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Extensions over Hereditary Artinian Rings with Self-Dualities, I

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In this paper we study the finitely generated indecomposable modules over an arbitrary extension over an Artinian ring with self-(Morita) duality. Let A be an Artinian ring with self-duality and T an extension over A with kernel Q (see [11, Chap. XIV, Sect. 2]) such that Q_A and ${}_AQ$ are isomorphic to injective hulls of $\mathrm{top}(A_A)$ and $\mathrm{top}({}_AA)$, respectively. Such an A-module Q will be called quasi-Frobenius. Then it will be proved that

(1) *T is a quasi-Frobenius ring.*

(2) *If A is of finite representation type and hereditary, then T is also of finite representation type. In this case, making use of almost split sequences in* mod T, *every finitely generated indecomposable nonprojective T-module is constructed from a finitely generated indecomposable projective A-module and, simultaneously from a finitely generated indecomposable injective A-module.*

Recently Tachikawa [25] has proved that results similar to the above hold for a hereditary Artin algebra A with the center C and with T its trivial extension $A \ltimes Q$ by $Q = \mathrm{Hom}_C(A, \mathrm{E}(\mathrm{top}(C)))$, where $\mathrm{E}(\mathrm{top}(C))$ is an injective hull of $\mathrm{top}(C)$ in mod C. He also suggested that his results will be still true for an arbitrary extension T (oral communication). Here it should be noted that, even if A is a hereditary algebra over a field K, there is a nontrivial extension T over A with kernel $Q = \mathrm{Hom}_K(A, K)$. This paper answers his question. Our proofs are quite different from the ones given by Tachikawa.

In Section 1 we first recall the definition of an extension over a ring according to [11, Chap. XIV]. Let A be an Artinian ring with a quasi-Frobenius module Q and T an extension over A with kernel Q. Then some fundamental relations between indecomposable projective T-modules and indepcomosable injective A-modules are examined. Using these results it will be proved that *T is a quasi-Frobenius ring which is not necessarily weakly symmetric*, while if A is an Artin algebra with the center C and

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$Q = \mathrm{Hom}_C(A, \mathrm{E}(\mathrm{top}(C)))$, then the trivial extension $A \ltimes Q$ is always weakly symmetric.

Let Ind R be the set of isomorphism classes of indecomposable R-modules. Let $\Phi\colon \mathrm{Ind}\, A \to \mathrm{Ind}\, T \backslash \mathrm{Ind}\, A$ be the map such that $\Phi([M])$ is $[\Omega_T(M)]$ for every nonprojective A-module M, and the isomorphism class of a projective cover of M in mod T for every projective A-module M, where $\Omega_T(M)$ is the first syzygy module of M in mod T. In Section 2, we prove that *Φ is injective* in general, and *Φ is bijective if and only if T satisfies the following property.*

(∗) *For every finitely generated indecomposable right T-module M, the annihilator $\ell_M(Q)$ of Q in M is M or injective* in mod A.

It is also shown that if A is hereditary, then T satisfies (∗), and in case that T is the trivial extension $A \ltimes Q$ of A by Q the converse holds. Therefore, combining these results we conclude that *in the case of $T = A \ltimes Q$, A is hereditary if and only if Φ is bijective.* In case A is of finite representation type, this shows that A is hereditary if and only if $T = A \ltimes Q$ is of finite representation type and the number of indecomposable right T-modules is two times the number of indecomposable right A-modules. Here we note that in the next paper it will be characterized the Artinian ring with an extension satisfying the above condition (∗).

Section 3 is a preparation for the following sections. We note some facts about almost split sequences. Some of them are well known, in fact, are due to the work by Auslander and Reiten [2–4].

In Section 4 we are devoted to a construction of indecomposable modules over an extension T, in case A is hereditary and of finite representation type. Assume that both A and T have almost split sequences. First we consider a relation between almost split sequences in mod A and in mod T. It will be shown that *T has property (∗) if and only if every almost split sequence in* mod A *is still almost split in* mod T, *if and only if every irreducible morphism between indecomposable modules in* mod A *is irreducible in* mod T. *In particular, these properties are valid for hereditary Artinian rings A.* Next we assume that A is hereditary, and for the quiver of A we define a "distance" from the sources to a vertex. After observing the properties of this distance, we prove that *if A is of finite representation type and hereditary, then every indecomposable nonprojective T-module M is isomorphic to $\omega_T^m(P)$ and $\omega_T^n(E)$ for some indecomposable projective A-module P and indecomposable injective A-module E and some integers m and n, where ω_T* is defined as follows: For an almost split sequence $0 \to Z \to Y \to X \to 0$ in mod T, we denote Z by $\omega_T(X)$ and X by $\omega_T^{-1}(Z)$. Then the meaning of ω_T^ℓ is clear for any integer ℓ. In view of [10], this construction theorem, as a corollary, implies the well known theorem [7, 12]: if A is a hereditary Artin algebra of finite representation type, then every indecomposable A-module is isomorphic to $(\mathrm{Tr}\,\mathrm{D})^m\,(P)$ and $(\mathrm{D}\,\mathrm{Tr})^n\,(E)$ for some indecomposable projective A-module P and indecomposable injective A-module E and m, $n \geqslant 0$, where Tr denotes the transpose and D the usual duality.

In the last section, 5, for a hereditary Artin algebra A of finite representation type we consider an almost split sequence in mod T such that the number of indecomposable summands of the middle term is maximal. Next, for a basic hereditary Artinian ring A with a quasi-Frobenius A-bimodule Q and of finite representation type we describe the Auslander–Reiten quiver in mod T. Here the Auslander–Reiten quiver in mod T is such a quiver that the vertices are the isomorphism classes of indecomposable T-modules, and for vertices $[X]$ and $[Y]$ there is an arrow $[X] \to [Y]$ if and only if there is an irreducible morphism $X \to Y$. As for a basic hereditary Artin algebra of finite representation type, it is associated with one of the Dynkin diagrams [12]: A_n $(n \geqslant 1)$, B_n $(n \geqslant 2)$, C_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$, E_6, E_7, E_8, F_4 and G_2. In our case, if we assume that the quiver of the ring A is one of the Dynkin diagrams, we will obtaine the Auslander–Reiten quiver in mod T which is similar to the corresponding Dynkin diagram.

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1. Preliminaries

Throughout this paper, rings will be assumed to be associative rings with identity elements. For a ring A, mod A means the category of finitely generated right A-modules. For a right A-module M we denote the socle of M by soc(M) and the top $M/M\,\mathrm{rad}(A)$ by top(M), where rad(A) denotes the Jacobson radical of A. An injective hull of M in mod A is explicitly denoted by $\mathrm{E}_A(M)$, but we usually denote it by $\mathrm{E}(M)$ for short, unless there is confusion. Let A and T be two rings and Q an A-bimodule such that there is an exact sequence

$$0 \to Q \xrightarrow{\kappa} T \xrightarrow{\rho} A \to 0$$

with a monomorphism κ and a ring epimorphism ρ. Then, T is said to be an *extension over A with kernel Q* [11, Chap. XIV, Sect. 2], provided that

$$\kappa(aq) = t\kappa(q) \qquad \text{and} \qquad \kappa(qa) = \kappa(q)\, t$$

for $q \in Q$, $a \in A$, $t \in T$ and $a = \rho(t)$. Clearly this condition is equivalent to saying that $\kappa(Q)^2 = 0$. Since ρ is a ring epimorphism, Ker ρ has the structure of a T-bimodule as a two-sided ideal in T, while the ρ and the given structure of Q as an A-bimodule canonically induce on Q the structure of a T-

bimodule. The condition that $\kappa(Q)^2 = 0$ then means that these T-bimodule structures coincide via κ. For this reason, in all that follows, κ will be identified as an inclusion map, if there is no confusion. An extension $0 \to Q \to^{\kappa} T \to^{\rho} A \to 0$ with a ring monomorphism $\iota: A \to T$ such that $\rho\iota = 1_A$ seems to have the simplest structure among the extensions over A with kernel Q. Such an extension is called a *trivial extension* of A by Q [15] and is equivalent to the extension

$$0 \to Q \xrightarrow{\kappa_0} A \ltimes Q \xrightarrow{\rho_0} A \to 0,$$

where the ring $A \ltimes Q$ is a direct sum $A \oplus Q$ as additive groups with

$$\kappa_0(q) = (0, q), \qquad \rho_0(a, q) = a$$

for $a \in A$ and $q \in Q$, and with multiplication defined by

$$(a_1, q_1)(a_2, q_2) = (a_1 a_2, a_1 q_2 + q_1 a_2)$$

for $a_1, a_2 \in A$ and $q_1, q_2 \in Q$. Here it should be remembered that for an algebra A over a field and an A-bimodule Q, as is well known, the set of all equivalence classes of extensions over A with kernel Q is in a one-to-one correspondence with the cohomology group $H^2(A, Q)$ of A with coefficients in Q. In particular, an extension whose equivalence class corresponds to the zero element in $H^2(A, Q)$ is a trivial extension of A with kernel Q (cf. [11, Chap. XIV, Theorem 2.1]).

