ON MINOR-MINIMALLY-CONNECTED MATROIDS

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By a well-known result of Tutte, if e is an element of a connected matroid M, then either the deletion or the contraction of e from M is connected. If, for every element of M, exactly one of these minors is connected, then we call M minor-minimally-connected. This paper characterizes such matroids and shows that they must contain a number of two-element circuits or cocircuits. In addition, a new bound is proved on the number of 2-cocircuits in a minimally connected matroid.

1. Introduction

A minimally connected matroid is a connected matroid for which every single-element deletion is disconnected. Several authors [5, 6, 9, 10] have shown that such matroids behave similarly to their graph-theoretic counterparts, minimally 2-connected graphs. In particular, just as minimally 2-connected graphs have a number of vertices of degree two [4, 8], minimally connected matroids have a number of 2-element cocircuits.

The following result of Tutte [11, 6.5] (see also [3, p. 410]) is well known.

Proposition 1.1. If e is an element of a connected matroid M, then either the deletion M \ e or the contraction M/e is also connected.

In this paper, we study the class of minor-minimally-connected matroids, that is, those connected matroids M with the property that, for every element e, exactly one of M \ e and M/e is connected. Such matroids arise naturally in induction arguments involving connected matroids since either a connected matroid M is minor-minimally-connected or else it has an element f such that both M \ f and M/f are connected. Evidently all minimally connected matroids and their duals are minor-minimally-connected and, in Section 2, we give a characterization of minor-minimally-connected matroids that is similar to the characterization of minimally connected matroids given in [6, Theorem 3.1]. This characterization is then used in Section 3 to prove several results on the occurrence of 2-element circuits and 2-element cocircuits in minor-minimally-connected matroids. Finally, in Section 4, we prove a new result on the number of 2-element cocircuits in a minimally connected matroid.
The terminology used here for matroids and graphs will in general follow [12] and [1], respectively. The ground set, rank and corank of a matroid $M$ will be denoted by $E(M)$, $\text{rk} M$ and $\text{cork} M$, respectively; $k$-element circuits and cocircuits of $M$ will be called $k$-circuits and $k$-cocircuits. A series class of $M$ is a maximal subset $X$ of $E(M)$ such that if $x$ and $y$ are distinct elements of $X$, then $x$ and $y$ are in series in $M$, that is, $\{x, y\}$ is a 2-cocircuit of $M$. A parallel class of $M$ is a series class of $M^\#$. We call a series or parallel class non-trivial if it contains at least two elements.

The main results of this paper use the operations of series and parallel connection of matroids. Let $M_1$ and $M_2$ be matroids on disjoint sets $S_1$ and $S_2$, respectively. Suppose that $p_i \in S_i$ for $i = 1, 2$ and let $p$ be an element which is in neither $S_1$ nor $S_2$. Then the series connection $S((M_1 ; p_1), (M_2 ; p_2))$ of $M_1$ and $M_2$ with respect to the basepoints $p_1$ and $p_2$ is the matroid on $S_1 \setminus p_1 \cup S_2 \setminus p_2 \cup p$ whose circuits are the circuits of $M_1$ not containing $p_1$, the circuits of $M_2$ not containing $p_2$, and all sets of the form $(C_1 \setminus p_1) \cup (C_2 \setminus p_2) \cup p$, where $C_i$ is a circuit of $M_i$ containing $p_i$. The parallel connection $P((M_1 ; p_1), (M_2 ; p_2))$ of $M_1$ and $M_2$ with respect to $p_1$ and $p_2$ is the matroid $S((M_1^\# ; p_1), (M_2^\# ; p_2))$. We shall make frequent reference to Brylawski's paper [2], where a detailed discussion of these operations and their properties can be found.

2. Minor-minimally-connected matroids

In this section we prove a characterization of minor-minimally-connected matroids closely resembling the characterization of minimally connected matroids given in [6].

Let $M$ be a connected matroid and $p$ be an element of $M$ for which $M \setminus p$ is disconnected. Then $M \setminus p = H_1 \oplus H_2$ for some non-empty matroids $H_1$ and $H_2$. If we let $N_1 = M / E(H_1)$ and $N_2 = M / E(H_2)$, then, by [2, Proposition 4.10], $M = S((N_1 ; p), (N_2 ; p))$. Furthermore, by [2, Proposition 4.6], $N_1$ and $N_2$ are connected. Now suppose that $M$ is minor-minimally-connected. Then, although $N_1$ and $N_2$ need not themselves be minor-minimally-connected matroids, by slightly modifying them we can obtain such matroids. For $i = 1, 2$, form $N_i'$ from $N_i$ by adding an element $u_i$ in parallel with $p$, and form $N_i''$ from $N_i$ by adding $u_i$ in series with $p$.

