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# Geometric properties of subclasses of starlike functions

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**Abstract**

We present some geometric characterization of the class $k$-$\mathcal{ST}$ consisting of the so-called $k$-starlike functions.



## 1. Introduction

Let $U(\zeta, r)$ denote the open disk with center at $\zeta$ and radius $r$, and $U = U(0,1)$ be the unit disk. By $\mathcal{S}$, as usual, we denote the class of functions $f$ that are analytic and univalent in $U$, normalized by $f(0) = f'(0) - 1 = 0$. The class of all starlike univalent functions will be denoted here by $\mathcal{ST}$. By $k$-$\mathcal{UCV}$, $0 \leqslant k < \infty$, we denote the class of all $k$-uniformly convex functions introduced in [3]. Recall that a function $f \in \mathcal{S}$ is said to be $k$-uniformly convex in $U$, if the image of every circular arc contained in $U$ with center at $\zeta$, where $|\zeta| \leqslant k$, is convex. Note that the class 1-$\mathcal{UCV}$ coincides with the class $\mathcal{UCV}$ of uniformly convex functions, introduced in [1]. Moreover, for $k = 0$ we get the class of all convex univalent functions. It is known that $f \in k$-$\mathcal{UCV}$ if and only if it satisfies the following condition:

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U, \ 0 \leqslant k < \infty. \tag{1}$$

For $k = 1$ we get one-variable characterization of $\mathcal{UCV}$ obtained in [5], and independently in [6].

We consider the class $k$-$\mathcal{ST}$, $0 \leqslant k < \infty$, of $k$-starlike functions (see [4]) which are associated with $k$-uniformly convex functions by the relation

$$f \in k\text{-}\mathcal{UCV} \Leftrightarrow zf'(z) \in k\text{-}\mathcal{ST}. \tag{2}$$

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Thus, the class $k\text{-}\mathcal{ST}$, $0 \leqslant k < \infty$, is the subfamily of $\mathcal{S}$, consisting of functions that satisfy the analytic condition

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in U. \tag{3}$$

The aim of this paper is to present some geometric characterization of $k$-starlike functions.

## 2. Main results

Note that $f \in \mathcal{S}$ is $k$-uniformly convex in $U$, if $f(U(\zeta, r) \cap U)$ is a convex domain for every $r > 0$ and each $\zeta$ such that $|\zeta| \leqslant k$.

To find a similar property for functions in the class $k\text{-}\mathcal{ST}$, we need the following two-variable analytic characterization of the class of $k$-uniformly convex functions.

**Theorem 1** (Kanas and Wiśniowska [3]). *Let $f \in \mathcal{S}$ and $0 \leqslant k < \infty$. Then $f \in k\text{-}\mathcal{UCV}$ if and only if*

$$\operatorname{Re}\left\{1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right\} \geqslant 0, \quad z \in U, \ |\zeta| \leqslant k.$$

Now, from relation (2) we immediately get

**Theorem 2.** *Let $f \in \mathcal{S}$ and $0 \leqslant k < \infty$. Then $f \in k\text{-}\mathcal{ST}$ if and only if*

$$\operatorname{Re}\left\{\frac{\zeta}{z} + \frac{(z-\zeta)f'(z)}{f(z)}\right\} \geqslant 0, \quad z \in U, \ |\zeta| \leqslant k.$$

The last inequality can be rewritten as

$$\operatorname{Re}\frac{(z-\zeta)f'(z)}{f(z)} \geqslant -\operatorname{Re}\frac{\zeta}{z}, \quad z \in U, \ |\zeta| \leqslant k.$$

Hence we have

**Corollary 1.** *Let $0 \leqslant k < \infty$. If $f \in k\text{-}\mathcal{ST}$, then*

$$\operatorname{Re}\frac{(z-\zeta)f'(z)}{f(z)} \geqslant 0$$

*for $z \in U$, $|\zeta| \leqslant k$ and $\pi/2 \leqslant \operatorname{Arg}\{\zeta/z\} \leqslant 3\pi/2$.*

Let $\gamma$ be a circular arc with center at $\zeta$, and let $f$ be analytic on $\gamma$. Then the arc $\Gamma = f(\gamma)$ is starlike with respect to a point $w_0$ if and only if (see [2])

$$\operatorname{Re}\frac{(z-\zeta)f'(z)}{f(z) - w_0} \geqslant 0, \quad z \in \gamma. \tag{4}$$

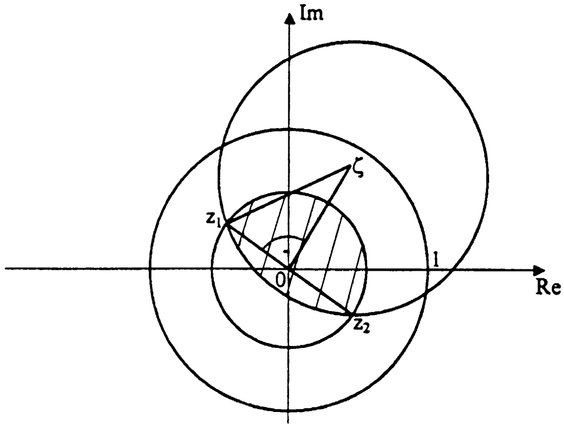


Fig. 1. $0 < R < 1$, $r = \sqrt{|\zeta|^2 + R^2} = |\zeta - z_1| = |\zeta - z_2|$.

It is known that every $f \in k\text{-}\mathcal{ST}$ has a continuous extension to $\bar{U}$, $f(U)$ is bounded and $f(\partial U)$ is a rectifiable curve (for details see [4]).

