \in \mathcal{C}_c^\infty$ compactly supported either in Ω_f or in Ω_p and such that $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω_f or in Ω_p respectively, and using the Stokes formula, we get with the De Rham theorem [30,29] the existence and uniqueness (Ω_f and Ω_p being connected) of the pressure restrictions $p_{0|\Omega_f}$ and $p_{0|\Omega_p}$ in $L_0^2(\Omega_f)$ and $L_0^2(\Omega_p)$ respectively. This defines the pressure field $p_0 = p_{0|\Omega_f} + p_{0|\Omega_p}$ in $L_0^2(\Omega)$ over the whole domain Ω such that $(\mathbf{v}, p_0)$ verifies the Stokes/Brinkman equations (3)–(5) a.e. in $\Omega_f \cup \Omega_p$.

Then, we can define the pressure field $p \in L^2(\Omega)$ with p_0 and the constant C^1 as in Corollary 3.1 such that the stress jump condition (10) is verified in $H^{-\frac{1}{2}}(\Sigma)^d$. Moreover, the constant C_ε^1 defined in Theorem 2.1 with $(\mathbf{v}_\varepsilon, p_{0\varepsilon})$ satisfies

$\lim_{\varepsilon\to 0} C^1_\varepsilon = C^1$ with the weak limits of $(\mathbf{v}_\varepsilon, p_{0\varepsilon})$ and the continuity of the trace applications. We can also give an interpretation of $\boldsymbol{\psi}$. On writing the difference between the weak form of problem (3)–(6) and (10) with test functions $\mathbf{w} \in \mathbf{W}$ using the Stokes formula and the limit weak form (14), this yields $\left\langle \overline{\boldsymbol{\sigma}(\mathbf{v}, p_0)\cdot \mathbf{n}}_{|\Sigma} - \boldsymbol{\psi}, [\![\mathbf{w}]\!]_\Sigma \right\rangle_{-\frac{1}{2},\Sigma} = 0, \forall \mathbf{w} \in \mathbf{W}$.

On constructing an ad hoc divergence-free extension in $\mathbf{W}$ of any function $\mathbf{u}$ in $H^{\frac{1}{2}}(\Sigma)^d$, as for [22, Theorem 1.1] (see also [29, Chapter III] for the Stokes/Neumann problem with a stress boundary condition), we define the constant $C^0 = \lim_{\varepsilon\to 0} C^0_\varepsilon$ below, with C^0_ε as defined in Theorem 2.1 with $(\mathbf{v}_\varepsilon, p_{0\varepsilon})$, such that we have $\boldsymbol{\psi} = \overline{\boldsymbol{\sigma}(\mathbf{v}, p_0 + C^0)\cdot \mathbf{n}}_{|\Sigma}$ in the sense of $H^{-\frac{1}{2}}(\Sigma)^d$:

$$C^0 = \frac{1}{|\Sigma|}\left\langle \overline{\boldsymbol{\sigma}(\mathbf{v}, p_0)\cdot \mathbf{n}}_{|\Sigma} - \boldsymbol{\psi}, \mathbf{n}\right\rangle_{-\frac{1}{2},\Sigma}, \quad \text{such that } \left\langle \overline{\boldsymbol{\sigma}(\mathbf{v}, p_0 + C^0)\cdot \mathbf{n}}_{|\Sigma} - \boldsymbol{\psi}, \mathbf{u}\right\rangle_{-\frac{1}{2},\Sigma} = 0, \quad \forall \mathbf{u} \in H^{\frac{1}{2}}(\Sigma)^d.$$

