Cameron–Martin formula for the $\sigma$-finite measure unifying Brownian penalisations

Kouji Yano

Graduate School of Science, Kobe University, Kobe, Japan

Received 30 September 2009; accepted 25 November 2009
Available online 9 December 2009
Communicated by Paul Malliavin

Abstract


© 2009 Elsevier Inc. All rights reserved.

Keywords: Cameron–Martin formula; Quasi-invariance; Penalisation; Wiener integral

1. Introduction

Let $\Omega = C([0, \infty) \to \mathbb{R})$. Let $(X_t: t \geq 0)$ denote the coordinate process and set $\mathcal{F}_\infty = \sigma(X_t: t \geq 0)$. We consider the following $\sigma$-finite measure on $(\Omega, \mathcal{F}_\infty)$:

\[ W = \int_0^\infty \frac{du}{\sqrt{2\pi u}} \Pi^{(u)} \bullet R \]  \hspace{1cm} (1.1)

where $\Pi^{(u)} \bullet R$ is given as follows:

E-mail address: kyano@math.kobe-u.ac.jp.
(i) $\Pi(u)$ denotes the law of the Brownian bridge from 0 to 0 of length $u$; 
(ii) $R$ denotes the law of the symmetrized 3-dimensional Bessel process; 
(iii) $\Pi(u) \cdot R$ denotes the concatenation of $\Pi(u)$ and $R$.

This measure $\mathcal{W}$ has been introduced by Najnudel, Roynette and Yor [11,12] so that it unifies various Brownian penalisations. The Brownian penalisations can be explained roughly as follows (we will discuss details in Section 2): For a “good” family $\{\Gamma_t(X)\}$ of non-negative $\mathcal{F}_\infty$-functionals such that $\Gamma_t(X) \to \Gamma(X)$ as $t \to \infty$, it holds that

$$\sqrt{\frac{\pi t}{2}} W\left[F_s(X)\Gamma_t(X)\right] \to_{t \to \infty} \mathcal{W}\left[F_s(X)\Gamma(X)\right]$$

for any bounded $\mathcal{F}_s$-measurable functional $F_s(X)$.

The purpose of this paper is to establish quasi-invariance of $\mathcal{W}$ under $h$-translation when $h$ belongs to the Cameron–Martin type space:

$$\left\{ h \in \Omega : h_t = \int_0^t f(s) \, ds \text{ for some } f \in L^2(ds) \cap L^1(ds) \right\}.$$  \hspace{1cm} (1.3)

Now we state our main theorem.

**Theorem 1.1.** Suppose that $h_t = \int_0^t f(s) \, ds$ with $f \in L^2(ds) \cap L^1(ds)$. Then, for any non-negative $\mathcal{F}_\infty$-measurable functional $F(X)$, it holds that

$$\mathcal{W}\left[F(X + h)\right] = \mathcal{W}\left[F(X)\mathcal{E}(f; X)\right]$$

where

$$\mathcal{E}(f; X) = \exp\left(\int_0^\infty f(s) \, dX_s - \frac{1}{2} \int_0^\infty f(s)^2 \, ds\right).$$  \hspace{1cm} (1.5)

Theorem 1.1 will be proved in Section 4.

Theorem 1.1 involves Wiener integral, i.e., the stochastic integral $\int_0^\infty f(s) \, dX_s$ of a deterministic function $f$. (To avoid confusion, we give the following remark: In [3,4], the Wiener integral means the integral with respect to the Wiener measure.) The author has proved in his recent work [18] that this Wiener integral is well defined if $f \in L^2(ds) \cap L^1(ds)$, i.e.,

$$\int_0^\infty |f(s)|^2 \, ds + \int_0^\infty |f(s)| \frac{ds}{1 + \sqrt{s}} < \infty.$$  \hspace{1cm} (1.6)

Note the obvious inclusion: $L^1(ds) \subset L^1\left(\frac{ds}{1 + \sqrt{s}}\right)$. We will discuss details in Section 3. One may conjecture that Theorem 1.1 is valid for $h_t = \int_0^t f(s) \, ds$ with $f \in L^2(ds) \cap L^1\left(\frac{ds}{1 + \sqrt{s}}\right)$, but we have not succeeded at this point.

We give several remarks which help us to understand Theorem 1.1 deeply.
1°). Rephrasing the main theorem. Let \( g(X) \) denote the last exit time from 0 for \( X \):

\[
g(X) = \sup\{u \geq 0 : X_u = 0\}.
\]

(1.7)

For \( u \geq 0 \), let \( \theta_u X \) denote the shifted process: \( (\theta_u X)_s = X_{u+s}, \ s \geq 0 \). Then the definition (1.1) says that the measure \( \mathcal{W} \) can be described as follows:

(i) \( \mathcal{W}(g(X) \in du) = \frac{du}{\sqrt{2\pi u}} \);

(ii) For (Lebesgue) a.e. \( u \in [0, \infty) \), it holds that, given \( g(X) = u \),

(iia) \( (X_s : s \leq u) \) is a Brownian bridge from 0 to 0 of length \( u \);

(iiib) \( ((\theta_u X)_s : s \geq 0) \) is a symmetrized 3-dimensional Bessel process.

In the same manner as this, Theorem 1.1 can be rephrased as the following corollary. We write \( T^*_h \mathcal{W} \) for the image measure of \( X + h \) under \( \mathcal{W} \). For \( u \in [0, \infty) \), we define

\[
\mathcal{E}_u(f; X) = \exp \left( \int_0^u f(s) \, dX_s - \frac{1}{2} \int_0^u f(s)^2 \, ds \right).
\]

(1.8)

Corollary 1.2. Suppose that \( h_t = \int_0^t f(s) \, ds \) with \( f \in L^2(ds) \cap L^1(ds) \). Then it holds that

\[
T^*_h \mathcal{W} = \int_0^\infty \, du \rho^f(u) \Pi^{(u),f} \bullet R^f(-u) \tag{1.9}
\]

where

\[
\rho^f(u) = \frac{1}{\sqrt{2\pi u}} \Pi^{(u),f} \left[ \mathcal{E}_u(f; \cdot) \right] R^f(f(\cdot + u); \cdot).
\]

(1.10)

\[
\Pi^{(u),f}(dX) = \frac{\mathcal{E}_u(f; X) \Pi^{(u)}(dX)}{\Pi^{(u)}[\mathcal{E}_u(f; \cdot)]},
\]

(1.11)

\[
R^f(-u)(dX) = \frac{\mathcal{E}(f(\cdot + u); X) R(dX)}{R[\mathcal{E}(f(\cdot + u); \cdot)]}.
\]

(1.12)

In other words, the law of the process \( X + h \) under \( \mathcal{W} \) may be described as follows:

(i') \( \mathcal{W}(g(X + h) \in du) = \rho^f(u) \, du \);

(ii') For a.e. \( u \in [0, \infty) \), it holds that, given \( g(X + h) = u \),

(iia') \( (X_s + h_s : s \leq u) \) has law \( \Pi^{(u),f} \);

(iiib') \( ((\theta_u (X + h))_s : s \geq 0) \) has law \( R^f(-u) \).

2°). Sketch of the proof. We will divide the proof of Theorem 1.1 into the following steps:

Step 1. \( \mathcal{W}[F(X + h,\cdot;T)] = \mathcal{W}[F(X)\mathcal{E}_T(f; X)] \) for 0 < \( T < \infty \);

Step 2. \( \mathcal{W}[F(X)\mathcal{E}_T(f; X)] \rightarrow \mathcal{W}[F(X)\mathcal{E}(f; X)] \) as \( T \rightarrow \infty \);
Step 3. \( \mathcal{W} [F(X + h_{\wedge T})] \rightarrow \mathcal{W} [F(X + h)] \) as \( T \rightarrow \infty \).

Note that, in Steps 2 and 3, we will confine ourselves to certain particular classes of test functions \( F \).