**Lemma 2.1.** If $p$ is not in a 2-circuit of $M$, then each of $N_1'$, $N_1''$, $N_2'$ and $N_2''$ is minor-minimally-connected having at least four elements.

**Proof.** It suffices to show that both $N_1'$ and $N_1''$ are minor-minimally-connected having at least four elements. Since $M$ has no 2-circuits containing $p$, $M \setminus p$ has no components having fewer than two elements. Thus $N_1'$ has at least three elements and therefore both $N_1'$ and $N_1''$ have at least four elements. As $N_1$ is connected, it
follows easily that $N_1'$ and $N_2''$ are connected. We now show that $N_1'$ is minor-minimally-connected noting that a similar argument shows that $N_2''$ has the same property. Firstly, as both $N_1'/p$ and $N_1'/u_1$ have loops, both are disconnected. Secondly, if $e \in E(N_1') \setminus \{p, u_1\}$, then either $M \setminus e$ or $M/e$ is disconnected. But, by [2, Proposition 4.7], $M \setminus e = S((N_1 \setminus e ; p), (N_2; p))$ and $M/e = S(N_1/e ; p), (N_2; p))$. Thus, by [2, Proposition 4.6], either $N_1 \setminus e$ or $N_1/e$ is disconnected. Hence either $N_1' \setminus e$ or $N_1'/e$ is disconnected and so $N_1'$ is minor-minimally-connected. □

**Theorem 2.2.** A matroid $M$ is minor-minimally-connected if and only if $|E(M)| \geq 3$ and either $M$ is connected and every element is in a 2-circuit or a 2-cocircuit; or one of $M$ and $M^*$ is isomorphic to the series connection $S((M_1/q_1 ; p_1), (M_2/q_2 ; p_2))$, where both $M_1$ and $M_2$ are minor-minimally-connected having at least four elements and $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are cocircuits of $M_1$ and $M_2$, respectively.

Since a matroid is connected if and only if its dual is, it follows easily from the definition of a minor-minimally-connected matroid that

**Proposition 2.3.** $M$ is minor-minimally-connected if and only if $M^*$ is.

A consequence of this and the link between series and parallel connections is that if $M^*$ is a series connection as described in Theorem 2.2, then $M$ is isomorphic to the parallel connection $P((M_1^* \setminus q_1 ; p_1), (M_2^* \setminus q_2 ; p_2))$, where both $M_1^*$ and $M_2^*$ are minor-minimally-connected having at least four elements and $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are circuits of $M_1^*$ and $M_2^*$, respectively.

**Proof of Theorem 2.2.** Evidently if $M$ is a connected matroid having at least three elements and every element is in a 2-circuit or a 2-cocircuit, then $M$ is minor-minimally-connected.

Now suppose that $M_1$ and $M_2$ are minor-minimally-connected matroids each having at least four elements and let $\{p_i, q_i\}$ be a cocircuit of $M_i$ for each $i$. Let $M$ be the series connection $S((M_1/q_1 ; p_1), (M_2/q_2 ; p_2))$. Then, as $M_i \setminus q_i$ is disconnected, Proposition 1.1 implies that $M_i/q_i$ is connected, so by [2, Proposition 4.6], $M$ is connected.

To show that $M$ is minor-minimally-connected, first note that, by [2, Proposition 4.9], $M \setminus p = (M_1/q_1) \setminus p_1 \Theta (M_2/q_2) \setminus p_2$, so $M \setminus p$ is disconnected. Now suppose that $e$ is an element of $M$ different from $p$. Then without loss of generality we may assume that $e \in E(M_1/q_1) \setminus \{p_1\}$. As $M_1$ is minor-minimally-connected, either $M_1 \setminus e$ or $M_1/e$ is disconnected. In the first case, consider $M \setminus e = S(M_1/q_1 \setminus e, M_2/q_2)$. If $M_1/q_1 \setminus e$ is disconnected, so is $M \setminus e$. Hence assume that $M_1/q_1 \setminus e$ is connected. Then, as $M_1 \setminus e$ is disconnected, it follows that $q_1$ is a loop or a coloop of $M_1 \setminus e$. Since $M_1$ is connected, we conclude that $\{e, q_1\}$ is a cocircuit
of $M_i$. But $\{p_1, q_1\}$ is also a cocircuit of $M_i$, hence so is $\{e, p_1\}$. Thus $p_1$ is a coo

of $M_i/q_1 \setminus e$ and so $M\setminus e$ is disconnected.