To prove the strong convergence and the error estimate, we first write the error equation as the difference between (9) satisfied by $\mathbf{v}_\varepsilon$ for all $\mathbf{w} \in \mathbf{W}$ and (14) using the fact that $[\![\mathbf{v}]\!]_\Sigma = 0$ and $\overline{\mathbf{v}}_{|\Sigma} = \mathbf{v}_{|\Sigma}$. Then, choosing $\mathbf{w} = \mathbf{v}_\varepsilon - \mathbf{v}$, we get, with the Cauchy–Schwarz inequality,

$$2\mu_0 \int_{\Omega_f\cup\Omega_p} |\mathbf{d}(\mathbf{v}_\varepsilon - \mathbf{v})|^2\, dx + \mu_0 K_0 \int_{\Omega_p} |\mathbf{v}_\varepsilon - \mathbf{v}|^2\, dx + M_0 \int_\Sigma |\overline{\mathbf{v}_\varepsilon}_{|\Sigma} - \mathbf{v}|^2\, ds$$
$$+ \frac{1}{\varepsilon}\int_\Sigma |[\![\mathbf{v}_\varepsilon - \mathbf{v}]\!]_\Sigma|^2\, ds \le \|\boldsymbol{\psi}\|_{0,\Sigma}\|[\![\mathbf{v}_\varepsilon - \mathbf{v}]\!]_\Sigma\|_{0,\Sigma}$$

which simply gives, using the Korn and Poincaré inequalities in Ω_f and Ω_p,

$$\|[\![\mathbf{v}_\varepsilon]\!]_\Sigma\|_{0,\Sigma} = \|[\![\mathbf{v}_\varepsilon - \mathbf{v}]\!]_\Sigma\|_{0,\Sigma} \le \|\boldsymbol{\psi}\|_{0,\Sigma}\, \varepsilon \quad \text{and} \quad \|\mathbf{v}_\varepsilon - \mathbf{v}\|_{\mathbf{W}} \le C(\Omega_f, \Omega_p, \mu_0)\|\boldsymbol{\psi}\|_{0,\Sigma}\sqrt{\varepsilon}. \tag{15}$$

If $\boldsymbol{\psi}$ belongs to $H^{\frac{1}{2}}(\Sigma)^d$, the last error estimate can be improved up to $\mathcal{O}(\varepsilon)$ by constructing some adequate extensions from $\boldsymbol{\psi}$ in the subdomains Ω_f and Ω_p. Finally, the pressure estimate is obtained using the Nečas theorem and we get

$$\|p_{0\varepsilon} - p_0\|_{0,\Omega} \le c(\Omega_f, \Omega_p)\left(\|\nabla(p_{0\varepsilon} - p_0)\|_{-1,\Omega_f} + \|\nabla(p_{0\varepsilon} - p_0)\|_{-1,\Omega_p}\right) \le C\,\|\mathbf{v}_\varepsilon - \mathbf{v}\|_{\mathbf{W}},$$

which completes the proof. □

4. The Stokes/Darcy problem with Beavers and Joseph interface conditions

We consider the problem (3)–(8) with the Dirichlet boundary condition (6) on Γ_p replaced by the stress boundary condition of Neumann where $\boldsymbol{\nu}$ is the outward unit normal vector on Γ_p and $\mathbf{q} \in H^{-\frac{1}{2}}(\Gamma_p)^d$ given (e.g. $\mathbf{q} = -p_e\,\boldsymbol{\nu}$):

$$\mathbf{v} = 0 \quad \text{on } \Gamma_f \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v}^p, p^p)\cdot\boldsymbol{\nu} = -p^p\,\boldsymbol{\nu} + \tilde{\mu}\nabla\mathbf{v}^p\cdot\boldsymbol{\nu} = \mathbf{q} \quad \text{on } \Gamma_p. \tag{16}$$

Let us define the Hilbert space $\mathbf{W}_N$ equipped with the natural inner product and norm in $H^1(\Omega_f \cup \Omega_p)^d$:

$$\mathbf{W}_N \equiv \{\mathbf{w} \in L^2(\Omega)^d, \mathbf{w}_{|\Omega_f} \in H^1_{0\Gamma_f}(\Omega_f)^d \text{ and } \mathbf{w}_{|\Omega_p} \in H^1(\Omega_p)^d;\ \nabla\cdot\mathbf{w} = 0 \text{ in } \Omega_f \cup \Omega_p\}.$$

Then, the following well-posedness result is ensured as a corollary of Theorem 2.1; see also [22, Theorem 2.1].