One may think that Step 1 should be immediate from the following rough argument using (1.2): For any “good” \( \mathcal{F}_T \)-measurable functional \( F_s(X) \),

\[
\mathcal{W} \left[ F_s(X + h_{\wedge T}) \Gamma(X + h_{\wedge T}) \right] = \lim_{t \to \infty} \left( \frac{\pi t}{2} \right)^{\frac{1}{2}} \mathcal{W} \left[ F_s(X) E_T(f; X) \Gamma(X) \right]
\]

This observation, however, should be justified carefully, because the functional \( E_T(f; X) \) is not bounded. We shall utilize Markov property for \( \{ (X_t), \mathcal{W} \} \) (see Section 2.4 for the details):

\[
\mathcal{W} \left[ F_T(X) G(\theta_T X) \right] = \mathcal{W} \left[ F_T(X) \mathcal{W}_{X_T} \left[ G(\cdot) \right] \right]
\]

where \( \mathcal{W}_{X} \) is the image measure of \( x + X \) under \( \mathcal{W}(dX) \). The identity (1.16) suggests, in a way, that \( \{ \mathcal{W}_{x} : x \in \mathbb{R} \} \) is a family of exit laws whose transition up to finite time is the Brownian motion, while the Markov property of the Brownian motion asserts that

\[
\mathcal{W} \left[ F_T(X) G(\theta_T X) \right] = \mathcal{W} \left[ F_T(X) \mathcal{W}_{X_T} \left[ G(\cdot) \right] \right].
\]

This makes a remarkable contrast with Itô’s excursion law \( n \) (see [8]), which satisfies the Markov property:

\[
n \left[ F_T(X) G(\theta_T X) \right] = n \left[ F_T(X) \mathcal{W}_{X_T} \left[ G(\cdot) \right] \right]
\]

where \( \{(X_t), (W^0_X)\} \) denotes the Brownian motion killed upon hitting the origin. In other words, \( n \) produces a family of entrance laws whose transition after positive time is the killed Brownian motion.

We remark again that the Wiener integral \( \int_0^\infty f(s) dX_s \) is not Gaussian. In order to prove necessary estimates involving Wiener integrals in Step 2, we utilize the theory of Wiener integrals for centered Bessel processes, which is due to Funaki, Harioya and Yor [5]. For the 3-dimensional Bessel process \( \{ (X_t), R^+_a \} \) starting from \( a \geq 0 \), we define

\[
\tilde{X}^{(a)}_t = X_t - R^+_a [X_t]
\]

and call \( \{ (\tilde{X}^{(a)}_t), R^+_a \} \) the centered Bessel process. We shall apply, to the convex function \( \psi(x) = (e^{\frac{|x|}{2}} - 1)^2 \), the following theorem, which was proved by Funaki, Harioya and Yor [5] via Brascamp–Lieb inequality [2], and from which we derive our necessary estimates.

**Theorem 1.3.** (See [5].) For any \( f \in L^2(ds) \) and any non-negative convex function \( \psi \) on \( \mathbb{R} \), it holds that
\[ R_\alpha^+ \left[ \psi \left( \int_0^\infty f(t) d\widehat{X}_t(a) \right) \right] \leq W \left[ \psi \left( \int_0^\infty f(t) dX_t \right) \right]. \tag{1.20} \]

For the proof of Theorem 1.3, see [5, Proposition 4.1].

3\textsuperscript{o}). \textbf{Comparison with the Brownian case.} Let us recall the well-known Cameron–Martin formula for Brownian motion (see [3,4]). Let \( W \) stand for the Wiener measure on \( \Omega \) with \( W(X_0 = 0) = 1 \).

It is well known that, if \( h_t = \int_0^t f(s) ds \) with \( f \in L^2(ds) \),

\[ W[F(X + h)] = W[F(X) \mathcal{E}(f; X)] \tag{1.21} \]

for any non-negative \( \mathcal{F}_\infty \)-measurable functional \( F(X) \). It is also well known that, if \( h \notin H \), the image measure of \( X + h \) under \( W(dX) \) is mutually singular on \( \mathcal{F}_\infty \) to \( W(dX) \).

It is immediate from (1.21) that, if \( h_t = \int_0^t f(s) ds \) with \( f \in L^2_{\text{loc}}(ds) \), then

\[ W[F_t(X + h)] = W[F_t(X) \mathcal{E}_t(f; X)] \tag{1.22} \]

for any non-negative \( \mathcal{F}_t \)-measurable functional \( F_t(X) \) where

\[ \mathcal{E}_t(f; X) = \exp \left( \int_0^t f(s) dX_s - \frac{1}{2} \int_0^t f(s)^2 ds \right). \tag{1.23} \]

Now we give some remarks about comparison between the two cases of \( W \) and \( \Psi \).

(i) Let \( f \in L^2(ds) \). As a corollary of (1.21), we see that \( W[\mathcal{E}(f; X)] < \infty \) and, consequently, that \( W[\mathcal{E}(f; X)^p] < \infty \) for any \( p \geq 1 \). This shows that, if \( F(X) \in L^p(W(dX)) \) for some \( p > 1 \), then \( F(X + h) \in L^1(W(dX)) \).

Let \( f \in L^2(ds) \cap L^1(ds) \). In the case of \( \Psi \), however, we see immediately by taking \( F \equiv 1 \) in (1.4) that

\[ \Psi[\mathcal{E}(f; X)] = \infty, \tag{1.24} \]

which we should always keep in mind. Now the following question arises:

\[ \Psi[\mathcal{E}(f; X) \Gamma(X)] < \infty \tag{1.25} \]

holds for what functional \( \Gamma(X) \)? The problem is that we do not know the distribution of the Wiener integral \( \int_0^\infty f(s) dX_s \) under \( \Psi \); in fact, it is no longer Gaussian! In Theorem 4.2, we will appeal to a certain penalisation result and establish (1.25) for Feynman–Kac functionals \( \Gamma(X) \), the class of which we shall introduce in Section 2.2.

(ii) In the Brownian case, we have the following criterion: The \( h \)-translation of \( W \) is quasi-invariant or singular with respect to \( W \) according as \( h \in L^2(ds) \) or \( h \notin L^2(ds) \), respectively.

In the case of \( \Psi \), however, we do not know what happens on \( \Psi(dX) \) when \( h \notin H \) or when \( f \notin L^1(ds) \).
(iii) Let \( f \in L^2_{\text{loc}}(ds) \). In the Brownian case, we have the quasi-invariance (1.22) on each \( \mathcal{F}_t \). In the case of \( \mathcal{W} \) (dX), however, we find a drastically different situation (see Theorem 2.5): For any non-negative \( \mathcal{F}_t \)-measurable functional \( F_t(X) \),

\[
\mathcal{W}[F_t(X + h)] = \mathcal{W}[F_t(X)] = 0 \text{ or } \infty
\]

according as \( W(F_t(X) = 0) = 1 \) or \( W(F_t(X) = 0) < 1 \).

4°). **Integration by parts formulae.** From the Cameron–Martin theorem (1.21) in the Brownian case, we immediately obtain the following integration by parts formula:

\[
W[\nabla_h F(X)] = W\left[F(X) \int_0^\infty f(s) \, dX_s\right]
\]

for \( h_t = \int_0^t f(s) \, ds \) with \( f \in L^2(ds) \) and for any good functional \( F(X) \), where \( \nabla \) denotes the Gross–Sobolev–Malliavin derivative (see, e.g., [17]). In the case of \( \mathcal{W} \), from Theorem 1.1, we may expect the following integration by parts formula:

\[
\mathcal{W}[\partial_h F(X)] = \mathcal{W}\left[F(X) \int_0^\infty f(s) \, dX_s\right]
\]

for \( h_t = \int_0^t f(s) \, ds \) with \( f \in L^2(ds) \cap L^1(ds) \) and for any good functional \( F(X) \), where \( \partial_h \) is in the Gâteaux sense. We have not succeeded in finding a reasonable class of functionals \( F \) for which both sides of (1.28) make sense and coincide.

Let us give a remark about 3-dimensional Bessel bridge of length \( u \) from 0 to 0, which we denote by \( \{(X_s; \ s \in [0,u]), \ R^+_u\} \). Although we do not have the Cameron–Martin formula for the bridge, there is a remarkable result due to Zambotti [20,21] that the following integration by parts formula holds:

\[
R^+.(1)[\partial_h F(X)] = R^+.(1)\left[F(X) \int_0^\infty f(s) \, dX_s\right] + (BC)
\]

for \( h_t = \int_0^t f(s) \, ds \) with \( f \) satisfying a certain regularity condition and for any good functional \( F(X) \), where \( \partial_h \) is in the Gâteaux sense and where

\[
(BC) = -\int_0^1 \frac{du \ h_u}{\sqrt{2\pi u^3(1-u)^3}} \left[ R^+(u) \cdot R^+,(1-u)\right][F(\cdot)].
\]

The remainder term (BC) may describe the *boundary contribution*. Indeed, the measure \( R^+.(1) \) is supported on the set of non-negative continuous paths on \([0,1]\), while the measure \( R^+(u) \cdot R^+,(1-u) \) is supported on the subset of paths which hit 0 once and only once; the latter set may be regarded in a certain sense as the boundary of the former. See also Bonaccorsi and
Zambotti [1], Zambotti [22], Hariya [7] and Funaki and Ishitani [6] for similar results about integration by parts formulae.

