Filtered colimits in the effective topos

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Abstract

It is shown that the “constant sheaves” functor $\nabla : \text{Sets} \to \text{Eff}$ does not preserve $\omega_1$-filtered colimits, and that as a consequence of this, the full subcategory of $\text{Eff}$ on the countable projective objects is not dense.

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0. Introduction

The present note aims to contribute to the study of the Effective Topos $\text{Eff}$, introduced in [1], is one of the prime examples of elementary topoi which are not Grothendieck.

In fact, $\text{Eff}$ is not cocomplete, and the global sections functor $\Gamma : \text{Eff} \to \text{Sets}$ does not have a left adjoint, but a right adjoint $\nabla : \text{Sets} \to \text{Eff}$.

A fundamental question is: how does $\text{Eff}$ compare to Grothendieck topoi? Is it possible to embed $\text{Eff}$ into a Grothendieck topos in a nice way? In [6], a functor from $\text{Eff}$ into the “recursive topos” of Mulry [4] is defined, but this functor does not preserve a lot of structure (it is, for example, not an embedding).

Good embeddings can be obtained by considering small full dense subcategories of $\text{Eff}$. Recall that for every category $\mathcal{E}$, a subcategory $\mathcal{C} \subset \mathcal{E}$ is dense if for every object $X$ of $\mathcal{E}$, the natural cocone with vertex $X$ for the diagram $\mathcal{C} \downarrow X \xrightarrow{\text{dom}} \mathcal{E} \to \mathcal{E}$ is colimiting. If this is the case, and $J$ is the Grothendieck topology on $\mathcal{C}$ induced by the canonical topology
on $\mathcal{C}$, then the left Kan extension of the Yoneda embedding on $\mathcal{C}$, the functor from $\mathcal{C}$ to $[\mathcal{C}^{op}, \text{Sets}]$ which sends $X$ to $\mathcal{C}(-, X)$, factors through the sheaf topos $\text{Sh}(\mathcal{C}, J)$ and this factorization is full and faithful, cartesian closed, and preserves all limits and colimits of $\mathcal{C}$; hence also the natural numbers object. This is standard topos theory, for which the most complete reference is now [2].

The category $\mathcal{C}^{ff}$ is an exact completion [5] and therefore, if a small full dense subcategory $\mathcal{C}$ of $\mathcal{C}^{ff}$ exists, we may assume that $\mathcal{C}$ consists of projective objects of bounded cardinality. In fact, I started out from the conjecture that the countable projectives might provide such a dense subcategory; to my surprise, this is wrong as this paper shows (it is fairly easy to see that the countable projectives form a separating set, i.e. that the natural cocone mentioned earlier is always an epimorphic family).

Basically, this note contains two theorems: Theorem 1.2 states an equivalent condition for the full subcategory of $\mathcal{C}^{ff}$ on the $\omega$-small (i.e., having underlying set of cardinality less than $\omega$) projectives to be dense in $\mathcal{C}^{ff}$, relating this to the preservation by $\nabla$ of $\omega$-filtered colimits. Then, after a few folklore results included for completeness’ sake, Theorem 1.5 states that $\nabla$ does not preserve $\omega_1$-filtered colimits.

I have not been able to settle the matter for higher cardinals such as $\mathcal{P}(\omega)^+$. However, the proof of Theorem 1.5 carries the suggestion that there is infinitary set-theoretic combinatorics at work here, and that any result might well depend on axioms independent of ZFC.

1. Filtered colimits and dense subcategories in $\mathcal{C}^{ff}$

For definitions and basic facts concerning $\mathcal{C}^{ff}$ the reader is referred to [1]. However there is one further fact, mentioned in [5], which is helpful to understand Theorem 1.2 and its proof. Let $N$ denote the natural numbers object of $\mathcal{C}^{ff}$. Then $\Gamma(N) = \mathbb{N}$; let $\eta_N : N \to \nabla(\mathbb{N})$ be the unit of the adjunction, and $\eta^*_N : \mathcal{C}^{ff}/\nabla(\mathbb{N}) \to \mathcal{C}^{ff}/N$ the pullback functor. There is a functor $\nabla_N : \text{Sets}/\mathbb{N} \to \mathcal{C}^{ff}/N$ obtained by composing with $\eta^*_N$. Furthermore denote, as usual, the forgetful (domain) functor $\mathcal{C}^{ff}/N \to \mathcal{C}^{ff}$ by $\Sigma_N$. Then

**Lemma 1.1 (Robinson and Rosolini [5]).** An object of $\mathcal{C}^{ff}$ is projective if and only if it is isomorphic to one in the image of $\Sigma_N \circ \nabla_N$.