Corollary 4.1 (*Global Solvability of the Stokes/Brinkman Model with J.E.B.C. and Stress B.C.*)**.** *With the assumptions of Theorem 2.1 and $\mathbf{q} \in H^{-\frac{1}{2}}(\Gamma_p)^d$, there exists a unique solution $(\mathbf{v}, p) \in \mathbf{W}_N \times L^2(\Omega)$ satisfying the weak form $a(\mathbf{v}, \mathbf{w}) = l(\mathbf{w}) + \langle \mathbf{q}\,, \mathbf{w}\rangle_{-\frac{1}{2},\Gamma_p}$ for all $\mathbf{w} \in \mathbf{W}_N$ with $p^f = p^f_0 + C^0 + C^1/2$ and $p^p = p^p_0 + C^0 - C^1/2$ where $p_0 \in L^2_0(\Omega)$ and the constants C^0, C^1 are defined as in Theorem 2.1 such that the Eqs. (3)–(5) hold almost everywhere in $\Omega_f \cup \Omega_p$ and (7)–(8) are satisfied in $H^{-\frac{1}{2}}(\Sigma)^d$. Then, if the following compatibility condition holds:*

$$C^0 - \frac{1}{2}C^1 = C^N \quad \textit{with } C^N = \frac{1}{|\Gamma_p|}\langle \boldsymbol{\sigma}(\mathbf{v}, p_0)\cdot\boldsymbol{\nu} - \mathbf{q}, \boldsymbol{\nu}\rangle_{-\frac{1}{2},\Gamma_p},$$

the stress boundary condition (16) is also satisfied in $H^{-\frac{1}{2}}(\Gamma_p)^d$ and $(\mathbf{v}, p) \in \mathbf{W}_N \times L^2(\Omega)$ is the unique solution of the problem (3)–(5), (7), (8) and (16).

For any $\varepsilon > 0$, let us now consider the solution $(\mathbf{v}_\varepsilon, p_\varepsilon) \in \mathbf{W}_N \times L^2(\Omega)$ of the problem (3)–(5), (7), (8) and (16) with a vanishing viscosity $\tilde{\mu} = \varepsilon$ for the Brinkman problem in Ω_p. The condition (16) avoids the creation of a spurious boundary layer along Γ_p for the Darcy problem when $\varepsilon \to 0$. The J.E.B.C. (7)–(8) are also calibrated as follows to obtain interface conditions of Beavers and Joseph type [13] with a jump of tangential velocity (2) allowing a possible pressure jump:

$$[\![\boldsymbol{\sigma}(\mathbf{v}, p)\cdot\mathbf{n}]\!]_\Sigma = \mathbf{M}\,\overline{\mathbf{v}}_{|\Sigma} \quad \text{with } M_{jj} = 0,\ j = 1, \ldots, d-1,\ M_{dd} = \frac{\mu\,\beta_n}{\sqrt{K_n}} \quad \text{on } \Sigma, \tag{17}$$

$$\overline{\boldsymbol{\sigma}(\mathbf{v}, p)\cdot\mathbf{n}}_{|\Sigma} = \mathbf{S}\,[\![\mathbf{v}]\!]_\Sigma \quad \text{with } S_{jj} = \frac{\mu\,\alpha_\tau}{\sqrt{K_\tau}},\ j = 1, \ldots, d-1,\ S_{dd} = \frac{1}{\varepsilon} \quad \text{on } \Sigma, \tag{18}$$

where $\mathbf{M}, \mathbf{S}$ are positive diagonal matrices with $\alpha_\tau = \alpha_{bj}$, $\beta_n \ge 0$ a.e. on Σ and K_τ, K_n permeability coefficients.

