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ON MINOR-MINIMALLY-CONNECTED MATROIDS

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By a well-known result of Tutte, if e is an element of a connected matroid M, then either the deletion or the contraction of e from M is connected. If, for every element of M, exactly one of these minors is connected, then we call M minor-minimally-connected. This paper characterizes such matroids and shows that they must contain a number of two-element circuits or cocircuits. In addition, a new bound is proved on the number of 2-cocircuits in a minimally connected matroid.

1. Introduction

A minimally connected matroid is a connected matroid for which every single-element deletion is disconnected. Several authors [5, 6, 9, 10] have shown that such matroids behave similarly to their graph-theoretic counterparts, minimally 2-connected graphs. In particular, just as minimally 2-connected graphs have a number of vertices of degree two [4, 8], minimally connected matroids have a number of 2-element cocircuits.

The following result of Tutte [11, 6.5] (see also [3, p. 410]) is well known.

Proposition 1.1. *If e is an element of a connected matroid M, then either the deletion $M\backslash e$ or the contraction M/e is also connected.*

In this paper, we study the class of *minor-minimally-connected* matroids, that is, those connected matroids M with the property that, for every element e, exactly one of $M\backslash e$ and M/e is connected. Such matroids arise naturally in induction arguments involving connected matroids since either a connected matroid M is minor-minimally-connected or else it has an element f such that both $M\backslash f$ and M/f are connected. Evidently all minimally connected matroids and their duals are minor-minimally-connected and, in Section 2, we give a characterization of minor-minimally-connected matroids that is similar to the characterization of minimally connected matroids given in [6, Theorem 3.1]. This characterization is then used in Section 3 to prove several results on the occurrence of 2-element circuits and 2-element cocircuits in minor-minimally-connected matroids. Finally, in Section 4, we prove a new result on the number of 2-element cocircuits in a minimally connected matroid.

The terminology used here for matroids and graphs will in general follow [12] and [1], respectively. The ground set, rank and corank of a matroid M will be denoted by $E(M)$, rk M and cork M, respectively; k-element circuits and cocircuits of M will be called *k-circuits* and *k-cocircuits*. A *series class* of M is a maximal subset X of $E(M)$ such that if x and y are distinct elements of X, then x and y are *in series* in M, that is, $\{x, y\}$ is a 2-cocircuit of M. A *parallel class* of M is a series class of M^*. We call a series or parallel class *non-trivial* if it contains at least two elements.

The main results of this paper use the operations of series and parallel connection of matroids. Let M_1 and M_2 be matroids on disjoint sets S_1 and S_2, respectively. Suppose that $p_i \in S_i$ for $i = 1, 2$ and let p be an element which is in neither S_1 nor S_2. Then the *series connection* $S((M_1 ; p_1), (M_2 ; p_2))$ of M_1 and M_2 with respect to the *basepoints* p_1 and p_2 is the matroid on $(S_1 \backslash p_1) \cup (S_2 \backslash p_2) \cup p$ whose circuits are the circuits of M_1 not containing p_1, the circuits of M_2 not containing p_2, and all sets of the form $(C_1 \backslash p_1) \cup (C_2 \backslash p_2) \cup p$, where C_i is a circuit of M_i containing p_i. The *parallel connection* $P((M_1 ; p_1), (M_2 ; p_2))$ of M_1 and M_2 with respect to p_1 and p_2 is the matroid $S((M_1^* ; p_1), (M_2^* ; p_2))$. We shall make frequent reference to Brylawski's paper [2], where a detailed discussion of these operations and their properties can be found.

2. Minor-minimally-connected matroids

In this section we prove a characterization of minor-minimally-connected matroids closely resembling the characterization of minimally connected matroids given in [6].