**Theorem 1.2.** Let $\lambda > \omega$ be a regular cardinal. Then the following two assertions are equivalent:

(i) The full subcategory of $\mathcal{C}^{ff}$ on the $\lambda$-small projectives is dense.

(ii) $\nabla : \text{Sets} \to \mathcal{C}^{ff}$ preserves $\lambda$-filtered colimits.

For $\lambda = \omega$, the implication (i) $\Rightarrow$ (ii) still holds.

**Proof.** (i) $\Rightarrow$ (ii): First observe that, since $\nabla$ preserves epi-mono factorizations, statement (ii) is equivalent to saying that for any set $X$, $\nabla(X)$ is the vertex of a colimiting cocone for the diagram consisting of all $\nabla(Y)$ for $Y \subseteq X \lambda$-small, and ($\nabla$-images of) inclusions. Now since for any $X$, any cocone to $\nabla(X)$ for a diagram of $\lambda$-small projectives also yields
a cone for a diagram of $\nabla$’s of $\lambda$-small subsets of $X$ (by sheafification), it is clear that (i) implies (ii).

For (ii) $\Rightarrow$ (i), observe that if $\nabla : \text{Sets} \to \delta ff$ preserves $\lambda$-small colimits then the same is true for the functor $\nabla/\mathbb{N} : \text{Sets}/\mathbb{N} \to \delta ff/\nabla(\mathbb{N})$ because the forgetful functors $\Sigma/\mathbb{N} : \text{Sets}/\mathbb{N} \to \text{Sets}$ and $\Sigma/\nabla(\mathbb{N}) : \delta ff/\nabla(\mathbb{N}) \to \delta ff/N$ preserve and create colimits. Since the pullback functor $\eta_N : \delta ff/\nabla(\mathbb{N}) \to \delta ff/N$ has a right adjoint, the composite functor $\nabla_N : \text{Sets}/\mathbb{N} \to \delta ff/N$ preserves $\lambda$-filtered colimits too.

In order to prove (i), it clearly suffices to prove that every projective object $X$ is a colimit of its $\lambda$-small sub-projectives. So suppose that for every $\lambda$-small sub-projective $Y$ of $X$ we are given a map $\phi_Y : Y \to (Z, =)$ in $\delta ff$, such that for $Y' \subset Y$, $\phi_Y|Y' = \phi_{Y'}$. Each such projective $Y$ is a set $Y$ together with a map $e : Y \to \mathbb{N}$; equivalently, an $\mathbb{N}$-indexed family of sets $(Y_n)_{n \in \mathbb{N}}$. Any map $(Y_n)_{n \in \mathbb{N}} \to (Z, =)$ is represented by a function $f : Y \to Z$, such that for some partial recursive function $p$ we have that for all $n$, such that $Y_n \neq \emptyset$, $p(n)$ is defined and

$$p(n) \in \bigcap_{y \in Y_n} \{f(y) = f(y)\}.$$

In such a case, one says that $p$ tracks $f$. Two such functions $f, g : Y \to Z$ represent the same morphism iff there is a partial recursive function $q$ such that for all $n$ with $Y_n \neq \emptyset$, $q(n) \in \bigcap_{y \in Y_n} \{f(y) = g(y)\}$.

Now I claim that for some partial recursive function $p$, it holds that for $Y \subset X$ $\lambda$-small, every $\phi_Y$ has a representative which is tracked by $p$; for otherwise choose for every $p$ a $\lambda$-small $Y_p \subset X$ for which no representative tracked by $p$ exists; since there are only countably many partial recursive functions the union $\bigcup_p Y_p$ is still $\lambda$-small (since $\lambda > \omega$); a contradiction is easily obtained.