Let us define the Hilbert spaces

$$\mathbf{W}_{S/D} \equiv \left\{ \mathbf{w} \in L^2(\Omega)^d, \mathbf{w}_{|\Omega_f} \in H^1_{0\Gamma_f}(\Omega_f)^d, \mathbf{w}_{|\Omega_p} \in L^2(\Omega_p)^d;\ \nabla\cdot\mathbf{w} = 0 \text{ in } \Omega_f \cup \Omega_p \right\}$$

equipped with the natural inner product and norm in $H^1(\Omega_f)^d \times L^2(\Omega_p)^d$ and

$$\mathbf{W}_{S-D} \equiv \left\{ \mathbf{w} \in \mathbf{W}_{S/D};\ \nabla\cdot\mathbf{w} \in L^2(\Omega), [\![\mathbf{w}]\!]_\Sigma \in L^2(\Sigma)^d, [\![\mathbf{w}\cdot\mathbf{n}]\!]_\Sigma = 0 \right\}$$

equipped with the norm defined by $\|\mathbf{w}\|^2_{\mathbf{W}_{S-D}} = \|\mathbf{w}\|^2_{1,\Omega_f} + \|\mathbf{w}\|^2_{0,\Omega_p} + \|\nabla\cdot\mathbf{w}\|^2_{0,\Omega} + \|[\![\mathbf{w}]\!]_\Sigma\|^2_{0,\Sigma}$.

We now prove the following convergence result which also ensures the well-posedness of the Stokes/Darcy problem with Beavers and Joseph type interface conditions (2) and (17) whatever the coefficients $\alpha_\tau, \beta_n \geq 0$ are a.e. on Σ.

Theorem 4.2 (*Convergence to the Stokes/Darcy Problem with B–J*). *With the data $\mathbf{f} \in L^2(\Omega)^d$ and $\mathbf{q} = 0$, the solution $(\mathbf{v}_\varepsilon, p_\varepsilon)$ in $\mathbf{W}_N \times L^2(\Omega)$ for any $\varepsilon > 0$ from Corollary 4.1 of the problem (3)–(5) and (16)–(18) with a vanishing viscosity $\tilde{\mu} = \varepsilon$ weakly converges to the solution $(\mathbf{v}, p)$ in $\mathbf{W}_{S/D} \times L^2(\Omega)$ of the Stokes/Darcy problem with the interface conditions (2) and (17) on Σ when $\varepsilon \to 0$. Indeed, in the porous domain Ω_p, $\mathbf{v}^p$ and p^p satisfy the Darcy equation, i.e. Eq. (4) with $\tilde{\mu} = 0$, and p^p belongs to $H^1(\Omega_p)$ such that $p^p = 0$ on Γ_p.*

With additional regularity assumptions such that $\mathbf{v}^p \in H^1(\Omega_p)^d$, then $\mathbf{v} \in \mathbf{W}_{S-D} \cap \mathbf{W}_N$ and we have the global error estimate with $C > 0$ depending on the data, $\|\nabla\mathbf{v}\|_{0,\Omega_p}$, $\|\psi\|_{0,\Sigma}$ and ψ defined as the weak limit of $\frac{1}{\varepsilon}[\![\mathbf{v}_\varepsilon\cdot\mathbf{n}]\!]_\Sigma$ in $L^2(\Sigma)$:

$$\|\mathbf{v}_\varepsilon - \mathbf{v}\|_{1,\Omega_f} + \sqrt{\varepsilon}\,\|\mathbf{v}_\varepsilon - \mathbf{v}\|_{1,\Omega_p} + \|\mathbf{v}_\varepsilon - \mathbf{v}\|_{0,\Omega_p} + \|p_{0\varepsilon} - p_0\|_{0,\Omega} \leq C\,\|\psi\|_{0,\Sigma}\sqrt{\varepsilon} \quad \text{and}$$

$$\|[\![\mathbf{v}_\varepsilon\cdot\mathbf{n}]\!]_\Sigma\|_{0,\Sigma} \leq \left(2\|\nabla\mathbf{v}\|^2_{0,\Omega_p} + \|\psi\|^2_{0,\Sigma}\right)^{\frac{1}{2}} \varepsilon.$$