Now let $0 \to Q \to^{\kappa} T \to^{\rho} A \to 0$ be an extension over a ring A with kernel ${}_AQ_A$. Since $Q^2 = 0$ in T, it then holds that $Q \subset \operatorname{rad}(T)$ and $\operatorname{rad}(T) = \rho^{-1}(\operatorname{rad}(A))$. Further, a right A-module M can be canonically regarded as a right T-module by ρ, so that it holds that $MQ = 0$. Conversely a right T-module M with $MQ = 0$ can be canonically regarded as an A-module by ρ. Particularly, every simple A-module is simple as a T-module and every simple T-module is also simple as an A-module, because $Q \subset \operatorname{rad}(T)$ and $\operatorname{rad}(T) = \rho^{-1}(\operatorname{rad}(A))$, In view of these facts, A-modules will be identified with the T-modules annihilated by Q, in all but the proofs of (2.2–3) and (2.8). Since Q is a nilpotent ideal in T, it is well known that a finite orthogonal set of primitive idempotents in A can be lifted to T. Hence, if an identity element 1_A is a sum of orthogonal primitive idempotents e_i $(1 \leqslant i \leqslant n)$, then there exists an orthogonal set of primitive idempotents $\mathbf{e}_i$ in T such that $1_T = \sum_{i=1}^n \mathbf{e}_i$ and $\rho(\mathbf{e}_i) = e_i$.

Now, let A be a rght Artinian ring and B a left Artinian ring such that there is a duality between the categories $\operatorname{mod} A$ and $\operatorname{mod} B^\circ$, where B° is the opposite ring of B. Such a duality is called a *Morita duality* and the duality functor is characterized as a functor $\operatorname{Hom}(\ , U)$, where U is a (B, A)-bimodule which has the properties that U_A and ${}_BU$ are finitely generated injective cogenerators and ${}_BU_A$ is balanced, i.e., $A \simeq \mathrm{End}({}_BU)$ and $B \simeq \mathrm{End}(U_A)$ canonically (see [8, 18]). Further, in this case, every indecomposable injective right A-module and every indecomposable injective left B-module are finitely generated. Let A be a left and right Artinian ring and Q an A-bimodule such that

$$Q_A \text{ and } {}_AQ \text{ are finitely generated,}$$

$$Q_A \simeq \mathrm{E}(\mathrm{top}(A_A)) \qquad \text{and} \qquad {}_AQ \simeq \mathrm{E}(\mathrm{top}({}_AA)).$$

In this paper we call such a finitely generated A-bimodule Q a *quasi-Frobenius module* or QF-*module* for short. (It must be noted that the same terminology is used in [8], but they do not coincide in general.) Then, since both Q_A and ${}_AQ$ are injective cogenerators, they are faithful. As is well known, a bimodule ${}_AU_A$ such that $\mathrm{Hom}_A(\ ,U)$ defines a duality between $\mathrm{mod}\,A$ and $\mathrm{mod}\,A^\circ$ is a QF-module. Hence an Artinian ring with a self-duality has always a QF-module. In fact, it will be shown in the future paper a QF-module is nothing but the bimodule which defines a self-duality. But we do not use this property in this paper. Let $1_A = \sum_{i=1}^n e_i$, where $\{e_i\}$ is an orthogonal set of primitive idempotents in A. Then we have that $Q_A = \bigoplus_{i=1}^n e_iQ$, ${}_AQ = \bigoplus_{i=1}^n Qe_i$, and each of e_iQ and Qe_i is indecomposable injective and hence its socle is simple. Therefore, since $\mathrm{soc}(Q_A) \simeq \bigoplus_{i=1}^n \mathrm{top}(e_iA)$ and $\mathrm{soc}({}_AQ) \simeq \bigoplus_{i=1}^n \mathrm{top}(Ae_i)$ by definition, there are two permutations π, π' on $\{1, 2, \ldots, n\}$ such that

$$\mathrm{soc}(e_iQ) \simeq \mathrm{top}(e_{\pi(i)}A) \qquad \text{and} \qquad \mathrm{soc}(Qe_i) \simeq \mathrm{top}(Ae_{\pi'(i)})$$

for all i. However, it is easily shown that $\pi' = \pi^{-1}$ (cf. [20, p. 8]). Such a permutation π is called the *Nakayama permutation* by ${}_AQ_A$. In case A is a quasi-Frobenius ring, π is nothing else than the usual permutation induced from A given in [20]. In case A is an Artin algebra over the center C, if we consider $\mathrm{Hom}_C(A, \mathrm{E}_C(\mathrm{top}(C)))$ as a QF-module Q, then it is well known that the Nakayama permutation by Q is identity. In the rest of this section, for an Artinian ring A with a QF-module Q we will prove that any extension T over A with kernel Q is always a quasi-Frobenius ring and the Nakayama permutation by ${}_TT_T$ coincides with that by ${}_AQ_A$.

Given a ring R, for a right R-module M and a subset X of R the (*left*) *annihilator* of X in M is usually defined as $\ell_M(X) = \{m \in M \mid mX = 0\}$, and for a left R-module N the (*right*) *annihilator* of X in N is $r_N(X) = \{n \in N \mid Xn = 0\}$.

LEMMA 1.1. *Let A be an Artinian ring and ${}_AQ_A$ a* QF-*module. Let $0 \to Q \to^{\kappa} T \to^{\rho} A \to 0$ be an extension. Then for an idempotent $\mathbf{e}$ in T and $e = \rho(\mathbf{e})$, the following statements hold.*

(1) $\mathbf{e}T/\mathbf{e}Q \simeq eA$,

(2) $\ell_{\mathbf{e}T}(Q) = \mathbf{e}Q$,

(3) $\operatorname{top}(\mathbf{e}T) \simeq \operatorname{top}(eA)$ as right A-modules and as right T-modules,

(4) $\operatorname{soc}(\mathbf{e}T) = \operatorname{soc}(\mathbf{e}Q_T) \simeq \operatorname{soc}(eQ_A)$.

Proof. Obvious from the definitions.

PROPOSITION 1.2. *Let A be an Artinian ring and ${}_AQ_A$ a QF-module with Nakayama permutation π. Then every extension T over A with kernel Q is a quasi-Frobenius ring, and π is coincident with the Nakayama permutation by ${}_TT_T$. Particularly, in case π is identity, T is weakly symmetric.*

Proof. First we show that T is left and right Artinian. For this, we note that a given A-module X with a finite (composition) length also has finite length as a T-module, which is an easy consequence of the fact that simple T-modules and simple A-modules are coincident. Then that T is Artinian follows form (1.1–1), because the lengths of A-modules eQ and Qe are finite. Thus to show that T is quasi-Frobenius we have only to show that there is a permutation between the set of $\operatorname{top}(\mathbf{e}_iT)$ and the set of $\operatorname{soc}(\mathbf{e}_jT)$, where $\{\mathbf{e}_i\}_{i=1}^n$ is a complete set of primitive idempotents in T, because $\operatorname{soc}(\mathbf{e}_iT)$ and $\operatorname{soc}(T\mathbf{e}_i)$ are simple [20]. For this, let ρ be the canonical epimorphism $T \to A$ and $e_i = \rho(\mathbf{e}_i)$ for $1 \leqslant i \leqslant n$. Then, $\operatorname{soc}(e_iQ) \simeq \operatorname{top}(e_{\pi(i)}A)$ and $\operatorname{soc}(\mathbf{e}_iT) \simeq \operatorname{soc}(e_iQ)$ by (1.1–4). Hence, it holds that $\operatorname{soc}(\mathbf{e}_iT) \simeq \operatorname{top}(e_{\pi(i)}A) \simeq \operatorname{top}(\mathbf{e}_{\pi(i)}T)$ by (1.1–3). This shows that π is the desired permutation, and also shows that the last assertion holds.

The following well known lemma is very useful for a study of indecomposable modules over a quasi-Frobenius ring.

LEMMA 1.3. *Let A be a quasi-Frobenius ring and let X and Y be finitely generated right A-modules each of which has no projective direct summands, and P a projective right A-module such that there is an exact sequence*

$$0 \to Y \to P \to X \to 0.$$

Then the following assertions hold.

(1) *P is a projective cover of X if and only if P is an injective hull of Y.*

(2) *X is indecomposable if and only if Y is indecomposable.*

Remark 1.4. In conclusion of this section, we will make a few remarks about extensions over hereditary algebras (over a field).

Generally speaking, it is not easy to see whether a given extension is splittable (i.e., trivial) or not. But, for algebras over a field K, if we restrict ring morphisms to K-algebra morphisms (i.e., K- and ring morphisms), we can show that, even for a semi-simple K-algebra A, there are nontrivial extensions over A with kernel $\mathrm{Hom}_K(A, K)$, as follows:

(1) Let A be an inseparable semi-simple algebra over a field K with $\dim_K A < \infty$. Then there is an extension

$$0 \to \mathrm{Hom}_K(A, K) \to T \xrightarrow{f} A \to 0$$

such that f is a K-algebra morphism and there is no K-algebra morphism $g: A \to T$ with $fg = 1_A$.