The organization of this paper is as follows. In Section 2, we recall several results of Brownian penalisations. In Section 3, we study Wiener integrals for the processes considered. Section 4 is devoted to the proofs of our main theorems.

2. Brownian penalisations

2.1. Notations

Let $X = (X_t: t \geq 0)$ denote the coordinate process of the space $\Omega = C([0, \infty); \mathbb{R})$ of continuous functions from $[0, \infty)$ to $\mathbb{R}$. Let $\mathcal{F}_t = \sigma(X_s: s \leq t)$ for $0 < t < \infty$ and $\mathcal{F}_\infty = \sigma(\bigcup_t \mathcal{F}_t)$. For $0 < u < \infty$, we write $X(u) = (X_t: 0 \leq t \leq u)$ and $\Omega(u) = C([0, u]; \mathbb{R})$.

1°). Brownian motion. For $a \in \mathbb{R}$, we denote by $W_a$ the Wiener measure on $\Omega$ with $W_a(X_0 = a) = 1$. We simply write $W$ for $W_0$.

2°). Brownian bridge. We denote by $\Pi^{(u)}$ the law on $\Omega^{(u)}$ of the Brownian bridge:

$$\Pi^{(u)}(\cdot) = W(\cdot | X_u = 0).$$

(2.1)

The process $X^{(u)}$ under $\Pi^{(u)}$ is a centered Gaussian process with covariance $\Pi^{(u)}[X_sX_t] = s - st/u$ for $0 \leq s \leq t \leq u$. As a realization of $\{X^{(u)}, \Pi^{(u)}\}$, we may take

$$\left\{ B_s - \frac{s}{u} B_u: s \in [0, u] \right\}.$$  

(2.2)

3°). 3-Dimensional Bessel process. For $a \geq 0$, we denote by $R^+_a$ the law on $\Omega$ of the 3-dimensional Bessel process starting from $a$, i.e., the law of the process $(\sqrt{Z_t})$ where $(Z_t)$ is the unique strong solution to the stochastic differential equation

$$dZ_t = 2\sqrt{|Z_t|} \, dB_t + 3 \, dt, \quad Z_0 = a^2$$  

(2.3)

with $(B_t)$ a one-dimensional standard Brownian motion. Under $R^+_a$, the process $X$ satisfies

$$dX_t = dB_t + \frac{1}{X_t} \, dt, \quad X_0 = a$$  

(2.4)

with $\{(B_t), R^+_a\}$ a one-dimensional standard Brownian motion. For $a > 0$, we denote by $R^-_a$ the law on $\Omega$ of $(-X_t)$ under $R^+_a$. We define

$$R_a = \begin{cases} R^+_a & \text{if } a > 0, \\ R^-_a & \text{if } a < 0 \end{cases}$$

(2.5)

and
\[ R = R_0 = \frac{R_0^+ + R_0^-}{2}; \]  

(2.6)
in other words, \( R \) is the law on \( \Omega \) of \( (\varepsilon X_t) \) under the product measure \( P(d\varepsilon) \otimes R_0^+(dX) \) where \( P(\varepsilon = 1) = P(\varepsilon = -1) = 1/2. \)

**4°. The \( \sigma \)-finite measure \( \mathcal{W} \).** For \( u > 0 \) and for two processes \( X^{(u)} = (X_t; 0 \leq t \leq u) \) and \( Y = (Y_t; t \geq 0) \), we define the concatenation \( X^{(u)} \cdot Y \) as

\[
(X^{(u)} \cdot Y)_t = \begin{cases} 
X_t & \text{if } 0 \leq t < u, \\
Y_{t-u} & \text{if } t \geq u \text{ and } X_u = Y_0, \\
X_u & \text{if } t \geq u \text{ and } X_u \neq Y_0.
\end{cases}
\]  

(2.7)

We define the concatenation \( \Pi^{(u)} \cdot R \) as the law of \( X^{(u)} \cdot Y \) under the product measure \( \Pi^{(u)}(dX^{(u)}) \otimes R(dY) \). Then we define

\[
\mathcal{W} = \int_0^\infty \frac{du}{\sqrt{2\pi u}} \Pi^{(u)} \cdot R.
\]  

(2.8)

For \( x \in \mathbb{R} \), we define \( \mathcal{W}_x \) as the image measure of \( x + X \) under \( \mathcal{W}(dX) \); in other words,

\[
\mathcal{W}_x[F(X)] = \mathcal{W}[F(x + X)]
\]  

(2.9)

for any non-negative \( \mathcal{F}_\infty \)-measurable functional \( F(X) \).

**5°. Random times.** For \( a \in \mathbb{R} \), we denote the first hitting time of \( a \) by

\[
\tau_a(X) = \inf\{t > 0: X_t = a\}.
\]  

(2.10)

We denote the last exit time from 0 by

\[
g(X) = \sup\{t \geq 0: X_t = 0\}.
\]  

(2.11)

**2.2. Feynman–Kac penalisations**

Let \( L_y^Y(X) \) denote the local time by time \( t \) of level \( y \): For \( W_x(dX) \)-a.e. \( X \), it holds that

\[
\int_0^t 1_A(X_s) \, ds = \int_A L_y^Y(X) \, dy, \quad A \in \mathcal{B}(\mathbb{R}), \ t \geq 0.
\]  

(2.12)

For a non-negative Borel measure \( V \) on \( \mathbb{R} \) and a process \( (X_t) \) under \( W(dX) \), we write

\[
\mathcal{K}_t(V; X) = \exp\left( -\int_{\mathbb{R}} L_y^X(X) \, V(dy) \right)
\]  

(2.13)
and

$$K(V; X) = \exp \left( - \int_{\mathbb{R}} L_{\infty}^X (X) V(dx) \right).$$

(2.14)

The following theorem is due to Roynette, Vallois and Yor [14], [15, Theorem 4.1] and [16, Theorem 2.1].

**Theorem 2.1.** (See [16, Theorem 2.1].) Let $V$ be a non-negative Borel measure on $\mathbb{R}$ and suppose that

$$0 < \int_{\mathbb{R}} (1 + |x|) V(dx) < \infty.$$  

(2.15)

Then the following statements hold:

(i) $\varphi_V(x) := \lim_{t \to \infty} \sqrt{\frac{2}{t}} W_x[K_t(V; X)]$ and the limit exists in $\mathbb{R}_+$;

(ii) $\varphi_V$ is the unique solution of the Sturm–Liouville equation

$$\varphi_{''}(x) = 2 \varphi_V(x) V(dx)$$

(2.16)

in the sense of distributions (see, e.g., [13, Appendix §8]) subject to the boundary condition:

$$\lim_{x \to -\infty} \varphi_V'(x) = -1 \quad \text{and} \quad \lim_{x \to \infty} \varphi_V'(x) = 1;$$

(2.17)

(iii) For any $0 < s < \infty$ and any bounded $\mathcal{F}_s$-measurable functional $F_s(X)$,

$$\frac{W_x[F_s(X)K_s(V; X)]}{W_x[K_s(V; X)]} \to W_x \left[ F_s(X) \frac{\varphi_V(X_s)}{\varphi_V(X_0)} K_s(V; X) \right] \quad \text{as} \quad t \to \infty;$$

(2.18)

(iv) $(M_s^{(V)}(X) := \frac{\varphi_V(X_s)}{\varphi_V(X_0)} K_s(V; X): s \geq 0)$ is a $(W_V, (\mathcal{F}_s))$-martingale which converges a.s. to 0 as $s \to \infty$;

(v) Under the probability measure $W^{(V)}_x$ on $\mathcal{F}_\infty$ induced by the relation

$$W^{(V)}_x[F_s(X)] = W_x[F_s(X) M^{(V)}_s(X)],$$

(2.19)

the process $(X_t)$ solves the stochastic differential equation

$$X_t = x + B_t + \int_0^t \frac{\varphi_V}{\varphi_V}(X_s) \, ds$$

(2.20)

where $(B_t)$ is a $(W^{(V)}_x, (\mathcal{F}_t))$-Brownian motion starting from 0; in particular, the process $(X_t)$ is a transient diffusion which admits the following function $\gamma_V(x)$ as its scale function:
\[ \gamma_V(x) = \frac{1}{\varphi_V(y)^2}. \]  
(2.21)

**Remark 2.2.** By (ii) of Theorem 2.1, we see that the function \( \varphi_V \) also enjoys the following properties:

1. \( \varphi_V(x) \sim |x| \) as \( x \to \infty \). This suggests that the process \((X_t, (W_x^{(V)})\) behaves like 3-dimensional Bessel process when the value of \( |X_t| \) is large.
2. \( \inf_{x \in \mathbb{R}} \varphi_V(x) > 0 \). This shows that the origin is regular for itself.