Sketch of proof. The proof is here abridged without explaining most of the arguments already detailed in the proof of Theorem 3.2. From (3)–(5) and (16)–(18) with the Stokes formula, the solution $\mathbf{v}_\varepsilon \in \mathbf{W}_N$ satisfies the weak form below:

$$2\int_{\Omega_f} \mu\,\mathbf{d}(\mathbf{v}_\varepsilon):\mathbf{d}(\mathbf{w})\,dx + 2\varepsilon\int_{\Omega_p} \mathbf{d}(\mathbf{v}_\varepsilon):\mathbf{d}(\mathbf{w})\,dx + \int_{\Omega_p} \mu\,\mathbf{K}^{-1}\,\mathbf{v}_\varepsilon\cdot\mathbf{w}\,dx + \int_\Sigma \mathbf{M}\,\overline{\mathbf{v}}_{\varepsilon|\Sigma}\cdot\overline{\mathbf{w}}_{|\Sigma}\,ds$$
$$+ \sum_{j=1}^{d-1}\int_\Sigma S_{jj}\,[\![\mathbf{v}_\varepsilon\cdot\boldsymbol{\tau}_j]\!]_\Sigma\,[\![\mathbf{w}\cdot\boldsymbol{\tau}_j]\!]_\Sigma\,ds + \frac{1}{\varepsilon}\int_\Sigma [\![\mathbf{v}_\varepsilon\cdot\mathbf{n}]\!]_\Sigma\,[\![\mathbf{w}\cdot\mathbf{n}]\!]_\Sigma\,ds = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx, \quad \forall\mathbf{w} \in \mathbf{W}_N. \tag{19}$$

On choosing $\mathbf{w} = \mathbf{v}_\varepsilon$, we get, using the Korn inequality in Ω_f, Ω_p and the Poincaré inequality in Ω_f,

$$\mu_0\int_{\Omega_f} |\nabla\mathbf{v}_\varepsilon|^2\,dx + 2\varepsilon\int_{\Omega_p} |\nabla\mathbf{v}_\varepsilon|^2\,dx + \frac{\mu_0 K_0}{2}\int_{\Omega_p} |\mathbf{v}_\varepsilon|^2\,dx + M_0\int_\Sigma |\overline{\mathbf{v}_\varepsilon\cdot\mathbf{n}}_{|\Sigma}|^2\,ds + S_0\sum_{j=1}^{d-1}\int_\Sigma [\![\mathbf{v}_\varepsilon\cdot\boldsymbol{\tau}_j]\!]^2_\Sigma\,ds$$
$$+ \frac{1}{\varepsilon}\int_\Sigma [\![\mathbf{v}_\varepsilon\cdot\mathbf{n}]\!]^2_\Sigma\,ds \leq c(\Omega_f, \Omega_p, \mu_0, K_0)\|\mathbf{f}\|^2_{0,\Omega}. \tag{20}$$