Fix such a $p$ as in the previous paragraph. Construct an object $(Z', =')$ from $(Z, =)$ by putting

$$Z' = \{(n, z) \mid p(n) \text{ defined and } p(n) \in [z = z]\}$$

and

$$[n, z] = [(m, z)'] = \begin{cases} \{n\} \land [z = z'] & \text{if } n = m, \\ \emptyset & \text{otherwise.} \end{cases}$$

Recall that $\{n\} \land [z = z']$ is $\{(n, a) \mid a \in [z = z']\}$, where $\langle -, -, \rangle$ is a recursive bijection $\mathbb{N}^2 \to \mathbb{N}$.

The object $Z'$ comes with maps $Z' \overset{\pi_2}{\to} N$ and $Z' \overset{\pi_1}{\to} Z$, such that every $\phi_Y : Y \to Z$ factors through some $\phi_Y' : Y \to Z'$ which has the property that if one regards $Y = (Y_n)_{n \in \mathbb{N}}$ as an object of $\delta ff/N$, $\phi_Y'$ is a map over $N$.

We have therefore a cone for the $\lambda$-filtered diagram of sub-projectives of $X$, regarded as objects of $\delta ff/N$, with vertex the object $Z \overset{\pi_1}{\to} N$. Since the diagram is in the image (under $\nabla_N$) of a $\lambda$-filtered diagram in $\text{Sets}/\mathbb{N}$ and $\nabla_N$ preserves $\lambda$-filtered colimits, its colimit is the projective $X$ (as object of $\delta ff/N$), and there is a unique mediating map $X \to Z'$ over $N$. But then the composite $X \to Z$ is the unique mediating map for the original cone of the $\phi_Y$’s. $\square$
Theorem 1.5. \( \text{morphisms be described as follows: objects are pairs } (X, E) \text{ where } X \text{ is a set and } E : X \to \mathcal{P}(\mathbb{N}); \) morphisms \((X, E) \to (Y, F)\) are functions \(f : \{x \in X \mid E(x) \neq \emptyset\} \to Y\) with the property that for some partial recursive function \(p\) it holds that whenever \(n \in E(x)\) then \(p(n)\) is defined and an element of \(F(f(x))\) (one says that \(p\) tracks \(f\), as before). The factorization \(\nabla : \text{Sets} \to \text{Ass}\) sends \(X\) to \((X, E_{\psi})\), where \(E_{\psi}(x) = \mathbb{N}\) for all \(x \in X\).

Clearly, if \(\nabla : \text{Sets} \to \mathcal{E}f\mathcal{f}\) preserves \(\lambda\)-filtered colimits then so does \(\nabla : \text{Sets} \to \text{Ass}\).

**Proposition 1.3.** \(\nabla : \text{Sets} \to \text{Ass}\) does not preserve \(\omega_1\)-filtered colimits.

**Proof.** Let \(e \mapsto [e] : \mathbb{N} \to \mathcal{P}_{\text{fin}}(\mathbb{N})\) be a bijective coding of finite subsets of \(\mathbb{N}\). Let \(A\) be the assembly \((\mathbb{N}, E)\), where \(E(n) = \{e \mid n \in [e]\}\). Then for any finite subset \([e]\) of \(\mathbb{N}\) there is a map of assemblies \(\nabla([e]) \to A\), tracked by the function which is constant \(e\); and this system of maps is clearly a cocone for the diagram of \(\nabla\)'s of finite subsets of \(\mathbb{N}\) and inclusions between them. But there is no mediating map: \(\nabla(\mathbb{N}) \to A\). □

**Proposition 1.4.** \(\nabla : \text{Sets} \to \text{Ass}\) preserves \(\omega_1\)-filtered colimits.

**Proof.** Easy. □

**Theorem 1.5.** The functor \(\nabla : \text{Sets} \to \mathcal{E}f\mathcal{f}\) does not preserve \(\omega_1\)-filtered colimits.