**Example 2.3.** (A key example for [14].) Suppose that \( V = \lambda \delta_0 \) with some \( \lambda > 0 \) where \( \delta_0 \) denotes the Dirac measure at 0. That is,

\[ K_t(\lambda \delta_0; X) = \exp\left(-\lambda L_0^0(X)\right). \]  
(2.22)

Then we can solve Eqs. (2.16)–(2.17) and consequently we obtain

\[ \varphi_{\lambda \delta_0}(x) = \frac{1}{\lambda} + |x|, \]  
(2.23)

\[ M_t^{(\lambda \delta_0)}(X) = \left(1 + \lambda |X_t|\right) \exp\left(-\lambda L_0^0(X)\right) \]  
(2.24)

and

\[ X_t = x + B_t + \int_0^t \frac{\operatorname{sgn}(X_s)}{\frac{1}{\lambda} + |X_s|} \, ds \quad \text{under } W_x^{(V)}. \]  
(2.25)

### 2.3. The universal \( \sigma \)-finite measure

Najnudel, Roynette and Yor [11,12] introduced the measure \( \mathcal{W} \) on \( \mathcal{F}_\infty \) defined by (2.8) to give a global view on the Brownian penalisations. It unifies the Feynman–Kac penalisations in the sense of the following theorem, which is due to Najnudel, Roynette and Yor [12, Theorem 1.1.2 and Theorem 1.1.6]; see also Yano, Yano and Yor [19, Theorem 8.1]. See also Najnudel and Nikeghbali [9,10] for careful treatment of augmentation of filtrations.

**Theorem 2.4.** (See [12].) Let \( x \in \mathbb{R} \) and let \( V \) be a non-negative measure on \( \mathbb{R} \) satisfying (2.15). Then it holds that

\[ \mathcal{W}_t[Z_t(X)K(V; X)] = W_t[Z_t(X)\varphi_V(X_t)K_t(V; X)] \]  
(2.26)

for any \( t \geq 0 \) and any non-negative \( \mathcal{F}_t \)-measurable functional \( Z_t(X) \), where \( K(V; X) \) has been defined as (2.14). Consequently, it holds that

\[ \varphi_V(x) = \mathcal{W}_x[K(V; X)] \]  
(2.27)

and that
\[ W_x^{(V)}(dX) = \frac{1}{\varphi_V(x)}K(V; X) \mathcal{W}_x(dX) \quad \text{on } F_\infty. \]  \hfill (2.28)

The following theorem can be found in [12, p. 6, point v) and Theorem 1.1.6]; see also [19, Theorem 5.1].

**Theorem 2.5.** (See [12].) The following statements hold:

(i) \( \mathcal{W}(g(X) \in du) = \frac{du}{\sqrt{2\pi u}} \) on \([0, \infty)\).

In particular, \( \mathcal{W} \) is \( \sigma \)-finite on \( F_\infty \);

(ii) For \( A \in F_t \) with \( 0 < t < \infty \),

\[ \mathcal{W}(A) = \begin{cases} 0 & \text{if } W(A) = 0, \\ \infty & \text{if } W(A) > 0. \end{cases} \]

In particular, \( \mathcal{W} \) is not \( \sigma \)-finite on \( F_t \).

We give the proof for completeness of this paper.

**Proof.** Claim (i) is obvious by definition (1.1) of \( \mathcal{W} \). Let us prove claim (ii). Let \( 0 < t < \infty \). Suppose that \( A \in F_t \) and \( W(A) = 0 \). Then we have \( \mathcal{W}[1_A K(\delta_0; X)] = 0 \) by (2.26), which implies that \( \mathcal{W}(A) = 0 \). Suppose in turn that \( A \in F_t \) and \( W(A) > 0 \). For \( \lambda > 0 \), we apply (2.26) for \( V = \lambda \delta_0 \) and we have

\[ \mathcal{W}(A) \geq \mathcal{W}[1_A e^{-\lambda L^0_t}] = \mathcal{W}[1_A \left( \frac{1}{\lambda} + |X_t| \right) e^{-\lambda L^0_t}] \geq \frac{1}{\lambda} \mathcal{W}[1_A e^{-\lambda L^0_t}]. \]  \hfill (2.29)

Letting \( \lambda \to 0^+ \), we obtain, by the monotone convergence theorem, that \( \mathcal{W}[1_A e^{-\lambda L^0_t}] \to W(A) > 0 \), and consequently, that \( \mathcal{W}(A) = \infty \). \( \Box \)

We also need the following property.

**Proposition 2.6.** For \( x \in \mathbb{R} \), it holds that

\[ \mathcal{W}_x(\tau_0(X) = \infty) = |x|. \]  \hfill (2.30)

**Proof.** By symmetry, we have only to prove the claim for \( x \geq 0 \). Let \( V = \delta_0 \) and \( F(X) = 1_{\{\tau_0(X) = \infty\}} \). Note that \( L^0_\infty(X) = 0 \) if \( \tau_0(X) = \infty \). Hence it follows from Example 2.3 and Theorem 2.4 that

\[ \mathcal{W}_x(\tau_0(X) = \infty) = \varphi_{\delta_0}(x) W_x^{(\delta_0)}(\tau_0(X) = \infty). \]  \hfill (2.31)

Since \( \varphi_{\delta_0}(x) = 1 + x \) and since \( \gamma_{\delta_0}(x) = \frac{x}{1+x} \), we have

\[ \mathcal{W}_x(\tau_0(X) = \infty) = (1 + x) \cdot \frac{\gamma_{\delta_0}(x) - \gamma_{\delta_0}(0)}{\gamma_{\delta_0}(\infty) - \gamma_{\delta_0}(0)} = x. \]  \hfill (2.32)

The proof is complete. \( \Box \)
2.4. Markov property of \((X_t), (\mathcal{F}_t), (\mathcal{W}_x)\)

We may say that \((X_t), (\mathcal{F}_t), (\mathcal{W}_x)\) possesses Markov property in the following sense.

**Theorem 2.7. (See \([11,12]\).)** Let \(x \in \mathbb{R}\) and \(t \geq 0\). Let \(F\) be a non-negative \(\mathcal{F}_\infty\)-measurable functional. Then it holds that

\[
\mathcal{W}_x[Z_t(X)F(\theta_t X)] = W_x[Z_t(X)\mathcal{W}_x[F(\cdot)]]
\]

(2.33)

for any non-negative \(\mathcal{F}_t\)-measurable functional \(Z_t(X)\). Moreover, the constant time \(t\) in (2.33) may be replaced by any finite \((\mathcal{F}_t)\)-stopping time \(\tau\).

**Proof.** Let \(V\) be as in Theorem 2.1. Then we have

\[
\mathcal{W}_x[Z_t(X)K_t(V; X)F(\theta_t X)K(V; \theta_t X)] = \mathcal{W}_x[Z_t(X)F(\theta_t X)]
\]

(by the multiplicativity property of \(K(V; X)\))

(2.35)

\[
= \varphi_V(X)W_x^{(V)}[Z_t(X)F(\theta_t X)]
\]

(by (2.28))

(2.36)

\[
= \varphi_V(X)W_x^{(V)}[Z_t(X)W_{\mathcal{W}_x}[F(\cdot)]]
\]

(by the Markov property of \(W_x^{(V)}\))

(2.37)

\[
= \varphi_V(X)W_x[Z_t(X)W_{\mathcal{W}_x}[F(\cdot)] \cdot \varphi_V(X)\mathcal{K}_t(V; X)]
\]

(by (2.19))

(2.38)

\[
= W_x[Z_t(X)\mathcal{K}_t(V; X)\mathcal{W}_x[F(\cdot)]]
\]

(by (2.28)).

(2.39)

Taking \(V = \lambda\delta_0\) and letting \(\lambda \to 0^+\), we obtain (2.33) by the monotone convergence theorem. In the same way, we can prove (2.33) also in the case where the constant time \(t\) is replaced by a finite stopping time \(\tau\). \(\square\)

Since the measure \(\mathcal{W}_x\) has infinite total mass, we cannot consider conditional expectation in the usual sense. But, by the help of Theorem 2.7, we can introduce a counterpart in the following sense.