Let M be a connected matroid and p be an element of M for which $M \backslash p$ is disconnected. Then $M \backslash p = H_1 \oplus H_2$ for some non-empty matroids H_1 and H_2. If we let $N_1 = M/E(H_1)$ and $N_2 = M/E(H_2)$, then, by [2, Proposition 4.10], $M = S((N_1 ; p), (N_2 ; p))$. Furthermore, by [2, Proposition 4.6], N_1 and N_2 are connected. Now suppose that M is minor-minimally-connected. Then, although N_1 and N_2 need not themselves be minor-minimally-connected matroids, by slightly modifying them we can obtain such matroids. For $i = 1, 2$, form N_i' from N_i by adding an element u_i in parallel with p, and form N_i'' from N_i by adding v_i in series with p.

Lemma 2.1. *If p is not in a 2-circuit of M, then each of N_1', N_1'', N_2' and N_2'' is minor-minimally-connected having at least four elements.*

Proof. It suffices to show that both N_1' and N_1'' are minor-minimally-connected having at least four elements. Since M has no 2-circuits containing p, $M \backslash p$ has no component having fewer than two elements. Thus N_1 has at least three elements and therefore both N_1' and N_1'' have at least four elements. As N_1 is connected, it follows easily that N_1' and N_1'' are connected. We now show that N_1' is minor-minimally-connected noting that a similar argument shows that N_1'' has the same property. Firstly, as both N_1'/p and N_1'/u_1 have loops, both are disconnected. Secondly, if $e \in E(N_1') \backslash \{p, u_1\}$, then either $M \backslash e$ or M/e is disconnected. But, by [2, Proposition 4.7], $M \backslash e = S((N_1 \backslash e; p), (N_2; p))$ and $M/e = S(N_1/e; p), (N_2; p))$. Thus, by [2, Proposition 4.6], either $N_1 \backslash e$ or N_1/e is disconnected. Hence either $N_1' \backslash e$ or N_1'/e is disconnected and so N_1' is minor-minimally-connected. □

Theorem 2.2. *A matroid M is minor-minimally-connected if and only if $|E(M)| \geq 3$ and either M is connected and every element is in a 2-circuit or a 2-cocircuit; or one of M and M^* is isomorphic to the series connection $S((M_1/q_1; p_1), (M_2/q_2; p_2))$, where both M_1 and M_2 are minor-minimally-connected having at least four elements and $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are cocircuits of M_1 and M_2, respectively.*

Since a matroid is connected if and only if its dual is, it follows easily from the definition of a minor-minimally-connected matroid that

Proposition 2.3. *M is minor-minimally-connected if and only if M^* is.*

A consequence of this and the link between series and parallel connections is that if M^* is a series connection as described in Theorem 2.2, then M is isomorphic to the parallel connection $P((M_1^* \backslash q_1; p_1), (M_2^* \backslash q_2; p_2))$, where both M_1^* and M_2^* are minor-minimally-connected having at least four elements and $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are circuits of M_1^* and M_2^*, respectively.

Proof of Theorem 2.2. Evidently if M is a connected matroid having at least three elements and every element is in a 2-circuit or a 2-cocircuit, then M is minor-minimally-connected.

Now suppose that M_1 and M_2 are minor-minimally-connected matroids each having at least four elements and let $\{p_i, q_i\}$ be a cocircuit of M_i for each i. Let M be the series connection $S((M_1/q_1; p_1), (M_2/q_2; p_2))$. Then, as $M_i \backslash q_i$ is disconnected, Proposition 1.1 implies that M_i/q_i is connected, so by [2, Proposition 4.6], M is connected.

To show that M is minor-minimally-connected, first note that, by [2, Proposition 4.9], $M \backslash p = (M_1/q_1) \backslash p_1 \oplus (M_2/q_2) \backslash p_2$, so $M \backslash p$ is disconnected. Now suppose that e is an element of M different from p. Then without loss of generality we may assume that $e \in E(M_1/q_1) \backslash \{p_1\}$. As M_1 is minor-minimally-connected, either $M_1 \backslash e$ or M_1/e is disconnected. In the first case, consider $M \backslash e = S(M_1/q_1 \backslash e, M_2/q_2)$. If $M_1/q_1 \backslash e$ is disconnected, so is $M \backslash e$. Hence assume that $M_1/q_1 \backslash e$ is connected. Then, as $M_1 \backslash e$ is disconnected, it follows that q_1 is a loop or a coloop of $M_1 \backslash e$. Since M_1 is connected, we conclude that $\{e, q_1\}$ is a cocircuit of M_1. But $\{p_1, q_1\}$ is also a cocircuit of M_1, hence so is $\{e, p_1\}$. Thus p_1 is a coloop of $M_1/q_1\backslash e$ and so $M\backslash e$ is disconnected.