Because, as is well known, semi-simple algebras are symmetric. Namely, $A \simeq \mathrm{Hom}_K(A, K)$ as A-bimodules [21, Theorems 55.6 and 55.10]. Hence $H^2(A, \mathrm{Hom}_K(A, K)) \simeq H^2(A, A)$ and so the desired result follows from the fact that $H^n(A, A) \neq 0$ for $n > 0$ [14, Proposition 14].

(2) For a hereditary algebra A over an algebraically closed field, all extensions over A are splittable. More generally, it is known that, for an algebra A over a field K all 2-cohomology groups $H^2(A, -) = 0$ if and only if $A/\mathrm{rad}\, A$ is separable and A is hereditary [17, Theorem].

2. Hereditary Artinian Rings with a Morita Duality

In all that follows, unless otherwise stated, all rings will be left and right Artinian and all modules will be *finitely generated* right modules. Let A be an Artinian ring with a quasi-Frobenius module Q and let T be an extension over A with kernel Q. In this section we consider some properties of indecomposable T-modules, each of which will be equivalent or close to the property that A is hereditary. In particular, they will imply very important information in the case of hereditary Artinian rings of finite representation type.

Throughout this section we fix once for all notations such that

$$0 \to Q \xrightarrow{\kappa} T \xrightarrow{\rho} A \to 0$$

is an extension over an Artinian ring A with kernel Q, where Q is a QF-module over A, ρ a ring epimorphism and κ an inclusion map. In this case, it should be remembered that T is a quasi-Frobenius ring by (1.2). To begin with, we study indecomposable T-modules without assuming that A is hereditary.

Lemma 2.1. *For a projective T-module P the following properties hold.*

(1) *$PQ = \ell_P(Q)$.*

(2) *PQ is injective as a right A-module.*

(3) $\operatorname{soc}(P_T) = \operatorname{soc}(PQ_T) = \operatorname{soc}(PQ_A)$.

(4) *P/PQ is projective as a right A-module.*

(5) *For any submodule M of P, $\ell_M(Q) = M \cap PQ$.*

Proof. Since T is Artinian by (1.2), P is a direct sum of indecomposable projectives each of which is isomorphic to a primitive right ideal. Hence, assertions (1)–(4) are easy consequences of (1.1).

(5) Since $\ell_M(Q) = M \cap \ell_P(Q)$, the result follows from (1).

LEMMA 2.2. *Let M be an A-module and P a projective T-module such that there is an embedding $j: M \to P/PQ$. Then there is a finitely generated T-module $\bar{M}$ which satisfies the following properties.*

(1) *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 \to & \ell_{\bar{M}}(Q) & \xrightarrow{v} & \bar{M} & \xrightarrow{u} & M & \to 0 \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle j} & \\ 0 \to & PQ & \xrightarrow{\kappa'} & P & \xrightarrow{\rho'} & P/PQ & \to 0, \end{array}$$

where the top row is an exact sequence with an inclusion v, the bottom is the canonical exact sequence and all vertical morphisms are monomorphic.

(2) $\ell_{\bar{M}}(Q) \subset \bar{M}\operatorname{rad}(T)$.

(3) *If T is a trivial extension of A by Q, then it further holds that*

$$\ell_{\bar{M}}(Q) = \bar{M}Q.$$

Proof. (1) Let $p: \mathrm{P}(M) \to M$ be a projective cover of M in $\operatorname{mod} T$. Then, for a given embedding $j: M \to P/PQ$, we have a morphism $f: \mathrm{P}(M) \to P$ which makes the following diagram commutative

$$\begin{array}{ccccc} \mathrm{P}(M) & \xrightarrow{p} & M & \longrightarrow 0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} & \\ P & \xrightarrow{\rho'} & P/PQ & \longrightarrow 0. \end{array}$$

Now let $\bar{M} = f(\mathrm{P}(M))$. Since $\rho'(\bar{M}) = j(M)$, there is then an epimorphism $u: \bar{M} \to M$, so that the following commutes:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \ell_{\bar{M}}(Q) & \xrightarrow{v} & \bar{M} & \xrightarrow{u} & M & \longrightarrow 0 \\ & & & \downarrow & & \downarrow{\scriptstyle j} & \\ 0 \longrightarrow & PQ & \xrightarrow{\kappa'} & P & \xrightarrow{\rho'} & P/PQ & \longrightarrow 0, \end{array}$$

where the morphisms $\bar{M} \to P$ and $v: \ell_{\bar{M}}(Q) \to \bar{M}$ are inclusions. Then, $\operatorname{Ker} u = \bar{M} \cap \operatorname{Ker} \rho' = \bar{M} \cap PQ$ and so $\operatorname{Ker} u = \bar{M} \cap \ell_P(Q) = \ell_{\bar{M}}(Q)$ by (2.1). Hence we can conclude that the top row in the above is an exact sequence and there is an inclusion $\ell_{\bar{M}}(Q) \to PQ$ which makes the above diagram commutative.

(2) By the construction, it is clear that $\operatorname{Ker} u = f(\operatorname{Ker}(\rho' f))$ and that $\operatorname{Ker}(\rho' f)$ is small in $P(M)$. Hence $\operatorname{Ker} u$ must be small in $\bar{M}$, because the image of a small submodule of a given module X is also small in the image of X. Since $\bar{M}\operatorname{rad}(T)$ contains all small submodules of $\bar{M}$, we thus have that $\operatorname{Ker} u \subset \bar{M}\operatorname{rad}(T)$.

(3) Our aim is to find the module $\bar{M}$ which satisfies all properties (1)–(3). In this proof we forget the agreement in Section 1 such that the A-modules are identified with the T-modules annihilated by Q. We denote by $X \dot{\oplus} Y$ the direct sum of given abelian groups X and Y. Now assume that $T = A \ltimes Q$ and ι is a ring monomorphism with $\rho\iota = 1_A$. Then $T = A \dot{\oplus} Q$ and $P = P' \dot{\oplus} PQ$, where P' is projective in $\operatorname{mod} A$ such that $P' \simeq P/PQ$ as A-modules, because the projective module P is isomorphic to a direct sum of direct summands of T_T. Then M may be regarded as a submodule of the right A-module P', so that MT is a T-submodule of P gerenated by M in P. On the other hand, $M \dot{\oplus} MQ$ is a T-submodule of P and of course contains M. It therefore holds that

$$MT = M \dot{\oplus} MQ, \qquad M \subset P', \qquad MQ \subset PQ.$$

Since $MQ \subset (MT)\operatorname{rad}(T)$ and $M \simeq MT/MQ$ as A-modules, it follows that $\operatorname{top}(MT_T) \simeq \operatorname{top}(M_A)_T$ and so $\operatorname{top}(MT_T) \simeq \operatorname{top}(\mathrm{P}(M)_T)$. Thus there exists an epimorphism $g: \mathrm{P}(M) \to MT$ such that

$$\begin{array}{ccccc} \mathrm{P}(M) & \xrightarrow{g} & MT & \longrightarrow & P \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow \\ j: M & \xrightarrow{\simeq} & MT/MQ & \longrightarrow & P/PQ \end{array}$$

is commutative, where the middle and right vertical morphisms are canonical epimorphisms and the composition of morphisms in the bottom is j. Hence if we put $\bar{M} = MT$ $(= g(\mathrm{P}(M)))$, then we see that assertion (1) holds for this $\bar{M}$ from the proof of (1). Noting that $M \subset P'$ and $MQ \subset PQ$, we have that

$$\ell_{\bar{M}}(Q) = MT \cap \ell_P(Q) = (M \dot{\oplus} MQ) \cap PQ = MQ,$$

in view of (1.1).

Now property (2) for his $\bar{M}$ is clear from the fact that $Q \subset \operatorname{rad}(T)$.

For an arbitrary Artinian ring R and an R-module M, $\Omega_R^n(M)$ means the

nth syzygy module for $n \geqslant 0$. Namely, for a minimal projective resolution $\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to M \to 0$, we put

$$\Omega_R^n(M) = \mathrm{Im}(P_{n+1} \to P_n).$$

Dually we define $\Omega_R^{-n}(M)$ by a minimal injective resolution of M for $n > 0$. We put $\Omega_R^0(M) = M$ and $\Omega_R^1(M) = \Omega_R(M)$. If the ring R is quasi-Frobenius and M an indecomposable nonprojective R-module, then it follows from (1.3) that $\Omega^{-1}(\Omega(M)) \simeq M \simeq \Omega(\Omega^{-1}(M))$. As we noted in Section 1, this is an important fact.

PROPOSITION 2.3. *Let M be a T-module without projective summands, P a projective cover of M and*

$$0 \to \Omega_T(M) \xrightarrow{v} P \xrightarrow{u} M \to 0$$

is exact. Then it holds that

(1) *PQ_A is an injective hull of $\ell_{\Omega_T(M)}(Q)_A$ and*

$$0 \to \ell_{\Omega_T(M)}(Q) \xrightarrow{v'} PQ \xrightarrow{u'} MQ \to 0$$

is exact, where u' and v' are canonically induced from u and v.