**Corollary 2.8. (See \([11,12]\); see also \([19]\).)** Let \(x \in \mathbb{R}\) and \(t \geq 0\). Let \(F\) be a \(\mathcal{F}_\infty\)-measurable functional which is in \(L^1(\mathcal{W}_x)\). Then there exists a unique \((\mathcal{F}_t), W_x\)\)-martingale \(M_t[F; X]\) such that

\[
\mathcal{W}_x[Z_t(X)F(X)] = W_x[Z_t(X)M_t[F; X]]
\]

(2.40)

for any bounded \(\mathcal{F}_t\)-measurable functional \(Z_t(X)\). Moreover, it is given as

\[
M_t[F; X] = \int_{\Omega} \mathcal{W}_x\,(dY)F(X^{(t)} \cdot Y), \quad W_x(dX)\text{-a.s.}
\]

(2.41)
Remark 2.9. If \( F \in L^1(W_x) \), then the family of the conditional expectations \( \{ W_x[F|\mathcal{F}_t] : t \geq 0 \} \) is a uniformly integrable martingale. In contrast with this fact, if \( F \in L^1(W_x) \), the martingale \( \{ M_t[F; X] : t \geq 0 \} \) under \( W_x \) converges to 0 as \( t \to \infty \), and consequently, it is not uniformly integrable.

Remark 2.10. Since \( M_t \) is an operator from \( L^1(W_x) \) to \( L^1(W_x) \), we do not have a counterpart of the tower property for the usual conditional expectation.

Example 2.11. Let \( V \) be a non-negative measure on \( \mathbb{R} \) satisfying (2.15). Then (iv) and (v) of Theorem 2.1 may be rewritten as

\[
M_t[K(V; \cdot); X] = \varphi_V(X_t)K_t(V; X). \tag{2.42}
\]

From this and from Remark 2.2, we see that

\[
M_t[K(V; \cdot); X] \in L^p(W) \quad \text{for any } p \geq 1. \tag{2.43}
\]

In particular, formula (2.24) may be rewritten as

\[
M_t[K(\lambda \delta_0; \cdot); X] = \left( \frac{1}{\lambda} + |X_t| \right)K_t(\lambda \delta_0; X). \tag{2.44}
\]

3. Wiener integrals

Let \( S \) denote the set of all step functions \( f \) on \([0, \infty)\) of the form:

\[
f(t) = \sum_{k=1}^{n} c_k 1_{[t_{k-1}, t_k)}(t), \quad t \geq 0 \tag{3.1}
\]

with \( n \in \mathbb{N} \), \( c_k \in \mathbb{R} \) \((k = 1, \ldots, n)\) and \( 0 = t_0 < t_1 < \cdots < t_n < \infty \). Note that \( S \) is dense in \( L^2(ds) \). For a function \( f \in S \) and a process \( X \), we define

\[
\int_0^\infty f(t) \, dX_t = \sum_{k=1}^{n} c_k (X_{t_k} - X_{t_{k-1}}). \tag{3.2}
\]

If \( \int_0^\infty f(t) \, dX_t \) can be defined as the limit in some sense of \( \int_0^\infty f_n(t) \, dX_t \) for an approximating sequence \( \{ f_n \} \) of \( f \), then we will call it Wiener integral of \( f \) for the process \( X \).

We have the following facts: If a sequence \( \{ f_n \} \subset S \) approximates \( f \) in \( L^2(ds) \), then it holds that

\[
\int_0^\infty f_n(s) \, dX_s \xrightarrow{n \to \infty} \int_0^\infty f(s) \, dX_s \quad \text{in } W\text{-probability} \tag{3.3}
\]

and that, for any \( u > 0 \),
\[
\int_0^u f_n(s) \, dX_s \quad \text{for} \quad n \to \infty \quad \int_0^u f(s) \, dX_s \quad \text{in } \mathcal{P}^{(u)}\text{-probability.}
\] (3.4)

3.1. Wiener integral for 3-dimensional Bessel process

Let \( p_t(x) \) denote the density of the Brownian semigroup:

\[
p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right), \quad t > 0, \quad x \in \mathbb{R}.
\] (3.5)

Let \( a \geq 0 \) be fixed. It is well known (see, e.g., [13, §VI.3]) that, for \( t > 0 \) and \( x > 0 \),

\[
R^+_a (X_t \in \text{d}x) = \begin{cases} \frac{1}{a} \int_a^x (p_t(x-a) - p_t(x+a)) \, dx, & a > 0, \\ 2x^2 \frac{1}{t} p_t(x) \, dx, & a = 0. \end{cases}
\] (3.6)

From this formula, it is straightforward that, for \( t > 0 \) and \( x > 0 \),

\[
\phi_a(t) := R^+_a \left[ \frac{1}{X_t} \right] = \begin{cases} \frac{1}{a} \int_0^a p_t(x) \, dx, & a > 0, \\ 2p_t(0) = \sqrt{\frac{2}{\pi t}}, & a = 0. \end{cases}
\] (3.7)

Since \( p_t(x) \leq p_t(0) \), it is obvious by definition that

\[
\phi_a(t) \leq \phi(0), \quad a > 0, \quad t > 0.
\] (3.8)

Note that \( \phi_a(t) \) has the following asymptotics as \( t \to 0^+ \):

\[
\phi_a(t) \sim \begin{cases} 1/a & \text{if } a > 0, \\ \sqrt{2/(\pi t)} & \text{if } a = 0. \end{cases}
\] (3.9)

By the stochastic differential equation (2.4), we see that

\[
R^+_a [X_t] = a + \int_0^t R^+_a \left[ \frac{1}{X_s} \right] \, ds = a + \int_0^t \phi_a(s) \, ds.
\] (3.10)

Now the following lemma is obvious.

**Lemma 3.1.** Let \( f \in L^2(ds) \cap L^1(\phi_a(s) \, ds) \). Then, according to the stochastic differential equation (2.4), the Wiener integral may be defined as

\[
\int_0^\infty f(s) \, dX_s = \int_0^\infty f(s) \, dB_s + \int_0^\infty \frac{f(s)}{X_s} \, ds.
\] (3.11)

If a sequence \( \{f_n\} \subset S \) approximates \( f \) both in \( L^2(ds) \) and in \( L^1(\phi_a(s) \, ds) \), i.e.,
\[ \int_{0}^{\infty} |f_n(s) - f(s)|^2 ds + \int_{0}^{\infty} |f_n(s) - f(s)| \phi_a(s) ds \to 0, \quad n \to \infty, \quad (3.12) \]

then it holds that

\[ \int_{0}^{\infty} f_n(s) dX_s \to \int_{0}^{\infty} f(s) dX_s \quad \text{in} \quad R^+_a \text{-probability.} \quad (3.13) \]

Following Funaki, Hariya and Yor [5], we may propose another way of constructing the Wiener integral. We define

\[ \hat{X}^{(a)}_s = X_s - R^+_a[X_s] \quad (3.14) \]

and we call \((\hat{X}^{(a)}_s), R^+_a\) the centered Bessel process. We simply write \(\hat{X}_s\) for \(\hat{X}^{(0)}_s\). By applying Theorem 1.3 with \(\psi(x) = x^2\), we obtain the following fact: If a sequence \(\{f_n\} \subset S\) approximates \(f\) in \(L^2(ds)\), then it holds that

\[ \int_{0}^{\infty} f_n(s) d\hat{X}^{(a)}_s \to \int_{0}^{\infty} f(s) d\hat{X}^{(a)}_s \quad \text{in} \quad R^+_a \text{-probability.} \quad (3.15) \]

We then obtain the following lemma.

**Lemma 3.2.** Let \(f \in L^2(ds) \cap L^1(\phi_a(s) ds)\). Then it holds that

\[ \int_{0}^{\infty} f(s) dX_s = \int_{0}^{\infty} f(s) d\hat{X}^{(a)}_s + \int_{0}^{\infty} f(s) \phi_a(s) ds \quad R^+_a \text{-a.s.} \quad (3.16) \]

### 3.2. Wiener integral for \(X\) under \(\mathcal{W}\)

Define

\[ L^1_+(\mathcal{W}) = \{ G : \Omega \to R_+, \mathcal{F}\text{-measurable, } \mathcal{W}(G = 0) = 0, \mathcal{W}[G] < \infty \}. \quad (3.17) \]

For \(G \in L^1_+(\mathcal{W})\), we define a probability measure \(\mathcal{W}^G\) on \((\Omega, \mathcal{F})\) by

\[ \mathcal{W}^G(A) = \frac{\mathcal{W}[1_A G]}{\mathcal{W}[G]}, \quad A \in \mathcal{F}. \quad (3.18) \]

We recall the following notion of convergence.