We now suppose that $M_i/e$ is disconnected and show that if this occurs, then $M\setminus e$ is also disconnected. Since $M_i/e = S(M_i/q_1, e, M_2/q_2)$, if $M_i/q_1, e$ is discon

nected, so is $M_i/e$. We may therefore assume that $M_i/q_1, e$ is connected. As $M_i/e$ is disconnected and $M_i$ is connected, it follows that $\{e, q_1\}$ is a circuit of $M_i$. But this circuit meets the cocircuit $\{p_1, q_1\}$ in a single element and this is a contradiction. We conclude that $M$ is minor-minimally-connected and, moreover, by Proposition 2.3, so is $M^*$.

For the converse, assume that $M$ is a minor-minimally-connected matroid having an element $p$ which is in neither a 2-circuit nor a 2-cocircuit. Then either $M\setminus p$ or $M/p$ is disconnected. In the first case, the result follows without difficulty from Lemma 2.1. In the second case, $M^*\setminus p$ is disconnected and the result again follows from Lemma 2.1, this time applied to $M^*$. This completes the proof of Theorem 2.2. □

Finally in this section we note that Theorem 2.2 remains true if one replaces $S((M_i/q_1; p_1), (M_2/q_2; p_2))$ by $S((M_i\setminus q_1; p_1), (M_2\setminus q_2; p_2))$, provided that $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are now required to be circuits rather than cocircuits of $M_1$ and $M_2$. To prove this one makes the obvious modifications to the proof of Theorem 2.2 and uses Lemma 2.1 again, this time applied to $\mathcal{N}_1'$ and $\mathcal{N}_2'$.

3. Two-element circuits and cocircuits

In this section we obtain two sets of results on the numbers of 2-circuits and 2-cocircuits in minor-minimally-connected matroids. One set is based on the preceding section's characterization of such matroids and the other set relies on results from [6].

We shall denote by $d_2(M)$ the maximum number of pairwise disjoint 2-circuits of $M$ and write $d_2^x(M)$ for $d_2(M^*)$. The set of elements $e$ of $M$ for which $M\setminus e$ is disconnected will be denoted by $\Delta(M)$. Thus, if $M$ is minor-minimally-connected, then $\Delta(M)$ is the disjoint union of $\Delta(M)$ and $\Delta(M^*)$.

Theorem 3.1. Let $M$ be a minor-minimally-connected matroid having at least four elements. Then

$$d_2(M) + d_2^x(M) \geq \text{rk}(M/\Delta(M)) + \text{rk}(M^*/\Delta(M^*)) + 1. \quad (3.1)$$

Note that, since a circuit and a cocircuit cannot have exactly one common element, if $M$ is connected having at least three elements, then $d_2(M) + d_2^x(M)$ equals the maximum number of pairwise disjoint 2-element subsets $X$ of $E(M)$ such that $X$ is either a circuit or a cocircuit.

If $M$ is minimally connected, then $\Delta(M) = E(M)$ and, furthermore, $M$ has no
2-circuits. Thus the preceding theorem generalizes the following result [6, Corollary 2.7].

**Corollary 3.2.** Let \( M \) be a minimally connected matroid having at least four elements. Then

\[
d_2^+(M) \geq \text{rk}(M^*) + 1 = \text{cork} M + 1.
\]

The next two results are used to prove Theorem 3.1. We note that Corollary 3.2 follows easily from the first of these. Indeed, this is how this result was derived in [6].

**Lemma 3.3** ([6, Corollary 2.6]). Let \( M \) be a connected matroid other than a single circuit. Suppose that \( A \subseteq E(M) \) such that for all \( a \) in \( A \), \( M \setminus a \) is disconnected. Then either \( A \) is independent or \( A \) contains at least \( \text{cork}(M | A) + 1 \) non-trivial series classes of \( M \).

**Lemma 3.4** ([6, Lemma 2.3]). Let \( M \) be a connected matroid having at least two elements and \( \{x_1, x_2, \ldots, x_m\} \) be a circuit of \( M \) such that \( M \setminus x_i \) is disconnected for all \( i \) in \( \{1, 2, \ldots, m-1\} \). Then \( \{x_1, x_2, \ldots, x_{m-1}\} \) contains a \( 2 \)-cocircuit of \( M \).