(2) *P/PQ_A is a projective cover of M/MQ_A and*

$$0 \to \Omega_T(M)/\ell_{\Omega_T(M)}(Q) \xrightarrow{\bar{v}} P/PQ \xrightarrow{\bar{u}} M/MQ \to 0$$

is exact, where $\bar{u}$ and $\bar{v}$ are canonically induced from u and v.

Proof. (1) It clearly holds that $\mathrm{soc}(\Omega_T(M)) \subset \ell_{\Omega_T(M)}(Q)$, because $Q \subset \mathrm{rad}(T)$. Since P is an injective hull of $\Omega_T(M)$ by (1.3), it further holds that $\mathrm{soc}(P) = \mathrm{soc}(\Omega_T(M))$. Hence it follows from (2.1) that $\mathrm{soc}(PQ) \subset \ell_{\Omega_T(M)}(Q)$. But $\mathrm{soc}(\ell_{\Omega_T(M)}(Q)) \subset \mathrm{soc}(PQ)$, because $\ell_{\Omega_T(M)}(Q) \subset PQ$. Therefore we have that

$$\mathrm{soc}(\ell_{\Omega_T(M)}(Q)) = \mathrm{soc}(PQ).$$

On the other hand, PQ is an injective A-module by (2.1). Thus we can conclude that PQ is an injective hull of $\ell_{\Omega_T(M)}(Q)$. The second assertion is an easy consequence of (2.1), because $\ell_{\Omega_T(M)}(Q) = \ell_P(Q) \cap \Omega_T(M)$.

(2) Since $Q \subset \mathrm{rad}(T)$, it holds that $\mathrm{top}(M) \simeq \mathrm{top}(M/MQ)$ and $\mathrm{top}(P) \simeq \mathrm{top}(P/PQ)$. Hence $\mathrm{top}(P/PQ) \simeq \mathrm{top}(M/MQ)$, because P is a projective cover of M and so $\mathrm{top}(P) \simeq \mathrm{top}(M)$. On the other hand, P/PQ is a projective A-module by (2.1). Therefore the canonical epimorphism

$\bar{u}: P/PQ \to M/MQ$ must be a projective cover. As for the kernel of $\bar{u}$, let X be a T-module such that $X/PQ = \operatorname{Ker} \bar{u}$. Then $X = u^{-1}(MQ)$ and $u(PQ) = MQ = u(X)$. Hence $X = PQ + \operatorname{Ker} u$. Thus we have that

$$X/PQ \simeq \Omega_T(M)/(\Omega_T(M) \cap PQ) = \Omega_T(M)/\ell_{\Omega_T(M)}(Q).$$

PROPOSITION 2.4. *For a T-module M without projective summands the following assertions hold.*

(1) *$MQ = 0$ if and only if $\ell_{\Omega_T(M)}(Q)$ is an injective A-module.*

(2) *$\Omega_T(M)\, Q = 0$ if and only if M/MQ is a projective A-module.*

(3) *If $MQ = 0$ and $\Omega_T(M)\, Q = 0$, then M is a projective A-module and $\Omega_T(M)$ is an injective A-module.*

Proof. Assertions (1) and (2) follow easily from (2.3). For (3), $\ell_{\Omega_T(M)}(Q)$ is injective from (1) and $\Omega_T(M) = \ell_{\Omega_T(M)}(Q)$ by assertion. Hence $\Omega_T(M)$ is an injective A-module. Similarly we can show that M is a projective A-module.

The above lemmas are general consequences in the sense that they imply no restrictions for the ring A. Next we prove some lemmas which are closely related to hereditary rings.

LEMMA 2.5. *For an idempotent* $\mathbf{e}$ *in T and $e = \rho(\mathbf{e})$, we assume that $M/\ell_M(Q)_A$ is projective for any submodule M of* $\mathbf{e}T$ *with $MQ \neq 0$. Then every submodule of eA is projective.*

Proof. Let eI be a nonzero submodule of eA. Then by (2.2) there exists a submodule M of $\mathbf{e}T$ such that

$$eI_A \simeq M/\ell_M(Q)_A.$$

Hence it follows from the assumption that eI_A is projective.

Here we recall the definition of torsionless modules. For an arbitrary ring R, a right R-module M is said to be *torsionless* provided that there is a monomorphism from M to some product of copies of R_R. In our case, since we assume that the ring A is right Artinian, every finitely generated torsionless A-module can be embedded into a finitely generated free A-module.

PROPOSITION 2.6. *Let M be a nonprojective torsionless A-module and $\bigoplus_{i=1}^n P_i$ a projective cover of M in* mod *T, where each P_i is indecomposable. Then it holds that $P_i Q \not\subset \Omega_T(M)\, Q$ for any i and*

$$\Omega_T(M)\, Q \subsetneqq \ell_{\Omega_T(M)}(Q).$$

Proof. Let $u: P \to M$ be a projective cover of M in $\operatorname{mod} T$ and $P = \bigoplus_{i=1}^{n} P_i$. It is then that $\operatorname{Ker} u = \Omega(M)$, $PQ = \ell_{\ker u}(Q)$ by (2.1–5) and

$$(\operatorname{Ker} u)\, Q \subset PQ = \ell_{\ker u}(Q).$$

Hence we must show that $(\operatorname{Ker} u)\, Q \subsetneq PQ$. For this, it suffices to show that

$$P_i Q \not\subset (\operatorname{Ker} u)\, Q \qquad \text{for any } i.$$

Since M can be embedded into some projective A-module by assumption, from (2.2) there is a T-module $\bar{M}$ such that

$$0 \to \ell_{\bar{M}}(Q) \to \bar{M} \xrightarrow{w} M \to 0$$

is exact. By the projectivity of P_T, there is then a morphism $v: P \to \bar{M}$ such that $u = wv$. Since $v(\operatorname{Ker} u) \subset \operatorname{Ker} w = \ell_{\bar{M}}(Q)$, it holds that

$$v((\operatorname{Ker} u)\, Q) = 0.$$

Now suppose that there is P_i such that $P_i Q \subset (\operatorname{Ker} u)\, Q$. Then $v(P_i Q) = 0$ from the above, i.e., $v(P_i) \subset \ell_{\bar{M}}(Q)$. Hence, $u(P_i) = wv(P_i) \subset w(\ell_{\bar{M}}(Q))$, and so it holds that $u(P_i) = 0$, because $\ell_{\bar{M}}(Q) = \operatorname{Ker} w$. But this contradicts that P is a projective cover of M in $\operatorname{mod} T$. Hence we have that $P_i Q \not\subset (\operatorname{Ker} u)\, Q$ for all i.

Making use of the above lemmas, we can now characterize a hereditary Artinian ring with respect to indecomposable T-modules.

THEOREM 2.7. *Let A be an Artinian ring with a* QF-*module Q and let T be an extension over A with kernel Q. Then the following statements are equivalent.*

(1) *A is hereditary.*

(2) *MQ is an injective A-module for every indecomposable T-module M with $MQ \neq 0$.*

(3) *$MQ = \ell_M(Q)$ for every indecomposable T-module M with $MQ \neq 0$.*

(4) *$M/\ell_M(Q)$ is a projective A-module for every indecomposable T-module M with $MQ \neq 0$.*

Proof. First we prove that (1) implies all the others. Let M be an indecomposable T-module with $MQ \neq 0$, and let $u: P' \to M$ be a projective cover of M. Then $u(P'Q) = MQ$ and $P'Q$ is injective in $\operatorname{mod} A$ by (2.1). Since A is hereditary, any factor of an injective module is also injective, and so MQ is injective in $\operatorname{mod} A$, which proves (2). Now let P be an injective hull of M. By the fact just proved above, MQ is injective in $\operatorname{mod} A$ and hence we have that $\ell_M(Q) = MQ \oplus X$ for some submodule X_A. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & X & \xrightarrow{u} & M & \longrightarrow M/X \longrightarrow 0 \\ & s\downarrow\wr & & \downarrow t & \\ 0 \longrightarrow & \ell_M(Q)/MQ & \xrightarrow{v} & M/MQ & \xrightarrow{w} P/PQ, \end{array}$$

where s, t, u, v and w are all canonical morphisms. In this diagram the two sequences are exact. For, we know that

$$\operatorname{Ker} w = (M \cap PQ)/MQ,$$

and by (2.1)

$$\ell_M(Q) = M \cap PQ.$$

By (2.1) we know also that P/PQ is projective in $\operatorname{mod} A$ and hence $\operatorname{Im} w$ is projective in $\operatorname{mod} A$, because A is hereditary. This shows that the exact sequence

$$0 \to \ell_M(Q)/MQ \xrightarrow{v} M/MQ \to \operatorname{Im} w \to 0$$

is splittable. Hence there is a morphism $v' : M/MQ \to \ell_M(Q)/MQ$ with $v'v$ is an identity on $\ell_M(Q)/MQ$. Hence $s = (v'v)\,s = (v't)\,u$, which implies that u is splittable, because s is an isomorphism. Therefore $M \simeq X \oplus M/X$. But $XQ = 0$ and M is indecomposable with $MQ \neq 0$. Hence X must be zero. This means that $MQ = \ell_M(Q)$ and w is monomorphic, that is, M/MQ is projective in $\operatorname{mod} A$. Thus (3) and (4) are proved.