**Proposition 3.3.** Let \(Z, Z_1, Z_2, \ldots\) be \(\mathcal{F}_\infty\text{-measurable functionals}. Then the following statements are equivalent:
(i) For any $\varepsilon > 0$ and any $A \in \mathcal{F}$ with $\mathcal{W}(A) < \infty$, it holds that $\mathcal{W}(A \cap \{ |Z_n - Z| \geq \varepsilon \}) \to 0$.

(ii) $Z_n \to Z$ in $\mathcal{W}^G$-probability for some $G \in L_+^1(\mathcal{W})$.

(iii) $Z_n \to Z$ in $\mathcal{W}^G$-probability for any $G \in L_+^1(\mathcal{W})$.

(iv) One can extract, from an arbitrary subsequence, a further subsequence $\{n(k) : k = 1, 2, \ldots\}$ along which $Z_{n(k)} \to Z$ $\mathcal{W}$-a.e.

If one (and hence all) of the above statements holds, then we say that

$$Z_n \to Z \quad \text{locally in } \mathcal{W} \text{-measure.}$$  \hspace{1cm} (3.19)

For the proof of Proposition 3.3, see, e.g., [18].

Wiener integral for $X$ under $\mathcal{W}(dX)$ may be defined with the help of the following theorem.

**Theorem 3.4.** (See [18].) Let $f \in L^2(ds) \cap L^1(\frac{ds}{1+\sqrt{s}})$. Suppose that a sequence $\{f_n\} \subset S$ approximates $f$ both in $L^2(ds)$ and in $L^1(\frac{ds}{1+\sqrt{s}})$, i.e.,

$$\int_0^\infty |f_n(s) - f(s)|^2 ds + \int_0^\infty |f_n(s) - f(s)| \frac{ds}{1 + \sqrt{s}} \to 0 \quad \text{as } n \to \infty. \hspace{1cm} (3.20)$$

(Note that this condition is strictly weaker than the condition (3.12).) Then it holds that

$$\int_0^\infty f_n(s) dX_s \to \int_0^\infty f(s) dX_s \quad \text{locally in } \mathcal{W} \text{-measure.} \hspace{1cm} (3.21)$$

Moreover, there exists a functional $J(f; u, X)$ measurable with respect to the product $\sigma$-field $\mathcal{B}([0, \infty)) \otimes \mathcal{F}_\infty$ such that

$$\int_0^\infty f(s) dX_s = J(f; g(X), X) \quad \mathcal{W} \text{-a.e.} \hspace{1cm} (3.22)$$

and that it holds $du$-a.e. that

$$J(f; u, X^{(u)} \cdot Y) = \int_0^u f(s) dX_s + \int_0^\infty f(s + u) dY_s \hspace{1cm} (3.23)$$

is valid a.e. with respect to $\Pi^{(u)}(dX^{(u)}) \otimes R(dY)$.

The following lemma allows us to use the same notation for Wiener integrals under $W(dX)$ and $\mathcal{W}(dX)$. Let us temporarily write $I^W(f; X)$ (resp. $I^\mathcal{W}(f; X)$) for the Wiener integral $I(f; X)$ under $W(dX)$ (resp. $\mathcal{W}(dX)$).
Lemma 3.5. Suppose that there exist $F \in L^1(W)$ and $G \in L^1(W)$ such that

$$H(X)F(X) = W[H(X)G(X)]$$

holds for any bounded measurable functional $H(X)$. Then, for any $f \in L^2(ds) \cap L^1(d\frac{ds}{1+\sqrt{s}})$, it holds that

$$W[\phi(I^W(f;X))H(X)F(X)] = W[\phi(I^W(f;X))H(X)G(X)]$$

for any bounded Borel function $\phi$ on $\mathbb{R}$.

Proof. This is obvious by Theorem 3.4 and by the dominated convergence theorem. □

3.3. Integrability lemma

For later use, we need the following lemma.

Lemma 3.6. Let $f \in L^1(ds)$. Define

$$\tilde{f}(t) = \int_0^\infty |f(s+t)| \frac{ds}{\sqrt{s}} = \int_t^\infty |f(s)| \frac{ds}{\sqrt{s-t}}, \quad t > 0.$$ (3.26)

Then the following statements hold:

(i) For any $a > 0$, it holds that

$$\int_0^a \tilde{f}(t) dt \leq 2\sqrt{a} \int_0^\infty |f(s)| ds.$$ (3.27)

(ii) There exists a sequence $t(n) \to \infty$ such that $\tilde{f}(t(n)) \to 0$.

Proof. (i) Let $a > 0$. Then we have

$$\int_0^a \tilde{f}(t) dt = \int_0^a dt \int_0^a |f(s)| \frac{ds}{\sqrt{s-t}} + \int_0^a dt \int_a^\infty |f(s)| \frac{ds}{\sqrt{s-t}}$$ (3.28)

$$= \int_0^a ds |f(s)| \int_0^s \frac{dr}{\sqrt{s-t}} + \int_0^\infty ds |f(s)| \int_0^a \frac{dr}{\sqrt{s-t}}$$ (3.29)

$$\leq \int_0^a |f(s)| (2\sqrt{s}) ds + \int_0^\infty ds |f(s)| \int_0^a \frac{dr}{\sqrt{s-t}}$$ (3.30)
\[ \leq 2\sqrt{a} \int_0^\infty |f(s)| \, ds. \] (3.31)

(ii) Let \( 0 < a < b < \infty \). Then we have

\[ \frac{(b - a)}{\sqrt{b}} \inf_{t : t > a} \tilde{f}(t) \leq \frac{1}{\sqrt{b}} \int_a^b \tilde{f}(t) \, dt \leq 2 \int_0^\infty |f(s)| \, ds. \] (3.32)

Since \( (b - a)/\sqrt{b} \to \infty \) as \( b \to \infty \) with \( a \) fixed, we see that \( \inf_{t : t > a} \tilde{f}(t) = 0 \) for any \( a > 0 \). This implies that

\[ \liminf_{t \to \infty} \tilde{f}(t) = 0. \] (3.33)

The proof is now complete. \( \square \)

4. Cameron–Martin formula

For a function \( h_t = \int_0^t f(s) \, ds \) with \( f \in L^2(ds) \cap L^1(\frac{ds}{1+\sqrt{s}}) \) and a process \((X_s)\) under \( \mathcal{W}_x \) for \( x \in \mathbb{R} \), we write

\[ \mathcal{E}_t(f; X) = \exp \left( \int_0^t f(s) \, dX_s - \frac{1}{2} \int_0^t f(s)^2 \, ds \right) \] (4.1)

and

\[ \mathcal{E}(f; X) = \exp \left( \int_0^\infty f(s) \, dX_s - \frac{1}{2} \int_0^\infty f(s)^2 \, ds \right). \] (4.2)

In what follows, let \( V \) be a non-negative Borel measure satisfying (2.15).

4.1. The first step

**Proposition 4.1.** Let \( h_t = \int_0^t f(s) \, ds \) with \( f \in L^2(ds) \) and let \( T > 0 \). Then, for any non-negative \( \mathcal{F}_\infty \)-measurable functional \( F(X) \), it holds that

\[ \mathcal{W} \left[ F(X + h_{\land T}) \right] = \mathcal{W} \left[ F(X) \mathcal{E}_T(f; X) \right]. \] (4.3)

If, moreover, \( M_T[F; X] \in L^p(W) \) for some \( p > 1 \), then \( F(X + h_{\land T}) \in L^1(\mathcal{W}) \).