We now suppose that M_1/e is disconnected and show that if this occurs, then M/e is also disconnected. Since $M/e = S(M_1/q_1, e, M_2/q_2)$, if $M_1/q_1, e$ is disconnected, so is M/e. We may therefore assume that $M_1/q_1, e$ is connected. As M_1/e is disconnected and M_1 is connected, it follows that $\{e, q_1\}$ is a circuit of M_1. But this circuit meets the cocircuit $\{p_1, q_1\}$ in a single element and this is a contradiction. We conclude that M is minor-minimally-connected and, moreover, by Proposition 2.3, so is M^*.

For the converse, assume that M is a minor-minimally-connected matroid having an element p which is in neither a 2-circuit nor a 2-cocircuit. Then either $M\backslash p$ or M/p is disconnected. In the first case, the result follows without difficulty from Lemma 2.1. In the second case, $M^*\backslash p$ is disconnected and the result again follows from Lemma 2.1, this time applied to M^*. This completes the proof of Theorem 2.2. □

Finally in this section we note that Theorem 2.2 remains true if one replaces $S((M_1/q_1\,;p_1),(M_2/q_2\,;p_2))$ by $S((M_1\backslash q_1\,;p_1),(M_2\backslash q_2\,;p_2))$, provided that $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are now required to be circuits rather than cocircuits of M_1 and M_2. To prove this one makes the obvious modifications to the proof of Theorem 2.2 and uses Lemma 2.1 again, this time applied to N_1'' and N_2''.

3. Two-element circuits and cocircuits

In this section we obtain two sets of results on the numbers of 2-circuits and 2-cocircuits in minor-minimally-connected matroids. One set is based on the preceding section's characterization of such matroids and the other set relies on results from [6].

We shall denote by $d_2(M)$ the maximum number of pairwise disjoint 2-circuits of M and write $d_2^*(M)$ for $d_2(M^*)$. The set of elements e of M for which $M\backslash e$ is disconnected will be denoted by $\Delta(M)$. Thus, if M is minor-minimally-connected, then $E(M)$ is the disjoint union of $\Delta(M)$ and $\Delta(M^*)$.

Theorem 3.1. *Let M be a minor-minimally-connected matroid having at least four elements. Then*

$$d_2(M) + d_2^*(M) \geq \text{rk}(M/\Delta(M)) + \text{rk}(M^*/\Delta(M^*)) + 1. \tag{3.1}$$

Note that, since a circuit and a cocircuit cannot have exactly one common element, if M is connected having at least three elements, then $d_2(M) + d_2^*(M)$ equals the maximum number of pairwise disjoint 2-element subsets X of $E(M)$ such that X is either a circuit or a cocircuit.

If M is minimally connected, then $\Delta(M) = E(M)$ and, furthermore, M has no

2-circuits. Thus the preceding theorem generalizes the following result [6, Corollary 2.7].

Corollary 3.2. *Let M be a minimally connected matroid having at least four elements. Then*

$$d_2^*(M) \geq \text{rk}(M^*) + 1 = \text{cork}\, M + 1.$$

The next two results are used to prove Theorem 3.1. We note that Corollary 3.2 follows easily from the first of these. Indeed, this is how this result was derived in [6].