**Proof of Theorem 3.1.** The result is easy to check if \( M \) is a single circuit or a single cocircuit. Next we note that if \( A \) is independent then \( \text{cork}(M | A) = 0 \). Thus, by applying Lemma 3.3 to \( M \) and \( M^* \) we obtain the required result, unless \( \Delta(M) \) is independent in \( M \) and \( \Delta(M^*) \) is independent in \( M^* \). But, in that case, \( \Delta(M) \) is a basis of \( M \) and \( \Delta(M^*) \) is a basis of \( M^* \). Hence, if \( x \in \Delta(M) \), then \( \Delta(M^*) \cup \{x\} \) contains a circuit \( C \) of \( M^* \). Now apply Lemma 3.4 to \( C \) to obtain that \( C \setminus \{x\} \) and hence \( \Delta(M^*) \) contains a \( 2 \)-cocircuit of \( M^* \), that is, a \( 2 \)-circuit of \( M \). This completes the proof of the theorem. \( \square \)

We note that, in the preceding proof, if \( \Delta(M) \) is a basis of \( M \), then not only does \( \Delta(M^*) \) contains a \( 2 \)-circuit of \( M \), but also \( \Delta(M) \) contains a \( 2 \)-cocircuit of \( M \).

It follows on combining this observation with the preceding proof that:

**Proposition 3.5.** For all minor-minimally-connected matroids \( M \) having at least four elements,

\[
d_2(M) + d_2^+(M) \geq 2.
\]

We now use the results of the preceding section to obtain an alternative set of results on the sum \( d_2(M) + d_2^+(M) \) when \( M \) is minor-minimally-connected.

**Theorem 3.6.** Let \( M \) be a minor-minimally-connected matroid having at least four
elements. Then
\[ d_2(M) + d_2^*(M) \geq \frac{1}{3} |E(M) - cork M| + 3. \]

**Proof.** We argue by induction on \(|E(M)|\). If \(M\) has rank or corank equal to one, then it is easy to check that the result holds, so assume that \(rk M, cork M \geq 2\). If every element of \(M\) is in a 2-circuit or a 2-cocircuit, then
\[ d_2(M) + d_2^*(M) \geq \frac{1}{3} |E(M)| \]
\[ \geq \frac{1}{3} (rk M - cork M| + 3), \]
where the second inequality holds since both \(rk M\) and \(cork M\) exceed one.

Now suppose that \(M\) has an element \(p\) which is not in a 2-circuit or a 2-cocircuit. Then \(M \setminus p\) or \(M^* \setminus p\) is disconnected. We shall assume the first possibility occurs. If not, the following argument may be applied using \(M^*\) in place of \(M\). From Section 2, \(M = S((M_1 \setminus q_1; p_1), (M_2 \setminus q_2; p_2))\) where, for \(i = 1, 2\), \(M_i\) is a minor-minimally-connected matroid having at least four elements, and \(\{p_i, q_i\}\) is a circuit of \(M_i\).

We now distinguish two cases:
(i) \(cork M \geq rk M\);
(ii) \(rk M > cork M\).

In the first case, since
\[ |E(M)| = |E(M_1)| + |E(M_2)| - 3 \]
and
\[ rk M = rk M_1 + rk M_2, \]
we have
\[ |rk M - cork M| = (cork M_1 - rk M_1) + (cork M_2 - rk M_2) - 3. \] (3.2)

But
\[ d_2(M) + d_2^*(M) \geq (d_2(M_i) + d_2^*(M_i)) - 1 + (d_2(M_2) + d_2^*(M_2)) - 1, \] (3.3)
and so, by the induction assumption,
\[ d_2(M) + d_2^*(M) \geq \frac{1}{3} (rk M_1 - cork M_1, \ldots) - 1 + \frac{1}{3} (rk M_2 - cork M_2| + 3) - 1. \] (3.4)
The result follows on combining (3.2) and (3.4).