As a consequence of the above facts, Theorem 2 and Corollary 1, we obtain

**Theorem 3.** *Let* $0 \leqslant k < \infty$. *If* $f \in k\text{-}\mathcal{ST}$, *then* $f(U(\zeta, r) \cap U(0, R))$ *is a starlike domain for every* $\zeta$, $r$, $R$ *such that*

$$0 < R \leqslant 1, \quad |\zeta| \leqslant k \quad \text{and} \quad r \geqslant \sqrt{|\zeta|^2 + R^2}.$$

**Proof.** If $k = 0$, then $0\text{-}\mathcal{ST} = \mathcal{ST}$ and since $\zeta = 0$ we get the well-known result: if $f \in \mathcal{ST}$, then $f(U(0, R))$ is a starlike domain for every $0 < R \leqslant 1$.

Let $0 < k < \infty$ and $|\zeta| \leqslant k$. For fixed $\zeta$ consider two cases:

1. Let $0 < R < 1$.

(a) Let $r \geqslant |\zeta| + R$. Then $U(\zeta, r) \cap U(0, R) = U(0, R)$. But $k\text{-}\mathcal{ST}$ as a subclass of $\mathcal{ST}$ maps every disk $U(0, R)$ onto a starlike domain so the result follows.

(b) If $\sqrt{|\zeta|^2 + R^2} \leqslant r < |\zeta| + R$, then in view of Corollary 1 and (4), $f$, as an element of $k - \mathcal{ST}$, maps every arc $|z - \zeta| = r$ lying in $U(0, R)$, connecting the intersection points $z_1$ and $z_2$ of $\partial U(0, R)$ and $U(\zeta, r)$ onto a starlike arc with respect to the origin. Hence and from starlikeness of $f(U(0, R))$ we obtain the thesis (see Fig. 1).

2. Let $R = 1$.

(a) It is clear that the result holds for $r \geqslant |\zeta| + 1$ since then $U(\zeta, r) \cap U = U$ (see Fig. 2).

(b) Let $\sqrt{|\zeta|^2 + 1} \leqslant r < |\zeta| + 1$. Then we have

$$U(\zeta, r) \cap U = \bigcup_{0 < \rho < 1} U(\zeta, r) \cap U(0, \rho).$$

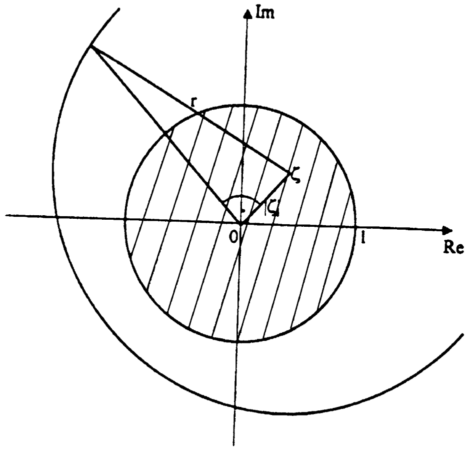


Fig. 2. $R = 1,\ r \geqslant |\zeta| + 1$.

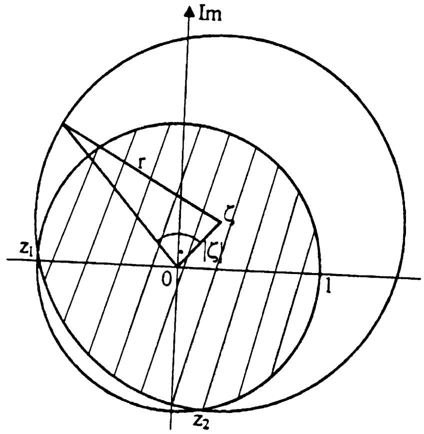


Fig. 3. $R = 1,\ \sqrt{|\zeta|^2 + 1} \leqslant r < |\zeta| + 1$.

From case 1(b) with $\rho$ instead of $R$ we see that every domain $f(U(\zeta, r) \cap U(0, \rho))$ is starlike, so is $f(U(\zeta, r) \cap U)$ as a union of starlike domains (see Fig. 3). □

It turns out that $k$-starlike functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geqslant 0, \tag{5}$$

are precisely those functions that are starlike of order $k/(k+1)$ (see [7]). Thus, from Theorem 3, we get the following geometric characterization for functions with negative coefficients that are starlike of order $\alpha$.

**Corollary 2.** *Fix* $\alpha \in (0,1)$. *If* $f$ *is of the form* (5) *and*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha, \quad z \in U,$$

*then* $f(U(\zeta,r) \cap U(0,R))$ *is a starlike domain for every* $\zeta$, $r$, $R$, *such that*

$$0 < R \leqslant 1, \quad |\zeta| \leqslant \frac{\alpha}{1-\alpha}, \quad r \geqslant \sqrt{|\zeta|^2 + R^2}.$$

## References

[1] A.W. Goodman, On uniformly convex functions, Ann. Polon. Math. 56 (1991) 87–92.
[2] A.W. Goodman, Univalent functions, Mariner Publishing Co, Tampa, FL, 1983.
[3] S. Kanas, A. Wiśniowska, Conic regions and $k$-uniform convexity, J. Comput. Appl. Math. 104 (1999) 327–336.
[4] S. Kanas, A. Wiśniowska, Conic regions and starlike functions, Rev. Roumaine Math. Pures Appl. 45 (4) (2000) 647–657.
[5] W. Ma, D. Minda, Uniformly convex functions, Ann. Polon. Math. 52 (2) (1992) 165–175.
[6] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. 118 (1993) 189–196.
[7] K.G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahmanyam, H. Silverman, Subclasses of uniformly convex and uniformly starlike functions, Math. Japon. 42 (3) (1995) 517–522.