Lemma 3.3 ([6, Corollary 2.6]). *Let M be a connected matroid other than a single circuit. Suppose that $A \subseteq E(M)$ such that for all a in A, $M \backslash a$ is disconnected. Then either A is independent or A contains at least* $\text{cork}(M \mid A) + 1$ *non-trivial series classes of M.*

Lemma 3.4 ([6, Lemma 2.3]). *Let M be a connected matroid having at least two elements and $\{x_1, x_2, \ldots, x_m\}$ be a circuit of M such that $M \backslash x_i$ is disconnected for all i in $\{1, 2, \ldots, m-1\}$. Then $\{x_1, x_2, \ldots, x_{m-1}\}$ contains a 2-cocircuit of M.*

Proof of Theorem 3.1. The result is easy to check if M is a single circuit or a single cocircuit. Next we note that if A is independent then $\text{cork}(M \mid A) = 0$. Thus, by applying Lemma 3.3 to M and M^* we obtain the required result, unless $\Delta(M)$ is independent in M and $\Delta(M^*)$ is independent in M^*. But, in that case, $\Delta(M)$ is a basis of M and $\Delta(M^*)$ is a basis of M^*. Hence, if $x \in \Delta(M)$, then $\Delta(M^*) \cup \{x\}$ contains a circuit C of M^*. Now apply Lemma 3.4 to C to obtain that $C \backslash \{x\}$ and hence $\Delta(M^*)$ contains a 2-cocircuit of M^*, that is, a 2-circuit of M. This completes the proof of the theorem. □

We note that, in the preceding proof, if $\Delta(M)$ is a basis of M, then not only does $\Delta(M^*)$ contains a 2-circuit of M, but also $\Delta(M)$ contains a 2-cocircuit of M. It follows on combining this observation with the preceding proof that:

Proposition 3.5. *For all minor-minimally-connected matroids M having at least four elements,*

$$d_2(M) + d_2^*(M) \geq 2.$$

We now use the results of the preceding section to obtain an alternative set of results on the sum $d_2(M) + d_2^*(M)$ when M is minor-minimally-connected.

Theorem 3.6. *Let M be a minor-minimally-connected matroid having at least four* *elements. Then*

$$d_2(M)+d_2^*(M)\geq \tfrac{1}{3}(|\mathrm{rk}\, M-\mathrm{cork}\, M|+3).$$

Proof. We argue by induction on $|E(M)|$. If M has rank or corank equal to one, then it is easy to check that the result holds, so assume that $\mathrm{rk}\, M$, $\mathrm{cork}\, M\geq 2$. If every element of M is in a 2-circuit or a 2-cocircuit, then

$$\begin{aligned} d_2(M)+d_2^*(M) &\geq \tfrac{1}{3}\,|E(M)| \\ &\geq \tfrac{1}{3}(|\mathrm{rk}\, M-\mathrm{cork}\, M|+3), \end{aligned}$$

where the second inequality holds since both $\mathrm{rk}\, M$ and $\mathrm{cork}\, M$ exceed one.

Now suppose that M has an element p which is not in a 2-circuit or a 2-cocircuit. Then either $M\backslash p$ or $M^*\backslash p$ is disconnected. We shall assume the first possibility occurs. If not, the following argument may be applied using M^* in place of M. From Section 2, $M=S((M_1\backslash q_1\,;p_1),(M_2\backslash q_2\,;p_2))$ where, for $i=1,2$, M_i is a minor-minimally-connected matroid having at least four elements, and $\{p_i, q_i\}$ is a circuit of M_i.

We now distinguish two cases:

(i) $\mathrm{cork}\, M\geq \mathrm{rk}\, M$;

(ii) $\mathrm{rk}\, M>\mathrm{cork}\, M$.

In the first case, since

$$|E(M)|=|E(M_1)|+|E(M_2)|-3$$

and

$$\mathrm{rk}\, M=\mathrm{rk}\, M_1+\mathrm{rk}\, M_2,$$

we have

$$\begin{aligned} |\mathrm{rk}\, M-\mathrm{cork}\, M| &= \mathrm{cork}\, M-\mathrm{rk}\, M \\ &= (\mathrm{cork}\, M_1-\mathrm{rk}\, M_1)+(\mathrm{cork}\, M_2-\mathrm{rk}\, M_2)-3. \end{aligned} \tag{3.2}$$