With this bound, there exist $\mathbf{v} \in \mathbf{W}_{S/D}$ and $\tilde{\mathbf{v}} \in H^1(\Omega_p)^d$ such that, up to a subsequence, $\mathbf{v}_\varepsilon$ tends to $\mathbf{v}$ in $\mathbf{W}_{S/D}$ or $H^1(\Omega_f)^d \times L^2(\Omega_p)^d$ weakly when $\varepsilon \to 0$ (strongly in $L^2(\Omega_f)^d$) and $\sqrt{\varepsilon}\,\mathbf{v}^p_\varepsilon$ tends to $\tilde{\mathbf{v}}$ in $H^1(\Omega_p)^d$ weakly. Indeed, since the trace application is continuous, we have $\mathbf{v}_{|\Gamma_f} = 0$. Moreover we have $\|[\![\mathbf{v}_\varepsilon\cdot\mathbf{n}]\!]_\Sigma\|_{0,\Sigma} \leq c(\Omega_f, \mu_0, K_0, \mathbf{f})\sqrt{\varepsilon}$ and thus $[\![\mathbf{v}\cdot\mathbf{n}]\!]_\Sigma = 0$, $\overline{\mathbf{v}\cdot\mathbf{n}}_{|\Sigma} = \mathbf{v}\cdot\mathbf{n}_{|\Sigma}$ in $L^2(\Sigma)$. Since $[\![\mathbf{v}_\varepsilon\cdot\boldsymbol{\tau}]\!]_\Sigma$ is bounded in $L^2(\Sigma)$ (for $\alpha_\tau > 0$ and thus $S_0 > 0$) and because $\mathbf{v}^f \in H^1(\Omega_f)^d$ has a trace in $H^{\frac{1}{2}}(\Sigma)^d$, there exists $\mathbf{v}^\star_\Sigma \in L^2(\Sigma)^d$ defined as the weak limit of the trace $\mathbf{v}^p_{\varepsilon|\Sigma}$ in $L^2(\Sigma)^d$. Hence we define the tangential velocity jump: $[\![\mathbf{v}\cdot\boldsymbol{\tau}]\!]_\Sigma = (\mathbf{v}^f_{|\Sigma} - \mathbf{v}^\star_\Sigma)\cdot\boldsymbol{\tau} \in L^2(\Sigma)$ and we have $\mathbf{v} \in \mathbf{W}_{S-D}$.

Then $p_{0\varepsilon}$ defined by Corollary 4.1 is bounded in $L^2_0(\Omega)$ because, using the Nečas theorem as for (13), we have $\|p_{0\varepsilon}\|_{0,\Omega} \leq c(\Omega_f, \Omega_p)\left(\|\nabla p_{0\varepsilon}\|_{-1,\Omega_f} + \|\nabla p_{0\varepsilon}\|_{-1,\Omega_p}\right) \leq C$, since $\|\mathbf{v}_\varepsilon\|_{1,\Omega_f}$, $\sqrt{\varepsilon}\,\|\mathbf{v}_\varepsilon\|_{1,\Omega_p}$ and $\|\mathbf{v}_\varepsilon\|_{0,\Omega_p}$ are all bounded. Thus, there exists $p_0 \in L^2_0(\Omega)$ such that, up to a subsequence, $p_{0\varepsilon}$ weakly tends to p_0 in $L^2(\Omega)$.

Now taking the limit of (19) when $\varepsilon \to 0$, there exists $\psi \in L^2(\Sigma)$ such that $\frac{1}{\varepsilon}[\![\mathbf{v}_\varepsilon\cdot\mathbf{n}]\!]_\Sigma$ weakly tends to ψ in $L^2(\Sigma)$ and we get that $\mathbf{v}$ is the unique solution in $\mathbf{W}_{S-D}$ satisfying the weak form

$$2\int_{\Omega_f} \mu\mathbf{d}(\mathbf{v}):\mathbf{d}(\mathbf{w})\,dx + \int_{\Omega_p} \mu\,\mathbf{K}^{-1}\,\mathbf{v}\cdot\mathbf{w}\,dx + \int_\Sigma \mathbf{M}\,\overline{\mathbf{v}}_{|\Sigma}\cdot\overline{\mathbf{w}}_{|\Sigma}\,ds + \sum_{j=1}^{d-1}\int_\Sigma S_{jj}\,[\![\mathbf{v}\cdot\boldsymbol{\tau}_j]\!]_\Sigma\,[\![\mathbf{w}\cdot\boldsymbol{\tau}_j]\!]_\Sigma\,ds$$
$$+ \int_\Sigma \psi\,[\![\mathbf{w}\cdot\mathbf{n}]\!]_\Sigma\,ds = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,dx, \quad \forall\mathbf{w} \in \mathbf{W}_N. \tag{21}$$