**Proof.** Let \( t \geq T \) be fixed. By the multiplicativity property of \( \mathcal{K}(\delta_0; \cdot) \) and since \( h_{(\land t)\land T} = h_T \), we have

\[ \mathcal{K}(\delta_0; X + h_{\land T}) = \mathcal{K}(\delta_0; X + h_{\land T}) \mathcal{K}(\delta_0; \theta_t X + h_T). \] (4.4)
Let $G_t(X)$ be a non-negative $\mathcal{F}_T$-measurable functional. Then, by the Markov property (2.41), by (2.9) and by (2.23), we have

$$M_t[K(\delta_0; \cdot + hT); X] = K_t(\delta_0; X + hT)W_X[K(\delta_0; X + hT)]$$  \hspace{1cm} (4.5)

$$= W[G_t(X + hT)K(\delta_0; X + hT)](1 + |X_t + hT|).$$  \hspace{1cm} (4.6)

Hence we obtain

$$W[G_t(X + hT)K(\delta_0; X + hT)E_T(f; X)] = W[G_t(X)K(\delta_0; X)E_T(f; X)].$$  \hspace{1cm} (4.7)

By the Cameron–Martin formula (1.21), by formula (2.44), and then by the Markov property (2.33), we have

$$W[G_t(X)K(\delta_0; X)E_T(f; X)] = W[G_t(X)Mt[K(\delta_0; \cdot); X]E_T(f; X)].$$  \hspace{1cm} (4.9)

Since $t \geq T$ is arbitrary, we see that

$$W[G(X + hT)K(\delta_0; X + hT)] = W[G(X)K(\delta_0; X)E_T(f; X)]$$  \hspace{1cm} (4.12)

holds for any non-negative $\mathcal{F}_\infty$-measurable functional $G(X)$. Replacing the functional $G(X)$ by $F(X)K(\delta_0; X)^{-1}$, we obtain (4.3).

Suppose that $M_T[F; X] \in L^p(W)$ for some $p > 1$. Since $E_T(h; X)$ is $\mathcal{F}_T$-measurable, we have

$$W[F(X)E_T(f; X)] = W[M_T[F; X]E_T(f; X)]$$  \hspace{1cm} (4.13)

$$\leq W[M_T[F; X]^p]^{1/p}W[E_T(f; X)^q]^{1/q} < \infty$$  \hspace{1cm} (4.14)

where $q$ is the conjugate exponent to $p$: $(1/p) + (1/q) = 1$. The proof is now complete.  \hspace{1cm} \square

4.2. Integrability under $\mathcal{W}$, when weighed by Feynman–Kac functionals

We need the following theorem.

**Theorem 4.2.** Let $h_t = \int_0^t f(s) \, ds$ with $f \in L^2(ds) \cap L^1(ds)$. Let $V$ be as in Theorem 2.1 and set $C_V = \inf_{x \in \mathbb{R}} \varphi_V(x) > 0$. Then it holds that

$$\mathcal{W}[K(V; X)E(f; X)] \leq \varphi_V(0)\exp\left(\frac{1}{C_V} |f|_{L^1(ds)}\right).$$  \hspace{1cm} (4.15)
Proof. By Theorem 2.4, we have
\[
\frac{1}{\varphi_V(0)} \mathcal{W} [K(V; X) \mathcal{E}(f; X)] = W^{(V)} [\mathcal{E}(f; X)].
\] (4.16)

By (v) of Theorem 2.1, we see that
\[
(4.16) = W^{(V)} \left[ \mathcal{E}(f; B) \exp \left( \int_0^\infty f(s) \frac{\varphi'_V}{\varphi_V} (X_s) \, ds \right) \right]
\] (4.17)
where \((B_t), W^{(V)}\) is a Brownian motion. Since \(|\varphi'_V(x)| \leq 1\) and \(\varphi_V(x) \geq C_V\) for any \(x \in \mathbb{R}\), we have
\[
(4.17) \leq W^{(V)} \left[ \mathcal{E}(f; B) \exp \left( \frac{1}{C_V} \int_0^\infty |f(s)| \, ds \right) \right].
\] (4.18)

Since \(W^{(V)}[\mathcal{E}(f; B)] = 1\), we obtain the desired inequality. \(\square\)

4.3. The second step

We utilize the following lemma.

Lemma 4.3. Let \(h_t = \int_0^t f(s) \, ds\) with \(f \in L^2(ds) \cap L^1(ds)\). Then, for any \(0 < s < \infty\), it holds that
\[
\mathcal{W} \left[ \mathcal{E}_t(f; X) e^{-g(X)}; g(X) > t \right] \xrightarrow{t \to \infty} 0.
\] (4.19)

Proof. By the Markov property (2.33), we see that
\[
\mathcal{W} \left[ \mathcal{E}_t(f; X) e^{-g(X)}; g(X) > t \right] = W_{\mathcal{E}_t(f; X) e^{-t}} \mathcal{W}_X \left[ e^{-g(X)}; \tau_0(X) < \infty \right].
\] (4.20)

By the strong Markov property (2.33), we see, for any \(x \in \mathbb{R}\), that
\[
\mathcal{W}_x \left[ e^{-g(X)}; \tau_0(X) < \infty \right] = W_x \left[ e^{-\tau_0(X)} \right] \mathcal{W}_0 \left[ e^{-g(X)} \right] \leq \int_0^\infty \frac{du}{\sqrt{2\pi u}} e^{-u} = \frac{1}{\sqrt{2}}.
\] (4.21)

Hence we obtain
\[
(4.20) \leq \frac{1}{\sqrt{2}} \exp(1/t) \mathcal{E}_t(f; X) \xrightarrow{t \to \infty} 0.
\] (4.22)

The proof is now complete. \(\square\)
Lemma 4.4. Let \( h_t = \int_0^t f(s) \, ds \) with \( f \in L^2(ds) \cap L^1(ds) \). Let \( V \) be as in Theorem 2.1. Then it holds that

\[
W[\mathcal{E}(f; X)K(V; X)e^{-g(X)}; g(X) > t] \to 0. 
\]

(4.23)

**Proof.** Since \( W[\mathcal{E}(f; X)K(V; X)] < \infty \) by Theorem 4.2. The desired conclusion is now obvious by the dominated convergence theorem. \( \square \)

Lemma 4.5. Let \( h_t = \int_0^t f(s) \, ds \) with \( f \in L^2(ds) \cap L^1(ds) \). Set

\[
\tilde{f}(t) = \int_0^\infty |f(s+t)| \frac{ds}{\sqrt{s}}, \quad t > 0,
\]

(4.24)

\[
\sigma_t = \|f(\cdot + t)\| = \left\{ \int_t^\infty f(s)^2 \, ds \right\}^{1/2}, \quad t > 0,
\]

(4.25)

and set

\[
E(t) = E\left[\exp\left\{\sigma_t |\mathcal{N}| + c \tilde{f}(t) + \frac{1}{2} \sigma_t^2 \right\} - 1\right], \quad t > 0
\]

(4.26)

where \( \mathcal{N} \) stands for the standard Gaussian variable and \( c = \sqrt{2/\pi} \). Then it holds that

\[
R_a[|\mathcal{E}(f(\cdot + t); \cdot) - 1|^2] \leq E(t) \quad \text{for any } t > 0 \text{ and any } a \in \mathbb{R}. 
\]

(4.27)

**Proof.** Let us write \( \langle f, g \rangle = \int_0^\infty f_1(s) f_2(s) \, ds \) for \( f_1, f_2 \in L^2(ds) \). Note that

\[
\mathcal{E}(f(\cdot + t); X) = \exp\left\{ \int_0^\infty f(s+t) \, d\tilde{X}_s^{(a)} + \langle f(\cdot + t), \phi_a \rangle - \frac{1}{2} \sigma_t^2 \right\} \quad \text{under } R_a^+. 
\]

(4.28)

Since \(|e^b - 1| \leq e^{|b|} - 1\) for any \( b \in \mathbb{R} \), we have

\[
|\mathcal{E}(f(\cdot + t); \cdot) - 1|^2 \leq \exp\left\{ \int_0^\infty f(s+t) \, d\tilde{X}_s^{(a)} + \langle f(\cdot + t), \phi_a \rangle - \frac{1}{2} \sigma_t^2 \right\} - 1 \right\}^2. 
\]

(4.29)

Since, for any constant \( b \in \mathbb{R}, \psi(x) = (e^{|x+b|} - 1)^2 \) is a convex function, we may apply Theorem 1.3 and obtain

\[
R_a^+\left[|\mathcal{E}(f(\cdot + t); \cdot) - 1|^2\right] \leq E\left[\exp\left\{\sigma_t |\mathcal{N}| + \langle f(\cdot + t), \phi_a \rangle - \frac{1}{2} \sigma_t^2 \right\} - 1\right]^2. 
\]

(4.30)
we obtain the desired result. \(\square\)

**Lemma 4.6.** Let \( h_t = \int_0^t f(s) \, ds \) with \( f \in L^2(ds) \cap L^1(ds) \). Then there exists a sequence \( t(n) \to \infty \) such that

\[
\mathcal{W} \big[ e^{-g(X)} K(V;X) | \mathcal{E}(f;X) - \mathcal{E}_{t(n)}(f;X) \big] \to 0. \quad (4.32)
\]

**Proof.** By Lemmas 4.3 and 4.4, it suffices to prove that

\[
\mathcal{W} \big[ e^{-g(X)} K(V;X) | \mathcal{E}(f;X) - \mathcal{E}_t(f;X) \big| : g(X) \leq t \big] \quad (4.33)
\]

converges to 0 along some sequence \( t = t(n) \to \infty \).