But

$$d_2(M)+d_2^*(M)\geq (d_2(M_1)+d_2^*(M_1))-1+(d_2(M_2)+d_2^*(M_2))-1, \tag{3.3}$$

and so, by the induction assumption,

$$\begin{aligned} d_2(M)+d_2^*(M) &\geq \tfrac{1}{3}(|\mathrm{rk}\, M_1-\mathrm{cork}\, M_1|+3)-1 \\ &\quad +\tfrac{1}{3}(|\mathrm{rk}\, M_2-\mathrm{cork}\, M_2|+3)-1. \end{aligned} \tag{3.4}$$

The result follows on combining (3.2) and (3.4).

In case (ii), by Theorem 2.2, $M=S((L_1/s_1\,;p_1),(L_2/s_2\,;p_2))$ where, for $i=1,2$, L_i is a minor-minimally-connected matroid having at least four elements and $\{p_i, s_i\}$ is a cocircuit. As p is not in a 2-cocircuit of M, $\{p_i, s_i\}$ is a series class of L_i. We now form L_i'' from L_i by adding t_i in series with s_i and p_i. Evidently L_i'' is minor-minimally-connected. Moreover,

$$|E(M)|=|E(L_1'')|+|E(L_2'')|-5$$

and

$$\operatorname{rk} M = \operatorname{rk} L_1'' + \operatorname{rk} L_2'' - 4, \tag{3.5}$$

so

$$\operatorname{cork} M = \operatorname{cork} L_1'' + \operatorname{cork} L_2'' - 1. \tag{3.6}$$

Thus, provided both $|E(L_1'')|$ and $|E(L_2'')|$ exceed 5, the induction assumption may be applied to both L_1'' and L_2'' to give that

$$\begin{aligned} d_2(M) + d_2^*(M) &\geq \sum_{i=1}^{2} (d_2(L_i'') + d_2^*(L_i'') - 1) \\ &\geq \sum_{i=1}^{2} [\tfrac{1}{3}(\operatorname{rk} L_i'' - \operatorname{cork} L_i'' + 3) - 1]. \end{aligned}$$

The result follows on combining (3.5) and (3.6). It remains to consider the case when $|E(L_1'')|$ or $|E(L_2'')|$ equals 5. Assume the first of these occurs. Then $|E(L_1)| = 4$ and, as p is not in a 2-cocircuit of M, it follows that L_1 is isomorphic to the matroid formed by adding an element in parallel to one of the elements of a 3-circuit. Thus

$$d_2(L_1) + d_2^*(L_1) = 2 = \tfrac{1}{3}(\operatorname{rk} L_1 - \operatorname{cork} L_1 + 6).$$

Moreover, by the induction assumption,

$$d_2(L_2) + d_2^*(L_2) \geq \tfrac{1}{3}(\operatorname{rk} L_2 - \operatorname{cork} L_2 + 3).$$

Therefore, as

$$d_2(M) + d_2^*(M) \geq \sum_{i=1}^{2} (d_2(L_i) + d_2^*(L_i) - 1)$$

and

$$\operatorname{rk} M - \operatorname{cork} M = (\operatorname{rk} L_1 + \operatorname{rk} L_2 - 2) - (\operatorname{cork} L_1 + \operatorname{cork} L_2 - 1),$$

we have

$$\begin{aligned} d_2(M) + d_2^*(M) &\geq \tfrac{1}{3}(\operatorname{rk} L_1 + \operatorname{rk} L_2 - \operatorname{cork} L_1 - \operatorname{cork} L_2 + 9) - 2 \\ &= \tfrac{1}{3}(\operatorname{rk} M - \operatorname{cork} M + 10) - 2 \\ &> \tfrac{1}{3}(|\operatorname{rk} M - \operatorname{cork} M| + 3). \end{aligned}$$

We conclude that if $|E(L_1'')| = 5$ or $|E(L_2'')| = 5$, then the required result holds. This completes the proof of the theorem. □

The next result is obtained by making the obvious modifications to the proof of Theorem 3.6. We omit the details.