**Proof.** Let \(D\) be the \(\omega_1\)-filtered diagram of countable subsets of \(\omega_1\) and inclusions between them; clearly, in \(\text{Sets}\), the cocone \(D \to \omega_1\) is colimiting. We shall see that \(\nabla(D) \to \nabla(\omega_1)\) is not colimiting in \(\mathcal{E}f\mathcal{f}\).

Recall the necessary ingredients of the construction of an \(\omega_1\)-Aronszajn tree (see [3] for the full story). If \(\alpha\) is a countable ordinal and \(s, \tau : \alpha \to \omega\), we write \(s \sim \tau\) if the set \(\{\xi \in \alpha \mid s(\xi) \neq \tau(\xi)\}\) is finite. If \(s \sim \tau\), let \(d(s, \tau)\) be the cardinality of this set.

It is possible to construct a sequence \(\{s_\beta : \alpha \in \omega_1\}\) such that for each \(\alpha\), \(s_\alpha\) is a 1-1 function from \(\alpha\) into \(\omega\), and such that for \(\alpha < \beta, s_\alpha \sim (s_\beta \upharpoonright \alpha)\).

Let \(T^*\) consist of all injective functions \(s : \alpha \to \omega\), defined on some countable \(\alpha\), such that \(s \sim s_\beta\). Note that for each \(\alpha \in \omega_1\), the set \(L_\alpha = \{s \in T^* \mid \text{dom}(s) = \alpha\}\) is countable.

Equip \(T^*\) with the structure of an object of \(\mathcal{E}f\mathcal{f}\), by defining

\[
[s = t] = \begin{cases} 
\emptyset & \text{if dom}(s) \neq \text{dom}(t), \\
\{n \mid d(s, t) \leq n\} & \text{otherwise}.
\end{cases}
\]

Clearly, if \(n \in [s = t]\) and \(m \in [t = u]\) then \(m + n \in [s = u]\), so this is a well-defined equality relation. \((T^*, =)\) is a uniform object since \(0 \in \bigcap_{s \in T^*} [s = s]\).

For each \(\alpha \in \omega_1\) let \(\phi_\alpha : \alpha \to T^*\) be defined by

\[
\phi_\alpha(\beta) = s_\beta|\beta.
\]
Then for each pair $\alpha < \alpha'$ in $\omega_1$, we have that

$$d(s_\alpha, s_{\alpha'}|\alpha) \in \bigcap_{\beta \in \alpha} [\phi_{s_\alpha}(\beta) = \phi_{s_{\alpha'}}(\beta)],$$

which means that the functions $\phi_{s_\alpha}$ and $\phi_{s_{\alpha'}}|\alpha$ define the same morphism from $\nabla(\alpha)$ to $(T^*, =)$ in $\mathcal{S}$; we shall denote this morphism also by $\phi_{s_\alpha}$.

If $A \subseteq \omega_1$ is a countable set, let $\phi_A : A \to T^*$ be the restriction of $\phi_\alpha$ to $A$, where $\alpha = \sup\{\beta + 1 \mid \beta \in A\}$. Clearly then, the system $\{\phi_A : \nabla(A) \to (T^*, =) \mid A \subseteq \omega_1$ countable$\}$ defines a cocone on $\nabla(D)$ with vertex $(T^*, =)$. I claim that this cocone does not factor through $\nabla(\omega_1)$.

Suppose, to the contrary, that there is a morphism $\Phi : \nabla(\omega_1) \to (T^*, =)$ such that for each $\alpha \in \omega_1$, $\Phi \circ \nabla(i_\alpha) = \phi_\alpha$, where $i_\alpha$ is the inclusion of $\alpha$ in $\omega_1$. Then $\Phi : \omega_1 \to T^*$ has the property that for every $\alpha$ there is an $n \in \omega$, such that

$$n \in \bigcap_{\beta \in \alpha} [\phi_{s_\beta}(\beta) = \Phi(\beta)].$$

Then there must be a number $n$, such that the set

$$A_n = \left\{ \alpha \in \omega_1 \mid n \in \bigcap_{\beta \in \alpha} [\phi_{s_\beta}(\beta) = \Phi(\beta)] \right\}$$

is unbounded in $\omega_1$. Fix such an $n$ for the rest of the proof. If $\alpha < \alpha'$ are elements of $A_n$, then

$$2n \in \bigcap_{\beta \in \alpha} [\phi_{s_\beta}(\beta) = \phi_{s'_{\beta}}(\beta)].$$

So for each $\beta < \alpha$ there are at most $2n$ ordinals $\xi \in \beta$ such that $s_\beta(\xi) \neq s_{\beta'}(\xi)$; it follows that $2n + 1 \in [s_\beta = s_{\beta'}|\alpha]$.