By the multiplicativity:

\[
\mathcal{E}(f;X) = \mathcal{E}_t(f;X) \mathcal{E}(f;+t;\theta_t X), \quad (4.34)
\]

we have

\[
(4.33) = \mathcal{W} \big[ e^{-g(X)} K(V;X) \mathcal{E}_t(f;X) | \mathcal{E}(f;+t;\theta_t X) - 1 | : g(X) \leq t \big]. \quad (4.35)
\]

By the Schwarz inequality, (4.35) is dominated by \( A^{1/2} B^{1/2} \) where

\[
A = \mathcal{W} \big[ K(V;X)^2 \mathcal{E}_t(f) \big] \quad (4.36)
\]

and

\[
B = \mathcal{W} \big[ e^{-2g(X)} | \mathcal{E}(f;+t;\theta_t X) - 1 |^2 : g(X) \leq t \big]. \quad (4.37)
\]

By Theorem 4.2, we see that

\[
A \leq \mathcal{W} \big[ K(2V;X) \mathcal{E}(2f1_{[0,t]};X) \big] \exp \left( \|f\|^2_{L^2(ds)} \right) \quad (4.38)
\]

\[
\leq \varphi_{2V}(0) \exp \left( \|f\|^2_{L^2(ds)} + \frac{2}{C_{2V}} \|f\|_{L^1(ds)} \right). \quad (4.39)
\]

By Lemma 4.5, we see that

\[
B = \int_0^t \frac{du}{\sqrt{2\pi u}} e^{-2u} \left( \mathcal{P}^{(u)} \cdot \mathcal{R} \right) \left[ |\mathcal{E}(f;+t;\theta_t X) - 1 |^2 \right] \quad (4.40)
\]

\[
= \int_0^t \frac{du}{\sqrt{2\pi u}} e^{-2u} \mathcal{R} \left[ \mathcal{E}_X (f;+t;\cdot) - 1 \right]^2 \quad (4.41)
\]
\[ E(t) \int_{0}^{\infty} \frac{du}{\sqrt{2\pi u}} e^{-2u}. \]  

Therefore we see that (4.33) is dominated by \( E(t) \) up to a multiplicative constant. The proof is now completed by (ii) of Lemma 3.6. \( \square \)

4.4. The third step

In what follows, we take and utilize a non-negative, bounded, continuous function \( v_0 \) on \( \mathbb{R} \) such that \( v_0(x) = 1 \) for \( |x| \leq 2 \) and \( v_0(x) = 0 \) for \( |x| \geq 3 \). We write \( v_1 = 1_{[-1,1]} \). We set \( V_0(\text{d}x) = v_0(\text{d}x) \) and \( V_1(\text{d}x) = v_1(\text{d}x) \). For any \( V \), we write

\[ \Gamma(V; X) = e^{-g(X)} K(V; X). \]  

(4.43)

Lemma 4.7. Let \( h_t = \int_{0}^{t} f(s) \text{d}s \) with \( f \in L^1(\text{d}s) \). Suppose that

\[ \int_{T}^{\infty} |f(s)| \text{d}s \leq 1 \]  

(4.44)

for some \( 0 < T < \infty \). Then it holds that

\[ K(V_0; X + h \cdot \wedge t) \leq K(V_1; X + h \cdot \wedge T), \quad t \geq T. \]  

(4.45)

Proof. Note that we have \( |h_t - h_T| \leq 1 \) for any \( t \geq T \). If \( s \geq T \) satisfies \( |X_s + h_T| \leq 1 \), then we have \( |X_s + h_{\wedge sT}| \leq 2 \). Hence we have

\[ \int_{0}^{\infty} v_0(X_s + h_{\wedge sT}) \text{d}s \geq \int_{0}^{\infty} v_1(X_s + h_{\wedge sT}) \text{d}s, \quad t \geq T. \]  

(4.46)

This completes the proof. \( \square \)

Lemma 4.8. Let \( h_t = \int_{0}^{t} f(s) \text{d}s \) with \( f \in L^2(\text{d}s) \cap L^1(\text{d}s) \). Let \( 0 < r < \infty \) and let \( G_r(X) \) be a non-negative, bounded, continuous \( \mathcal{F}_r \)-measurable functional. Then it holds that

\[ \mathbb{W} \left[ G_r(X + h \cdot \wedge t) \Gamma(V_0; X + h \cdot \wedge t) \right] \rightarrow_{t \rightarrow \infty} \mathbb{W} \left[ G_r(X + h) \Gamma(V_0; X + h) \right]. \]  

(4.47)

Proof. Note that \( g(X + h \cdot \wedge t) \rightarrow g(X + h) \) as \( t \rightarrow \infty \), because \( h \cdot \wedge t \rightarrow h \) uniformly. By the continuity assumptions on \( G_r \) and \( v \), we have

\[ G_r(X + h \cdot \wedge t) \Gamma(V_0; X + h \cdot \wedge t) \rightarrow G_r(X + h) \Gamma(V_0; X + h) \]  

(4.48)

for \( \mathbb{W}(\text{d}X) \)-almost every path \( X \). Since \( G_r(X) \) is bounded, it suffices to find \( Z \in L^1(\mathbb{W}) \) such that \( \Gamma(V_0; X + h \cdot \wedge t) \leq Z(X) \), \( \mathbb{W} \)-a.e. for any large \( t \); in fact, we may obtain (4.47) by the dominated convergence theorem.
Since \( f \in L^1(ds) \), we may take \( T > 0 \) such that (4.44) holds. By Lemma 4.7, we have (4.45), and hence we have

\[
\Gamma(V_0; X + h_{\cdot T}) \leq K(V_1; X + h_{\cdot T}), \quad t \geq T. \tag{4.49}
\]

Since \( M_T[K(V_1; \cdot); X] \in L^2(W) \) by (2.43), we see, by Proposition 4.1, that

\[
K(V_1; X + h_{\cdot T}) \in L^1(W). \tag{4.50}
\]

Therefore this functional \( K(V_1; X + h_{\cdot T}) \) is as desired. \( \square \)

### 4.5. Proof of Theorem 1.1

We now proceed to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( h_t = \int_0^t f(s) \, ds \) with \( f \in L^2(ds) \cap L^1(ds) \). Let \( 0 < s < \infty \) and let \( G_s(X) \) be a non-negative, bounded, continuous \( \mathcal{F}_s \)-measurable functional. Let \( T > 0 \). Then, by Proposition 4.1, we have

\[
\mathbb{W}[G_s(X + h_{\cdot T}) \Gamma(V_0; X + h_{\cdot T})] = \mathbb{W}[G_s(X) \Gamma(V_0; X) \mathcal{E}_T(f; X)]. \tag{4.51}
\]

By Lemma 4.8, we have

\[
\mathbb{W}[G_s(X + h_{\cdot T}) \Gamma(V_0; X + h_{\cdot T})] \xrightarrow{T \to \infty} \mathbb{W}[G_s(X + h) \Gamma(V_0; X + h)]. \tag{4.52}
\]

By Lemma 4.6, we have

\[
\mathbb{W}[G_s(X) \Gamma(V_0; X) \mathcal{E}_T(f; X)] \to \mathbb{W}[G_s(X) \Gamma(V_0; X) \mathcal{E}(f; X)] \tag{4.53}
\]

along some sequence \( T = t(n) \to \infty \). Thus, taking the limit as \( T = t(n) \to \infty \) in both sides of (4.51), we obtain

\[
\mathbb{W}[G_s(X + h) \Gamma(V_0; X + h)] = \mathbb{W}[G_s(X) \Gamma(V_0; X) \mathcal{E}(f; X)]. \tag{4.54}
\]

Hence we obtain

\[
\mathbb{W}[G(X + h) \Gamma(V_0; X + h)] = \mathbb{W}[G(X) \Gamma(V_0; X) \mathcal{E}(f; X)] \tag{4.55}
\]

for any non-negative \( \mathcal{F}_\infty \)-measurable functional \( G(X) \). Replacing \( G(X) \) by \( F(X) \Gamma(V_0; X)^{-1} \), we obtain the desired conclusion. \( \square \)

### Acknowledgments

The author would like to thank Professors Marc Yor, Tadahisa Funaki, Shinzo Watanabe and Ichiro Shigekawa for their useful comments which helped improve this paper.
References