(2) $\Rightarrow$ (1) Let E be a nonsimple indecomposable injective A-module. To show (1) it suffices to show that any factor of E is also injective (see [1, Corollary 11]). By the definition of the QF-module, there is an idempotent e in A such that $E \simeq eQ$. Then it is sufficient to show that any factor of eQ is injective. Let $\mathbf{e}$ be an idempotent in T with $e = \rho(\mathbf{e})$, and $M = \mathbf{e}T/\mathbf{e}I$ for an arbitrary proper submodule $\mathbf{e}I$ of $\mathbf{e}Q$. Then, clearly M is indecomposable, because $\operatorname{top}(M)$ is simple, and $MQ \neq 0$ for $MQ = \mathbf{e}Q/\mathbf{e}I$. Hence, by assumption (2), MQ is injective in $\operatorname{mod} A$, that is, $\mathbf{e}Q/\mathbf{e}I$ is injective in $\operatorname{mod} A$.

(3) $\Rightarrow$ (1) It suffices to show that $\operatorname{rad}(eA)$ is projective for any idempotent e in A. Suppose that $\operatorname{rad}(eA)$ is not projective for some idempotent e. There is then an indecomposable nonprojective summand M of $\operatorname{rad}(eA)$. Then $\Omega_T(M)$ is indecomposable by (1.3) and, since M is nonprojective in $\operatorname{mod} A$, $\Omega_T(M)$ is not annihilated by Q, in view of (2.4). Hence, by

assumption (3), we have that $\Omega_T(M)\,Q = \ell_{\Omega_T(M)}(Q)$. However, since M is torsionless in mod A, this contradicts (2.6). Thus A must be hereditary.

(4) $\Rightarrow$ (1). Let I be a nonzero right ideal in A. Then, by (2.2) there is a T-module M such that

$$0 \to \ell_M(Q) \to M \to I \to 0$$

is exact. For a decomposition $M = \bigoplus_{i=1}^n M_i$ with M_i indecomposable T-modules, it is clear that $\ell_M(Q) = \bigoplus_{i=1}^n \ell_{M_i}(Q)$. Hence,

$$I_A \simeq M/\ell_M(Q)_A \simeq \bigoplus_{i=1}^{n} M_i/\ell_{M_i}(Q)_A.$$

By assumption (4), the right hand side is projective and so I is projective in mod A. Thus we are done.

LEMMA 2.8. *Let A be an Artinian ring with a* QF*-module Q and T a trivial extension of A by Q. For a primitive idempotent* $\mathbf{e}$ *in T let $M = \mathbf{e}T/\mathbf{e}I$, where $\mathbf{e}I$ is a proper submodule of $\mathbf{e}Q$. Then, if $\ell_M(Q)$ is injective in* mod A, *it holds that $MQ = \ell_M(Q)$.*

Proof. In this proof we again forget the agreement in Section 1 for the A-modules (see the proof of (2.2–3)). Since T is a trivial extension of A by Q, there is a splittable ring monomorphism $\iota: A \to T$ with $\rho\iota = 1_A$ by definition. By this morphism ι, every T-module may be canonically regarded as an A-module. Then M can be considered as an A-module, so that it is decomposed into a direct sum of $\ell_M(Q)$ and some A-module X, because $\ell_M(Q)$ is an injective A-module:

$$M_A = X_A \oplus \ell_M(Q)_A. \tag{1}$$

Here, $X \neq 0$ because $MQ = \mathbf{e}Q/\mathbf{e}I \neq 0$. Since M has the unique maximal submodule in mod T, it therefore holds that M is a T-module generated by X, i.e., $M = XT$, while $X \oplus XQ$ is a T-submodule of M. Hence

$$M = X \oplus XQ \tag{2}$$

as right A-modules. Comparing (1) with (2), we conclude that $XQ = \ell_M(Q)$ because $XQ \subset \ell_M(Q)$. On the other hand, it is clear that $MQ = XQ$, thus we have proved that $MQ = \ell_M(Q)$.

THEOREM 2.9. *Let A be an Artinian ring with a* QF*-module Q and T an extension over A with kernel Q.*

(I) *The following statements are equivalent.*

(1) *M/MQ is a projective A-module for any indecomposble T-module M with $MQ \neq 0$.*

(2) *$\ell_M(Q)$ is an injective A-module for any indecomposable T-module M with $MQ \neq 0$.*

(3) *$\Omega_T(M)\,Q = 0$ for any indecomposable T-module M with $MQ \neq 0$.*

(4) *$\Omega_T^{-1}(M)\,Q = 0$ for any indecomposable T-module M with $MQ \neq 0$.*

(II) *If A is hereditary, the assertions of* (I) *hold, and conversely if T is a trivial extension of A by Q.*

Proof. (I) Since T is a quasi-Frobenius ring by (1.2), (1.3) asserts that $\Omega_T(\Omega_T^{-1}(M)) \simeq M$ and $\Omega_T^{-1}(\Omega_T(M)) \simeq M$ for any nonprojective indecomposable T-module M. Hence, (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) are obtained from (2.4).

(3) $\Rightarrow$ (4) Let M be a nonprojective indecomposable T-module with $MQ \neq 0$. The $\Omega_T^{-1}(M)$ is indecomposable nonprojective by (1.3). By assumption, if $\Omega_T^{-1}(M)\,Q \neq 0$, then $\Omega_T(\Omega_T^{-1}(M))\,Q = 0$. But this means that $MQ = 0$, because $M \simeq \Omega_T(\Omega_T^{-1}(M))$, a contradiction.

(4) $\Rightarrow$ (3) This is proved by the same argument as in the above and we omit the proof.

(II) By (2.7) we know that if A is hereditary, (I) holds. To show the converse for a trivial extension T, we show that (I–2) implies that A is hereditary. Now let e be a primitive idempotent in A such that eQ is not simple. Then it suffices to show that all factors of eQ are also injective [1, Corollary 11]. Let eI be a proper submodule of eQ and $\mathbf{e}$ a primitive idempotent in T with $e = \rho(\mathbf{e})$. Let $M = \mathbf{e}T/\mathbf{e}I$. Then, since M is indecomposable, $MQ = \ell_M(Q)$ by (2.8). Hence, MQ is injective in $\operatorname{mod} A$, which shows that eQ/eI is injective in $\operatorname{mod} A$.

EXAMPLE 2.10. Here we note that if T is not a trivial extension of A by Q, then assertion (II) in (2.9) does not hold in general, even if T is weakly symmetric.

Let T be a serial quasi-Frobenius algebra over a field such that $1_T = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, where $\{\mathbf{e}_i\}$ is an orthogonal set of primitive idempotents such that each composition length $|\mathbf{e}_i T|$ is 4 and

$$\operatorname{top}(\mathbf{e}_1 \operatorname{rad}(T)) \simeq \operatorname{top}(\mathbf{e}_2 T),$$

$$\operatorname{top}(\mathbf{e}_2 \operatorname{rad}(T)) \simeq \operatorname{top}(\mathbf{e}_3 T),$$

and

$$\operatorname{top}(\mathbf{e}_3 \operatorname{rad}(T)) \simeq \operatorname{top}(\mathbf{e}_1 T).$$

Let $Q = \operatorname{rad}(T)^2$, $A = T/Q$ and $e_i = \mathbf{e}_i + Q$ in A. Then $Q^2 = 0$ in T and T is an extension over A with kernel Q, but not of course trivial. For, if $T = A \ltimes Q$, $\operatorname{rad}(T) = \operatorname{rad}(A) \oplus \Omega$ as additive groups and so $\operatorname{rad}(T)^3 = 0$, a contradiction. Further, A is a serial quasi-Frobenius algebra with a Nakayama permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

by ${}_A A_A$ and it is not difficult to see that

$$e_1 Q \simeq e_3 A, \qquad e_2 Q \simeq e_1 A, \qquad e_3 Q \simeq e_2 A,$$

$$Qe_1 \simeq Ae_2, \qquad Qe_2 \simeq Ae_3, \qquad Qe_3 \simeq Ae_1.$$

This shows that $Q_A \simeq A_A$ and ${}_A Q \simeq {}_A A$. Hence Q is a QF-module. On the other hand, since T is serial, by Nakayama's theorem [20, Theorem 17] all indecomposable right T-modules are factors of primitive right ideals in T. Using this fact, we can easily check that each property in (2.9, I) holds. Furthermore, clearly A is not hereditary but T is weakly symmetric.

Example 2.10 is generalized as follows.

PROPOSITION 2.11. *Let A be an Artinian ring and Q a* QF-*module and let $0 \to Q \to T \to A \to 0$ be an extension of A with kernel Q. Assume that T is a serial ring. Then for every indecomposable T-module M with $MQ \neq 0$ it holds that M/MQ is projective in* $\operatorname{mod} A$.