In case (ii), by Theorem 2.2, \(M = S((L_1/s_1; p_1), (L_1/s_2; p_2))\) where, for \(i = 1, 2\), \(L_i\) is a minor-minimally-connected matroid having at least four elements and \(\{p_i, s_i\}\) is a cocircuit. As \(p\) is not in a 2-cocircuit of \(M\), \(\{p_i, s_i\}\) is a series class of \(L_i\). We now form \(L_i''\) from \(L_i\) by adding \(t_i\) in series with \(s_i\) and \(p_i\). Evidently \(L_i''\) is minor-minimally-connected. Moreover,
\[ |E(M)| = |E(L_1'')| + |E(L_2'')| - 5 \]
and
\[ \text{rk } M = \text{rk } L_1'' + \text{rk } L_2'' - 4, \]
so
\[ \text{cork } M = \text{cork } L_1'' + \text{cork } L_2'' - 1. \]

Thus, provided both \( |E(L_1'')| \) and \( |E(L_2'')| \) exceed 5, the induction assumption may be applied to both \( L_1'' \) and \( L_2'' \) to give that
\[
\sum_{i=1}^{2} (d_2(L_i'') + \text{rk } L_i'' + 3) - 1.
\]

The result follows on combining (3.5) and (3.6). It remains to consider the case when \( |E(L_1'')| \) or \( |E(L_2'')| \) equals 5. Assume the first of these occurs. Then \( |E(L_1')| = 4 \) and, as \( p \) is not in a 2-cocircuit of \( M \), it follows that \( L_1 \) is isomorphic to the matroid formed by adding an element in parallel to one of the elements of a 3-circuit. Thus
\[ \text{d}_2(L_1) + \text{d}_2^*(L_1) = 2 = \frac{1}{3}(\text{rk } L_1 - \text{cork } L_1 + 6). \]

Moreover, by the induction assumption,
\[ \text{d}_2(L_2) + \text{d}_2^*(L_2) \geq \frac{1}{3}(\text{rk } L_2 - \text{cork } L_2 + 3). \]

Therefore, as
\[ \sum_{i=1}^{2} (d_2(L_i) + \text{d}_2^*(L_i) - 1) \]
and
\[ \text{rk } M - \text{cork } M = (\text{rk } L_1 + \text{rk } L_2 - 2) - (\text{cork } L_1 + \text{cork } L_2 - 1), \]
we have
\[ \sum_{i=1}^{2} (d_2(L_i) + \text{d}_2^*(L_i) - 1) \]
and
\[ \text{rk } M - \text{cork } M \geq \frac{1}{3}(\text{rk } L_1 + \text{rk } L_2 - \text{cork } L_1 - \text{cork } L_2 + 9) - 2 \]
\[ = \frac{1}{3}(\text{rk } M - \text{cork } M + 10) - 2 \]
\[ > \frac{1}{3}(\text{rk } M - \text{cork } M) + 3. \]

We conclude that if \( |E(L_1'')| = 5 \) or \( |E(L_2'')| = 5 \), then the required result holds. This completes the proof of the theorem. \( \square \)

The next result is obtained by making the obvious modifications to the proof of Theorem 3.6. We omit the details.

**Theorem 3.7.** Let \( M \) be a minor-minimally-connected matroid having at least four elements and suppose that no series or parallel class of \( M \) has more than two elements. Then
\[ \text{d}_2(M) + \text{d}_2^*(M) \geq \frac{1}{3}(\text{rk } M - \text{cork } M) + 3. \]
The matroids which attain equality in the last result have been characterized, as have the matroids attaining equality in Theorems 3.1 and 3.6. The details of these characterizations will not be given here.

The bounds in Theorems 3.1 and 3.6 are not directly comparable. To see that the difference between these two bounds may become arbitrarily large in either direction, consider the cycle matroids of the graphs $A_k$ and $B_k$ shown in Figs. 1 and 2.

Evidently,
\[ \frac{1}{3}(|rk(M(A_k)) - cork(M(A_k))| + 3) = \frac{1}{3}(|k + 1 - 3k| + 3) = \frac{1}{3}(2k + 2), \]
whereas
\[ rk(M(A_k)/\Delta(M(A_k))) + rk(M^*(A_k)/\Delta(M^*(A_k))) + 1 = 0 + 0 + 1 = 1. \]

In this case, the bound in Theorem 3.6 exceeds the bound in Theorem 3.1. The
reverse is true for $M(B_k)$ as

$$\frac{1}{3}(\text{rk}(M(B_k)) - \text{cork}(M(B_k))) + 3 = \frac{1}{3}((3k - 1 - (3k + 1)) + 3) = \frac{5}{3},$$

while

$$\text{rk}(M(B_k)/\Delta(M(B_k))) + \text{rk}(M^*(B_k)/\Delta(M^*(B_k))) + 1 = k + 1 + 1 = k + 2.$$

4. Minimally connected matroids

In this section we shall prove a new bound on the number of pairwise disjoint 2-cocircuits in a minimally connected matroid. This new bound is sometimes better and sometimes worse than the bound given in Corollary 3.2. It was suggested by Dirac’s result [4; (6), (5)] that a minimally 2-connected graph $G$ has at least $\frac{1}{3}(|V(G)| + 4)$ vertices of degree two.