The existence and uniqueness of the solution $\mathbf{v} \in \mathbf{W}_{S-D}$ to the above problem can also be ensured *a priori* by the generalized Lax–Milgram theorem of Nečas [26] with an inf-sup stability inequality. Also, using test functions $\mathbf{w} = \boldsymbol{\varphi} \in \mathcal{C}^\infty_c$ compactly supported either in Ω_f or in Ω_p and such that div $\boldsymbol{\varphi} = 0$ in Ω_f or in Ω_p respectively, and using the Stokes formula, we get with the De Rham theorem the existence and uniqueness (Ω_f and Ω_p being connected) of the pressure restrictions $p_{0|\Omega_f}$ and $p_{0|\Omega_p}$ in $L_0^2(\Omega_f)$ and $L_0^2(\Omega_p)$ respectively. This defines the pressure field $p_0 = p_{0|\Omega_f} + p_{0|\Omega_p}$ in $L_0^2(\Omega)$ over the whole domain Ω such that $(\mathbf{v}, p_0)$ verifies the Stokes/Darcy equations (3)–(5) a.e. in $\Omega_f \cup \Omega_p$ with $\tilde{\mu} = 0$ in (4), i.e. the Darcy equation. Because of the uniqueness, the whole sequence $(\mathbf{v}_\varepsilon, p_{0\varepsilon})$ weakly converges to $(\mathbf{v}, p_0)$ in $\mathbf{W}_{S/D} \times L_0^2(\Omega)$.

Then, to satisfy the interface conditions (17) and (18) on Σ, i.e. in $H^{-\frac{1}{2}}(\Sigma)^d$, the pressure field $p \in L^2(\Omega)$ must be adjusted from the zero-average pressure $p_0 \in L_0^2(\Omega)$ such that $\overline{(p - p_0)_{|\Sigma}} = C^0$ and $[\![p - p_0]\!]_\Sigma = C^1$, where the constants C^0, C^1 are calculated as in Theorem 2.1 with $(\mathbf{v}, p_0)$ defined above. Since $\mathbf{f}^p$, $\mathbf{v}^p \in L^2(\Omega_p)^d$, we have by the Darcy equation that p^p belongs to $H^1(\Omega_p)$. The limit boundary condition (16) which reduces to $p^p_{|\Gamma_p} = 0$ in $H^{\frac{1}{2}}(\Gamma_p)$ can be also satisfied if the following compatibility condition holds:

$$C^0 - \frac{1}{2}C^1 = C^N \quad \text{with} \quad C^N = -\frac{1}{|\Gamma_p|}\int_{\Gamma_p} p_0 \,\mathrm{d}s, \tag{22}$$

such that $p^f = p_0^f + C^0 + C^1/2$ and $p^p = p_0^p + C^N$ define the pressure solution $p \in L^2(\Omega_f) \times H^1_{0\Gamma_p}(\Omega_p)$. We can also interpret ψ in a similar way to $\boldsymbol{\psi}$ in the proof of Theorem 3.2.

Now, if $\mathbf{v}^p$ belongs to $H^1(\Omega_p)^d$ with sufficient regularity assumptions, then $\mathbf{v} \in \mathbf{W}_{S-D} \cap \mathbf{W}_N$, $\mathbf{v}^\star_\Sigma = \mathbf{v}^p_{|\Sigma} \in H^{\frac{1}{2}}(\Sigma)^d$ and we prove the strong convergence and a global error estimate in Ω. The equation giving the difference between (19) and (21) reads: for all $\mathbf{w} \in \mathbf{W}_N$,