Theorem 3.7. *Let M be a minor-minimally-connected matroid having at least four elements and suppose that no series or parallel class of M has more than two elements. Then*

$$d_2(M) + d_2^*(M) \geq \tfrac{1}{2}(|\operatorname{rk} M - \operatorname{cork} M| + 3).$$

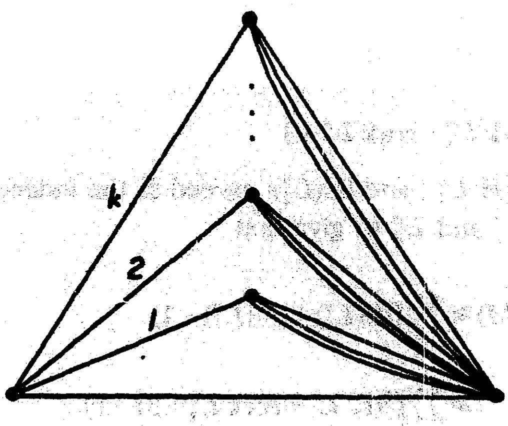

Fig. 1. A_k.

The matroids which attain equality in the last result have been characterized, as have the matroids attaining equality in Theorems 3.1 and 3.6. The details of these characterizations will not be given here.

The bounds in Theorems 3.1 and 3.6 are not directly comparable. To see that the difference between these two bounds may become arbitrarily large in either direction, consider the cycle matroids of the graphs A_k and B_k shown in Figs. 1 and 2.

Evidently,

$$\tfrac{1}{3}(|\mathrm{rk}(M(A_k))-\mathrm{cork}(M(A_k))|+3)=\tfrac{1}{3}(|k+1-3k|+3)$$
$$=\tfrac{1}{3}(2k+2),$$

whereas

$$\mathrm{rk}(M(A_k)/\Delta(M(A_k)))+\mathrm{rk}(M^*(A_k)/\Delta(M^*(A_k)))+1=0+0+1=1.$$

In this case, the bound in Theorem 3.6 exceeds the bound in Theorem 3.1. The

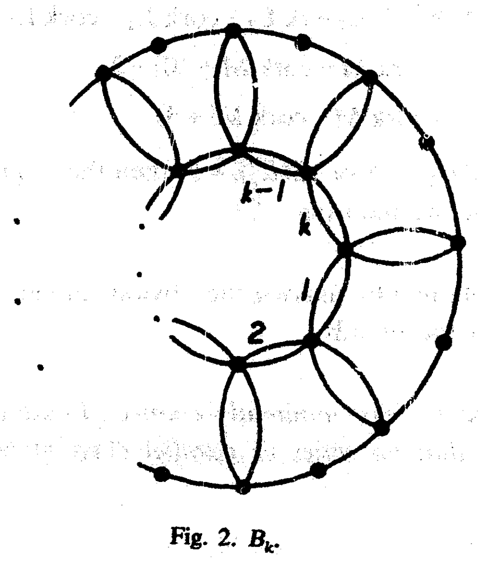

Fig. 2. B_k.

reverse is true for $M(B_k)$ as

$$\tfrac{1}{3}(|\text{rk}(M(B_k)) - \text{cork}(M(B_k))| + 3) = \tfrac{1}{3}(|3k-1-(3k+1)|+3) = \tfrac{5}{3},$$

while

$$\text{rk}(M(B_k)/\Delta(M(B_k))) + \text{rk}(M^*(B_k)/\Delta(M^*(B_k))) + 1 = k+1+1 = k+2.$$

4. Minimally connected matroids

In this section we shall prove a new bound on the number of pairwise disjoint 2-cocircuits in a minimally connected matroid. This new bound is sometimes better and sometimes worse than the bound given in Corollary 3.2. It was suggested by Dirac's result [4; (6), (5)] that a minimally 2-connected graph G has at least $\frac{1}{3}(|V(G)|+4)$ vertices of degree two.