Proof. Let M be an indecomposable T-module with $MQ \neq 0$. By Nakayama's theorem there is an idempotent $\mathbf{e}$ such that $\mathbf{e}T \to^f M \to 0$ is exact. Since $f(\mathbf{e}Q) = MQ \neq 0$, it holds that $\mathbf{e}Q \not\subset \operatorname{Ker} f = \Omega_T(M)$. Hence $\Omega_T(M) \subsetneq \mathbf{e}Q$ because the set of right ideals of T is totally ordered, so that $\Omega_T(M)$ is annihilated by Q. Hence the conclusion follows from (2.4–2).

In conclusion of this section, we apply our theorems to the representation theory of Artinian rings. Especially properties (3) and (4) in (2.9, I) are remarkable, which will give a correspondence between indecomposable A-modules and indecomposable T-modules which are not annihilated by Q. To say it more explicitly, for an Artinian ring R we denote by $\operatorname{Ind} R$ the set of isomorphism classes of finitely generated indecomposable right R-modules. Now let A be an Artinian ring with a QF-module Q and T an extension over A with kernel Q. Then $\operatorname{Ind} T \setminus \operatorname{Ind} A$ is the set of finitely generated indecomposable right T-modules which are not annihilated by Q. We define a map Φ

from $\operatorname{Ind} A$ to $\operatorname{Ind} T \backslash \operatorname{Ind} A$ as follows. (i) $\Phi([M]) = [\Omega_T(M)]$ for a nonprojective indecomposable module M in $\operatorname{mod} A$ and (ii) $\Phi([M])$ is the isomorphism class of a projective cover of M in $\operatorname{mod} T$ for a projective module M in $\operatorname{mod} A$ (cf. (2.4)). Then we have

Theorem 2.12. *Let A be an Artinian ring with a* QF-*module Q and T an extension over A with kernel Q. Then it holds that*

(1) *Φ is an injection.*

(2) *Φ is bijective if and only if $\ell_M(Q)$ is injective in* $\operatorname{mod} A$ *for every indecomposable right T-module M with $MQ \neq 0$.*

(3) *If A is hereditary, Φ is bijective. Moreover, in case T is a trivial extension of A by Q, the converse holds.*

Proof. (1) Let $\mathbf{P}(A)$ be the set of isomorphism classes of indecomposable projective right A-modules. Then it is clear that Φ is injective on $\mathbf{P}(A)$ by (2.1) and on $\operatorname{Ind} A \backslash \mathbf{P}(A)$ by (1.3). Moreover, since T is quasi-Frobenius, it is clear that $\Omega_T(M)$ is nonprojective in $\operatorname{mod} T$ for $[M] \in \operatorname{Ind} A \backslash \mathbf{P}(A)$. Hence $\Phi(\mathbf{P}(A)) \cap \Phi(\operatorname{Ind} A \backslash \mathbf{P}(A)) = \varnothing$. This shows that Φ is injective on $\operatorname{Ind} A$.

(2) Φ is a bijection if and only if for $[M] \in \operatorname{Ind} T \backslash \operatorname{Ind} A$ there is an $[N] \in \operatorname{Ind} A$ such that $M \simeq \Omega_T(N)$, i.e., $[\Omega_T^{-1}(M)] \in \operatorname{Ind} A$, which means condition (4) in (2.9). Thus we are done.

(3) This is an immediate consequence of (2.9-II) and the above result (2).

Corollary 2.13. *Let A be an Artinian ring of finite representation type with a* QF-*module Q and T an extension over A with kernel Q. Then, if A is hereditary, T is of finite representation type and the number of isomorphism classes of indecomposable right T-modules is two times the number of isomorphism classes of indecomposable right A-modules. In case T is a trivial extension of A by Q, the converse holds.*

Proof. This is obvious from (2.12).

Example 2.14. In (2.12–3) the condition that T is a trivial extension of A by Q cannot be removed. For example, we consider the rings A and T given in (2.10). Then, since both T and A are serial, it is easy to see that $\mathbf{n}(T) = 12$ and $\mathbf{n}(A) = 6$ (see [20, Theorem 17]), where $\mathbf{n}$ means the number of isomorphism classes of indecomposable modules. But neither A is hereditary nor T is a trivial extension of A by Q.

Remark 2.15. In the case $T = A \ltimes Q$, (2.13) has already been proved in [25] as noted in the introduction.

3. Almost Split Sequences over a QF-ring

In this section we remark some results for almost split sequences, which will be used in the following sections. We are concerned with setting up a principle of construction of indecomposable modules over a quasi-Frobenius ring.

We first recall definitions from [2]. Let A be an Artinian ring. Let $\mathbb{E}: 0 \to X \to Y \to Z \to 0$ be a nonsplittable exact sequence in $\operatorname{mod} A$ with X and Z indecomposable modules. $\mathbb{E}$ is called *almost split* if any morphism $f: X \to W$ in $\operatorname{mod} A$ which is not a splittable monomorphism can be extended to Y, or equivalently any morphism $g: W \to Z$ in $\operatorname{mod} A$ which is not a splittable epimorphism can be factored through Y. A nonsplittable morphism $f: X \to Y$ in $\operatorname{mod} A$ is called *irreducible* provided that if $f = hg$ for some morphisms $g: X \to W$ and $h: W \to Y$ in $\operatorname{mod} A$, then g is a splittable monomorphism or h is a splittable epimorphism. The ring A is said to *have almost split sequences* if there are almost split sequences in $\operatorname{mod} A$

$$0 \to X \to X' \to X'' \to 0,$$
$$0 \to Y'' \to Y' \to Y \to 0$$

for any indecomposable noninjective module X and any indecomposable nonprojective module Y. For an almost split sequence $0 \to X \to Y \to Z \to 0$ we denote X by $\omega_A(Z)$ and Z by $\omega_A^{-1}(X)$. For an indecomposable module M, let

$$\omega_A^n(M) = \omega_A(\omega_A^{n-1}(M)),$$
$$\omega_A^{-n}(M) = \omega_A^{-1}(\omega_A^{-n+1}(M)) \qquad \text{for } n > 0$$

if the right hand sides are well defined. By the uniqueness of the almost split sequences, it holds that for any indecomposable module X, $X \simeq \omega_A^{-1}\omega_A(X)$ if X is not projective and $\omega_A(X)$ is defined, and $X \simeq \omega_A \omega_A^{-1}(X)$ if X is not injective and $\omega_A^{-1}(X)$ is defined. If there is no confusion, ω_A^n will be denoted by ω^n for short.

The following lemma is very fundamental and it is proved in [3]. (See (2.4), (2.15) and (3.1) in [3].)

Lemma 3.1. *Let A be an Artinian ring with almost split sequences. Then the following statements hold in* $\operatorname{mod} A$.

(1) *For indecomposable modules X and Z, a nonsplittable exact sequence $0 \to Z \to^{v} Y \to^{u} X \to 0$ is almost split if and only if both u and v are irreducible.*

(2) *For an indecomposable nonprojective module X and a nonzero*

morphism $u: Y \to X$ *in* $\operatorname{mod} A$, *u is irreducible if and only if there is an almost split sequence*

$$0 \to \omega(X) \to Y \oplus Y' \xrightarrow{f} X \to 0$$

such that $f = u \oplus u'$ *for some morphism* $u': Y' \to X$. *In this case, there is an irreducible morphism* $\omega(X) \to Y$.

(3) *For an indecomposable noninjective module X and a nonzero morphism* $u: X \to Y$ *in* $\operatorname{mod} A$, *u is irreducible if and only if there is an almost split sequence*

$$0 \to X \xrightarrow{f} Y \oplus Y' \to \omega^{-1}(X) \to 0$$

such that $f = u \oplus u'$ *for some morphism* $u': X \to Y'$. *In this case, there is an irreducible morphism* $Y \to \omega^{-1}(X)$.

(4) *For a nonzero morphism* $u: X \to Y$,

(i) *if X is injective, u is irreducible if and only if the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \| & & \downarrow{\scriptstyle m} \\ X & \xrightarrow{p} & X/\operatorname{soc}(X), \end{array}$$

where p is the canonical epimorpohism and m is a splittable monomorphism.

(ii) *if Y is projective, u is irreducible if and only if the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle m}\downarrow & & \| \\ \operatorname{rad}(Y) & \xrightarrow{k} & Y, \end{array}$$

where k is the inclusion and m a splittable monomorphism.

The following is a special case of [3, 4.11].

LEMMA 3.2. *Let A be a quasi-Frobenius ring and P an indecomposable projective A-module. Then the exact sequence*

$$\mathbb{E}: 0 \to \operatorname{rad}(P) \xrightarrow{u} P \oplus (\operatorname{rad}(P)/\operatorname{soc}(P)) \xrightarrow{v} P/\operatorname{soc}(P) \to 0$$

is almost split, where $u(x) = (x, x + \mathrm{soc}(P))$ *and* $v(y, z + \mathrm{soc}(P)) = (y - z) + \mathrm{soc}(P)$ *for* $x, z \in \mathrm{rad}(P)$ *and* $y \in P$. *Moreover,* $\mathrm{rad}(P)/\mathrm{soc}(P)$ *does not contain a projective submodule.*

For a module M over an Artinian ring A, $\mathbf{L}(M)$ denotes the upper Loewy length, that is, it is the smallest number n such that $M\,\mathrm{rad}(A)^n = 0$. Let A be an indecomposable quasi-Frobenius ring with $\mathbf{L}(A) = 2$. Then it is well known that A is serial and so nonprojective indecomposable modules are simple. This is a trivial case. The following lemma gives information for a quasi-Frobenius ring A with $\mathbf{L}(A) > 2$ [5, Lemma 4.3].