$$\begin{aligned}
&2\int_{\Omega_f} \mu\, \mathbf{d}(\mathbf{v}_\varepsilon - \mathbf{v}):\mathbf{d}(\mathbf{w})\,\mathrm{d}x + 2\varepsilon\int_{\Omega_p} \mathbf{d}(\mathbf{v}_\varepsilon - \mathbf{v}):\mathbf{d}(\mathbf{w})\,\mathrm{d}x + \int_{\Omega_p} \mu\,\mathbf{K}^{-1}(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\mathbf{w}\,\mathrm{d}x + \int_\Sigma \mathbf{M}\,\overline{(\mathbf{v}_\varepsilon - \mathbf{v})}_{|\Sigma}\cdot\overline{\mathbf{w}}_{|\Sigma}\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{d-1}\int_\Sigma S_{jj}\,[\![(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\boldsymbol{\tau}_j]\!]_\Sigma\,[\![\mathbf{w}\cdot\boldsymbol{\tau}_j]\!]_\Sigma\,\mathrm{d}s + \frac{1}{\varepsilon}\int_\Sigma [\![\mathbf{v}_\varepsilon\cdot\mathbf{n}]\!]_\Sigma\,[\![\mathbf{w}\cdot\mathbf{n}]\!]_\Sigma\,\mathrm{d}s \\
&\quad = -2\varepsilon\int_{\Omega_p}\mathbf{d}(\mathbf{v}):\mathbf{d}(\mathbf{w})\,\mathrm{d}x - \int_\Sigma \psi\,[\![\mathbf{w}\cdot\mathbf{n}]\!]_\Sigma\,\mathrm{d}s.
\end{aligned} \tag{23}$$

Then, choosing $\mathbf{w} = (\mathbf{v}_\varepsilon - \mathbf{v}) \in \mathbf{W}_N$ with $[\![\mathbf{v}\cdot\mathbf{n}]\!]_\Sigma = 0$, we get the error estimate for the velocity:

$$\begin{aligned}
&2\mu_0\,\|\mathbf{d}(\mathbf{v}_\varepsilon - \mathbf{v})\|^2_{0,\Omega_f} + \varepsilon\,\|\mathbf{d}(\mathbf{v}_\varepsilon - \mathbf{v})\|^2_{0,\Omega_p} + \mu_0 K_0\,\|\mathbf{v}_\varepsilon - \mathbf{v}\|^2_{0,\Omega_p} + M_0\,\|\overline{(\mathbf{v}_\varepsilon - \mathbf{v})}_{|\Sigma}\|^2_{0,\Sigma} \\
&\quad + S_0\sum_{j=1}^{d-1}\|[\![(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\boldsymbol{\tau}_j]\!]_\Sigma\|^2_{0,\Sigma} + \frac{1}{2\varepsilon}\|[\![\mathbf{v}_\varepsilon\cdot\mathbf{n}]\!]_\Sigma\|^2_{0,\Sigma} \le \frac{1}{2}\left(2\|\nabla\mathbf{v}\|^2_{0,\Omega_p} + \|\psi\|^2_{0,\Sigma}\right)\varepsilon
\end{aligned} \tag{24}$$

which yields the result with the Korn and Poincaré inequalities in Ω_f or Ω_p. Finally, the pressure estimate is obtained using the Nečas theorem and we get, with the Stokes and Darcy equations,

$$\|p_{0\varepsilon} - p_0\|_{0,\Omega} \le C\left(\|\mathbf{v}_\varepsilon - \mathbf{v}\|_{1,\Omega_f} + \|\mathbf{v}_\varepsilon - \mathbf{v}\|_{0,\Omega_p} + \varepsilon\,\|\nabla\mathbf{v}_\varepsilon\|_{0,\Omega_p}\right), \tag{25}$$

which concludes the proof with (24) since $\sqrt{\varepsilon}\,\|\nabla\mathbf{v}_\varepsilon\|_{0,\Omega_p}$ is bounded with (20). We thus obtain the given error estimate, typical of the existence of a spurious boundary layer in this singular perturbation problem (see e.g. [32]), as for the L^2-penalty method analysed in [7,33]. □

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