**Theorem 4.1.** Let $M$ be a minimally connected matroid having at least four elements. Then

$$d_2^x(M) \geq \frac{1}{3}(\text{rk } M + 2).$$

**Proof.** We argue by induction on $|E(M)|$. If cork $M = 1$, then $M$ is a circuit and, since $|E(M)| \geq 4$, the result follows easily. We may therefore assume that cork $M \geq 2$. Now suppose that every element of $M$ is in a non-trivial series class. Then, since a $k$-element series class contains $[\frac{1}{2}k]$ pairwise disjoint 2-cocircuits and the series classes of $M$ partition $E(M)$, it follows that $d_2^x(M) \geq \frac{1}{3} |E(M)| \geq \frac{1}{3}(\text{rk } M + 2)$, the second inequality being a consequence of the fact that cork $M \geq 2$. Thus, if every element of $M$ is in a 2-cocircuit, then the result holds. We now assume that $M$ has an element $p$ which is not in a 2-cocircuit. Then, by the characterization of minimally connected matroids [6, Theorem 3.1], $M = S((M_i/q_i ; p_i), (M_2/q_2 ; p_2)$ where, for $i = 1, 2$, $M_i$ is minimally connected having at least four elements, and $\{p_i, q_i\}$ is a cocircuit. Furthermore, since $p$ is not a 2-cocircuit of $M$, $\{p_i, q_i\}$ is a series class of $M_i$ and $M_i$ has at least five elements. We form the matroid $M_i'$ from $M_i$ by adding $s_i$ in series with $p_i$ and $q_i$. Then

$$d_2^x(M) = d_2^x(M_i') - 1 + d_2^x(M_2) - 1.$$

Thus, by the induction assumption,

$$d_2^x(M) \geq \frac{1}{3}(\text{rk } M_i' + 2) - 1 + \frac{1}{3}(\text{rk } M_2 + 2) - 1 = \frac{1}{3}(\text{rk } M_i' + \text{rk } M_2 + 2 - 2) = \frac{1}{3}(\text{rk } M + 2),$$

and the theorem is proved. □

A minimally connected matroid $M$ of rank $r$ is not free and therefore has at
least \( r+1 \) elements. Moreover, Murty [5, Theorem 3.2] has shown that, provided \( M \) is not a 3-circuit, \(|E(M)| \leq 2r - 2\). A comparison of Corollary 3.2 and Theorem 4.1 shows that the size of \(|E(M)|\) determines which of these results provides the sharper bound; this is specified in the next result.

**Corollary 4.2.** Let \( M \) be a minimally connected matroid. Then

\[
d_2^M(M) \geq \begin{cases} \lfloor \frac{1}{3}(rk M + 2) \rfloor & \text{for } rk M + 1 \leq |E(M)| < \lfloor \frac{1}{3}(4rk M - 1) \rfloor, \\ \text{cork } M + 1 & \text{for } \lfloor \frac{1}{3}(4rk M - 1) \rfloor \leq |E(M)| \leq 2rk M - 2. \end{cases}
\]

The matroids for which equality is attained in Corollary 3.2 were determined in [7, Theorem 3.2] using the characterization of minimally connected matroids. The same result can also be used to characterize those matroids for which equality is attained in Theorem 4.1. We omit the routine proof. We shall denote by \( U_{n,n+2,k} \) the matroid obtained from the uniform matroid \( U_{n,n+2} \) by replacing each element of the latter by \( k \) elements in series.

**Proposition 4.3.** Let \( M \) be a minimally connected matroid. Then \( M \) has exactly \( \lfloor \frac{1}{3}(rk M + 2) \rfloor \) pairwise disjoint 2-cocircuits if and only if \( M \) is a 5-circuit or \( M = U_{n,n+2,3} \) for some \( n \geq 1 \), or \( M = S((M_1/q_1, s_1 ; p_1), (M_2/q_2, s_2 ; p_2)) \) where, for \( i = 1, 2 \), \( M_i \) is a minimally connected matroid having \( \lfloor \frac{1}{3}(rk M_i + 2) \rfloor \) pairwise disjoint 2-cocircuits and having \( \{p_i, q_i, s_i\} \) as a series class.

**References**