LEMMA 3.3. *Let* A *be an indecomposable quasi-Frobenius ring with* $\mathbf{L}(A) \geqslant 3$. *Then* $\mathbf{L}(P) \geqslant 3$ *for every indecomposable projective module* P.

The construction of indecomposable modules considered in this paper is closely connected with irreducible morphisms between indecomposable modules. For an Artinian ring A with almost split sequences, a set $\{M_i\}_{i \in I}$ of finitely generated indecomposable modules is called an *ω-generating set* for a given class $\mathbf{C}$ of finitely generated indecomposable modules provided that for any module M belonging to $\mathbf{C}$ there is some M_i such that $M \simeq \omega^n(M_i)$ for an integer n. Further the set $\{M_i\}_{i \in I}$ is called an *ω-basis* [16] in case that it is an ω-generating set for the class of all finitely generated indecomposable modules and that it is not ω-generated by any proper subset. It is the problem that we find an ω-basis for an Artinian ring of finite representation type. As example, ω-bases are known for those Artinian rings of finite representation type that are serial rings [3, 20], algebras with squared zero radical [19], hereditary algebras [12] and algebras of local–colocal representation type [16]. For the others, see [9, 13].

Before stating the main result in this section, we recall notation from [27]. Let A be an Artinian ring with almost split sequences. For an indecomposable module M, $\mathbf{E}_n(M)$ is defined for any integer $n \geqslant 0$ as follows (for $n \leqslant 0$ it is similarly defined);

(i) $\mathbf{E}_0(M) = \{[M]\}$,

(ii) $[X] \in \mathbf{E}_{n+1}(M)$ if and only if X is indecomposable and there is an irreducible morphism $X \to Y$ for some $[Y] \in \mathbf{E}_n(M)$, where $[\ ]$ denotes the isomorphism class of a given module. Let $\bar{\mathbf{E}}_n(M) = \{[X] \mid [X] \in \mathbf{E}_n(M)$ and X is not projective or injective$\}$. For two sets $\mathbf{E}_0$, $\mathbf{E}_1$ of isomorphism classes of indecomposable modules, the ordered pair $(\mathbf{E}_1, \mathbf{E}_0)$ is called *reflexive* provided, for any irreducible morphism $X_1 \to X_0$ between nonprojective, noninjective and indecomposable modules X_i, $[X_0] \in \mathbf{E}_0$ if and only if $[X_1] \in \mathbf{E}_1$.

PROPOSITION 3.4. *Let* A *be an indecomposable quasi-Frobenius ring with almost split sequences and* $\mathbf{L}(A) \geqslant 3$. *Let* $\bar{\mathbf{E}}_0$ *be a set of isomorphism classes* *of indecomposable nonprojective A-modules* M_i $(i \in I)$, *and let* $\bar{\mathbf{E}}_n = \bigcup_{i \in I} \bar{\mathbf{E}}_n(M_i)$ *for any integer n. If* $(\bar{\mathbf{E}}_1, \bar{\mathbf{E}}_0)$ *is a reflexive pair, then it holds that*

(1) $(\bar{\mathbf{E}}_n, \bar{\mathbf{E}}_{n-1})$ *is reflexive for any integer n.*

(2) $\bar{\mathbf{E}}_0 \cup \bar{\mathbf{E}}_1$ *is an* ω*-generating set for* $\bigcup_{n \in \mathbb{Z}} \bar{\mathbf{E}}_n$.

Proof. (a) We will note that every almost split sequence has a nonprojective middle term. For this, suppose that $0 \to \omega(X) \to Y \to X \to 0$ is an almost split sequence such that Y has an indecomposable projective summand P. Then $\mathbf{L}(P) \geqslant 3$ by (3.3), hence $\mathrm{rad}(P)/\mathrm{soc}(P) \neq 0$. Moreover there is an irreducible morphism $P \to X$ by (3.1). Again by (3.1), since P is injective, X must be isomorphic to a direct summand of $P/\mathrm{soc}(P)$. However, since P has the simple top, $P/\mathrm{soc}(P)$ is indecomposable. Hence X is isomorphic to $P/\mathrm{soc}(P)$. Therefore, from the uniqueness of almost split sequences and (3.2), we know that $Y \simeq P \oplus \mathrm{rad}(P)/\mathrm{soc}(P)$ and $\mathrm{rad}(P)/\mathrm{soc}(P)$ is not projective, so Y is not projective. It follows from this fact that if $\bar{\mathbf{E}}_n$ is not empty, then neither $\bar{\mathbf{E}}_{n+1}$ nor $\bar{\mathbf{E}}_{n-1}$ is empty. Hence, in fact, it holds that $\bar{\mathbf{E}}_n$ is not empty for every integer n, because $\bar{\mathbf{E}}_0 \neq \varnothing$ by assumption.

(b) We will show the following properties by induction on $n \geqslant 1$.

$(1)_n$: $(\bar{\mathbf{E}}_n, \bar{\mathbf{E}}_{n-1})$ is reflexive.

$(2)_n$: $\bar{\mathbf{E}}_n$ has $\bar{\mathbf{E}}_0 \cup \bar{\mathbf{E}}_1$ as an ω-generating set.

For this, first of all we will observe that for every $n \geqslant 1$ and $[X] \in \bar{\mathbf{E}}_{n+1}$ there exists $[Y_2] \in \bar{\mathbf{E}}_n$ with an irreducible morphism $X \to Y_2$, i.e., $[X] \in \mathbf{E}_1(Y_2)$. By the definition of $\bar{\mathbf{E}}_i$, there is $[Y] \in \mathbf{E}_n$ with $[X] \in \mathbf{E}_1(Y)$. Here, if Y is nonprojective, there is nothing to prove. Now then, assume that Y is projective. By definition there is $[Y_3] \in \mathbf{E}_{n-1}$ with $[Y] \in \mathbf{E}_1(Y_3)$. Since Y is indecomposable projective, it follows from (3.2) that there is an almost split sequence $0 \to X \to Y \oplus Y' \to Y_3 \to 0$ (cf. the proof of (a)). In particular, Y_3 is nonprojective and Y' has a nonprojective indecomposable summand Y_2. Hence $[Y_3] \in \bar{\mathbf{E}}_{n-1}$ and so $[Y_2] \in \bar{\mathbf{E}}_n$. Moreover, $[X] \in \mathbf{E}_1(Y_2)$ clearly.

Now, $(1)_1$, $(2)_0$ and $(2)_1$ hold clearly. Assume that assertions $(1)_k$ and $(2)_k$ are valid for all integers k such that $0 < k \leqslant n$. Let $[X] \in \bar{\mathbf{E}}_{n+1}$ and Y_1 be an indecomposable nonprojective module with $[X] \in \mathbf{E}_1(Y)$. From the above observation there exists $[Y_2] \in \bar{\mathbf{E}}_n$ with $[X] \in \mathbf{E}_1(Y_2)$. Since A is quasi-Frobenius, X is noninjective and so there is an irreducible morphism $Y_i \to \omega^{-1}(X)$ for $i = 1, 2$ by (3.1). Then by the induction hypothesis $(1)_n$ we have that $[\omega^{-1}(X)] \in \bar{\mathbf{E}}_{n-1}$, because $[Y_2] \in \bar{\mathbf{E}}_n$ and $\omega^{-1}(X)$ is nonprojective. Hence $[Y_1] \in \bar{\mathbf{E}}_n$, because Y_1 is nonprojective. This shows that $(1)_{n+1}$ holds. On the other hand, by the induction hypothesis $(2)_{n-1}$, there exists $[M]$ in $\bar{\mathbf{E}}_0 \cup \bar{\mathbf{E}}_1$ such that $\omega^{-1}(X) \simeq \omega^m(M)$ for some integer m and hence $X \simeq \omega(\omega^{-1}(X)) \simeq \omega^{m+1}(M)$. This shows that $(2)_{n+1}$ holds.

For $n \leqslant 1$ we can similarly prove $(1)_n$ and $(2)_n$.

4. Construction of Indecomposable Modules

The aim of this section is to find an ω-basis for every extension over a hereditary Artinian ring of finite representation type with kernel Q a QF-module. First we study some relations between irreducible morphisms in $\operatorname{mod} A$ and in $\operatorname{mod} T$.

THEOREM 4.1. *Let A be an Artinian ring with a* QF-*module Q and T an extension over A with kernel Q, and consider the following properties:*

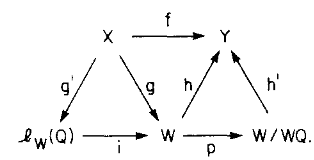
(1) *$\ell_M(Q)$ is injective in* $\operatorname{mod} A$ *for every finitely generated indecomposable T-module M with $MQ \neq 0$.*

(2) *Every irreducible morphism between indecomposable modules in* $\operatorname{mod} A$ *is irreducible in* $\operatorname{mod} T$.