Theorem 4.1. *Let M be a minimally connected matroid having at least four elements. Then.*

$$d_2^*(M) \geq \tfrac{1}{3}(\text{rk}\, M + 2).$$

Proof. We argue by induction on $|E(M)|$. If cork $M = 1$, then M is a circuit and, since $|E(M)| \geq 4$, the result follows easily. We may therefore assume that cork $M \geq 2$. Now suppose that every element of M is in a non-trivial series class. Then, since a k-element series class contains $\lfloor \frac{1}{2}k \rfloor$ pairwise disjoint 2-cocircuits and the series classes of M partition $E(M)$, it follows that $d_2^*(M) \geq \frac{1}{3}|E(M)| \geq \frac{1}{3}(\text{rk}\, M + 2)$, the second inequality being a consequence of the fact that cork $M \geq 2$. Thus, if every element of M is in a 2-cocircuit, then the result holds. We now assume that M has an element p which is not in a 2-cocircuit. Then, by the characterization of minimally connected matroids [6, Theorem 3.1], $M = S((M_1/q_1; p_1), (M_2/q_2; p_2)$ where, for $i = 1, 2$, M_i is minimally connected having at least four elements, and $\{p_i, q_i\}$ is a cocircuit. Furthermore, since p is not a 2-cocircuit of M, $\{p_i, q_i\}$ is a series class of M_i and M_i has at least five elements. We form the matroid M_i'' from M_i by adding s_i in series with p_i and q_i. Then

$$d_2^*(M) = d_2^*(M_1'') - 1 + d_2^*(M_2'') - 1.$$

Thus, by the induction assumption,

$$\begin{aligned} d_2^*(M) &\geq \tfrac{1}{3}(\text{rk}\, M_1'' + 2) - 1 + \tfrac{1}{3}(\text{rk}\, M_2'' + 2) - 1 \\ &= \tfrac{1}{3}(\text{rk}\, M_1'' + \text{rk}\, M_2'' - 2) \\ &= \tfrac{1}{3}(\text{rk}\, M + 2), \end{aligned}$$

and the theorem is proved. □

A minimally connected matroid M of rank r is not free and therefore has at least $r+1$ elements. Moreover, Murty [5, Theorem 3.2] has shown that, provided M is not a 3-circuit, $|E(M)| \leq 2r-2$. A comparison of Corollary 3.2 and Theorem 4.1 shows that the size of $|E(M)|$ determines which of these results provides the sharper bound; this is specified in the next result.

Corollary 4.2. *Let M be a minimally connected matroid. Then*

$$d_2^*(M) \geq \begin{cases} \lceil \frac{1}{3}(\operatorname{rk} M+2) \rceil & \text{for } \operatorname{rk} M+1 \leq |E(M)| < \lceil \frac{1}{3}(4 \operatorname{rk} M-1) \rceil, \\ \operatorname{cork} M+1 & \text{for } \lceil \frac{1}{3}(4 \operatorname{rk} M-1) \rceil \leq |E(M)| \leq 2 \operatorname{rk} M-2. \end{cases}$$

The matroids for which equality is attained in Corollary 3.2 were determined in [7, Theorem 3.2] using the characterization of minimally connected matroids. The same result can also be used to characterize those matroids for which equality is attained in Theorem 4.1. We omit the routine proof. We shall denote by $U_{n,n+2,k}$ the matroid obtained from the uniform matroid $U_{n,n+2}$ by replacing each element of the latter by k elements in series.

Proposition 4.3. *Let M be a minimally connected matroid. Then M has exactly $\frac{1}{3}(\operatorname{rk} M+2)$ pairwise disjoint 2-cocircuits if and only if M is a 5-circuit or $M \cong U_{n,n+2,3}$ for some $n \geq 1$, or $M = S((M_1/q_1, s_1; p_1), (M_2/q_2, s_2; p_2))$ where, for $i = 1, 2$, M_i is a minimally connected matroid having $\frac{1}{3}(\operatorname{rk} M_i+2)$ pairwise disjoint 2-cocircuits and having $\{p_i, q_i, s_i\}$ as a series class.*

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