However, this is a contradiction once we have proved the following

**Claim 1.** Let $A$ be unbounded in $\omega_1$; then there exist, for each $k \in \omega$, elements $\alpha < \alpha'$ of $A$, such that $d(s_\alpha, s_{\alpha'}|\alpha) \geq k$.

**Proof of Claim 1.** First, observe that if $A \subseteq \omega_1$ is unbounded, then for each $\xi \in \omega_1$ there is at least one $n$ such that the set

$$A_{\xi, n} = \{ \alpha \in A \mid \alpha > \xi$ and $s_\alpha(\xi) = n \}$$

is unbounded.

**Claim 2.** Let $A$ be unbounded. Then for each $\eta \in \omega_1$ there is a $\xi > \eta$, such that there are $n, m$ with $n \neq m$ and both $A_{\xi, n}$ and $A_{\xi, m}$ unbounded.

**Proof of Claim 2.** Suppose Claim 2 is false; then by the remark preceding it, there is $\eta \in \omega_1$ such that for each $\xi > \eta$ there is exactly one $n$ such that $A_{\xi, n}$ is unbounded. Then
for every $\zeta > \eta$ there is a $\beta_\zeta \in A$ such that for all $x, x' \in A$ that are $\triangleright_{\zeta} \beta_\zeta, s_x(\zeta) = s_{x'}(\zeta)$.

But then the function $\zeta \mapsto s_{\beta_\zeta}(\zeta)$ is easily seen to be a 1-1 function from $\{\zeta \mid \eta < \zeta\}$ to $\omega$, which is impossible.

**Proof of Claim 1 (continued).** we construct, for each $k \in \omega$, sequences $(\xi_1, \ldots, \xi_k)$ and $((n_1, m_1), \ldots, (n_k, m_k))$ with $\xi_1 < \cdots < \xi_k < \omega_1, n_i, m_i \in \omega$ such that $n_i \neq m_i$ and the sets

\[ A_{\xi, m} = \{x \in A \mid x > \xi_k \text{ and } \forall i \leq k(s_x(\xi_i) = n_i)\}, \]

\[ B_{\xi, m} = \{x \in A \mid x > \xi_k \text{ and } \forall i \leq k(s_x(\xi_i) = m_i)\} \]

are both unbounded.

For $k = 1$ simply apply Claim 2. Inductively, suppose $(\xi_1, \ldots, \xi_k)$ and $((n_1, m_1), \ldots, (n_k, m_k))$ have been defined satisfying the conditions. Apply Claim 2 with $A = A_{\xi, m}$ and $\eta = \xi_k$. One finds $\xi_{k+1} > \xi_k$ and $a \neq b$ such that both $A_{\xi_{k+1}, a}$ and $A_{\xi_{k+1}, b}$ are unbounded.

If $B_{\xi, \xi_{k+1}, m, a}$ is unbounded, let $n_{k+1} = b, m_{k+1} = a$. If $B_{\xi, \xi_{k+1}, m, b}$ is unbounded, let $n_{k+1} = a, m_{k+1} = b$. If neither of these two, let $n_{k+1} = a$ and pick $m_{k+1}$ arbitrary, such that $B_{\xi, \xi_{k+1}, m, m_{k+1}}$ is unbounded. $\square$

**Remark.** Echoing the remark following the proof of Theorem 1.2, it is worth noting that Theorem 1.5 holds for every realizability topos based on a partial combinatory algebra $A$, whatever its cardinality; since the object $(T^*, =)$ can be constructed in every such topos.

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**References**