(3) *Every almost split sequence in* $\operatorname{mod} A$ *is almost split in* $\operatorname{mod} T$.

Then the implications $(1) \Rightarrow (2) \Rightarrow (3)$ *always hold. Moreover if A has almost split sequences, then the implication* $(3) \Rightarrow (1)$ *holds.*

Proof. $(1) \Rightarrow (2)$. Suppose for a contradiction, that $f: X \to Y$ is irreducible in $\operatorname{mod} A$, but that $f = hg$ in $\operatorname{mod} T$, where $g: X \to W$ is not a splittable monomorphism and $h: W \to Y$ is not a splittable epimorphism. Since $XQ = 0$ in $\operatorname{mod} T$, $g(X) \subset \ell_W(Q)$, so that $g = ig'$, where $g': X \to \ell_W(Q)$, and $i: \ell_W(Q) \to W$ is the inclusion. Similarly, since $h(WQ) = 0$, we have that $h = h'p$, where $p: W \to W/WQ$ is the projection and $h': W/WQ \to Y$.



Now $f = (hi)\, g'$ in $\operatorname{mod} A$, so the irreducibility in $\operatorname{mod} A$ of f and the assumption that h is not a splittable epimorphism imply that $g': X \to \ell_W(Q)$ is a splittable monomorphism. Similarly, the factorisation $f = h'(pg)$ implies that $h': W/WQ \to Y$ is a splittable epimorphism. Next we will show that X is injective and Y is projective in $\operatorname{mod} A$, which of course contradicts the irreducibility of f (cf. (3.1)).

Now let $W = W_1 \oplus W_2$, where W_1 has no direct summands annihilated by Q and $W_2 Q = 0$. Then

$$\ell_W(Q) = \ell_{W_1}(Q) \oplus W_2$$

and $\ell_{W_1}(Q)$ is injective by assumption. Let

$$\ell_W(Q) = g(X) \oplus W'$$

for some module W'. Since $g(X)$ has the exchange property (cf. [26]), we have that

$$\ell_W(Q) = g(X) \oplus W_1' \oplus W_2'$$

for some $W_1' \subseteq \ell_{W_1}(Q)$ and $W_2' \subseteq W_2$. If X is not injective in mod A, it must be that $W_1' = \ell_{W_1}(Q)$. On the other hand, clearly $\operatorname{soc}(W_1) \subset \ell_{W_1}(Q)$. Therefore $(g(X) \oplus W_2') \cap W_1 = 0$. Consequently, we have that

$$\begin{aligned} W = W_1 + \ell_W(Q) &= W_1 + (g(X) \oplus \ell_{W_1}(Q) \oplus W_2') \\ &= W_1 \oplus g(X) \oplus W_2'. \end{aligned}$$

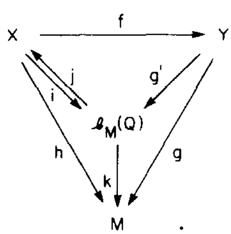
This shows that g is a splittable monomorphism, which contradicts the assumption for g. Hence X is injective in mod A. Moreover, since $W/WQ = (W_1/W_1Q) \oplus W_2$ and W_1/W_1Q is projective in mod A by (2.9), it will be shown that Y is projective in mod A by the similar argument. Thus we conclude the proof.

(2) $\Rightarrow$ (3). This is an immediate consequence of (3.1–1).

(3) $\Rightarrow$ (1). Let M be an indecomposable T-module with $MQ \neq 0$, and suppose that $\ell_M(Q)$ is not injective in mod A. Let X be a noninjective indecomposable summand of $\ell_M(Q)$ and

$$\mathbb{E}: 0 \to X \xrightarrow{f} Y \to \omega_A^{-1}(X) \to 0$$

an almost split sequence in mod A. Then by assumption, $\mathbb{E}$ is also almost split in mod T. Hence, for an inclusion $h: X \to M$, there is a morphism $g: Y \to M$ such that $h = gf$. Since $g(Y)Q = 0$, g induces a morphism $g': Y \to \ell_M(Q)$ such that $g = kg'$, where $k: \ell_M(Q) \to M$ the inclusion. Let $i: X \to \ell_M(Q)$ and $j: \ell_M(Q) \to X$ be the canonical injection and projection, respectively. Thus the following diagram is commutative:



Since $ki = h = gf = k(g'f)$, $i = g'f$. Hence $1_X = ji = (jg')f$. This shows that f is a splittable monomorphism, a contradiction.

Combining (4.1) with (2.9), we have

COROLLARY 4.2. *Let A be an Artinian ring with a* QF-*module Q and T an extension over A with kernel Q. Then*

(1) *If A is hereditary, then all properties in* (4.1) *hold.*

(2) *If A has almost split sequences and T is a trivial extension of A by Q, then each property in* (4.1) *implies that A is hereditary.*

In the following we assume that A is a *hereditary* Artinian ring with a QF-module Q and T an extension over A with kernel Q.

LEMMA 4.3. *For an indecomposable projective T-module P, it holds that*

(1) *If* $\operatorname{top}(P)$ *is an injective A-module, then PQ is a simple injective A-module.*

(2) *If* $\operatorname{soc}(P)$ *is a projective A-module, then P/PQ is a simple projective A-module.*

Proof. We prove only (1), then (2) will be obtained by the dual argument. Let K be a submodule of PQ such that PQ/K is simple and let $S = PQ/K$. We must show that $K = 0$. Since A is hereditary, S is an injective A-module by (2.1). Let P' be an injective hull of S in $\operatorname{mod} T$. Then since $P'Q$ is injective indecomposable in $\operatorname{mod} A$ (2.1), $S = P'Q$ and hence P'/S is a projective A-module. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & PQ/K & \xrightarrow{u} & P/K & \xrightarrow{v} & P/PQ & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & S & \xrightarrow{u'} & P' & \xrightarrow{v'} & P'/S & \longrightarrow & 0, \end{array}$$

where u, v, u' and v' are canonical, and f, g are morphisms induced from the injectivity of P'_T. It then holds that $f(P/K) \supsetneqq u'(S)$. For, suppose that $f(P/K) = u'(S)$. Then it is clear that u is a splittable monomorphism, that is, in $\operatorname{mod} T$

$$P/K \simeq PQ/K \oplus P/PQ.$$

But the right hand side is annihilated by Q, so that $(P/K)\,Q = 0$. Hence $PQ \subset K$, which contradicts the choice of K. Consequently, $v'f \neq 0$ and hence $g \neq 0$ by the commutativity of the above diagram. Since A is hereditary and both P/PQ and P'/S are indecomposable projective A-modules, g is then a monomorphism. Hence g induces a monomorphism $\bar{g}\colon \operatorname{top}(P) \to (P'/S)/g(\operatorname{rad}(P)/PQ)$ in $\operatorname{mod} A$. Since $\operatorname{top}(P)$ is injective in $\operatorname{mod} A$, $\bar{g}$ is splittable, and so $\bar{g}$ must be an isomorphism because $(P'/S)/g(\operatorname{rad}(P)/PQ)$ is indecomposable. This implies that g is an isomorphism. Thus f is an isomorphism. Since P' is projective in $\operatorname{mod} T$, P/K is aso projective in $\operatorname{mod} T$. Since P is indecomposable, this implies that $K = 0$.

LEMMA 4.4. *Let eA be an indecomposble projective A-module such that* top(eA) *is injective in* mod A, *and let* $f: M \to eA$ *be an irreducible morphism in* mod T *with M an indecomposable T-module. Then either*

(1) $MQ = 0$ *and f is irreducible in* mod A, *or*

(2) *M is projective in* mod T *such that f induces an isomorphism from* M/MQ *to eA.*

Proof. If $MQ = 0$, then M is an A-module. Hence it is clear from the definition of irreducible morphism that f is irreducible in mod A. Assume that $MQ \neq 0$. Then f is not monomorphic, because $(eA)Q = 0$ regarding eA as a T-module. Therefore f is epimorphic by the definition of irreducible morphism. Let P be an indecomposable projective T-module with $P/PQ \simeq eA$. Then there is a morphism $g: P \to M$ such that $p = fg$, where p is the canonical morphism $P \to eA$. By (4.3–1), $PQ = \mathrm{soc}(P)$. Since P is injective, it follows that $p: P \to P/\mathrm{soc}(P) \simeq eA$ is irreducible, and then clearly g is a splittable monomorphism. Since M is indecomposable, g is an isomorphism and (2) follows at once.

Here we will recall the definition of the quiver of A from [12]. Let Γ be a set in a one-to-one correspondence with the set of isomorphism classes of indecomposable projective right A-modules, and we denote by P_v an indecomposable projective module corresponding to a vertex $v \in \Gamma$. For $v_1, v_2 \in \Gamma$, d_{v_1,v_2} denotes the multiplicity of top(P_{v_2}) in composition factors of top(P_{v_1} rad(A)), and d'_{v_1,v_2} is the multiplicity of top($P^*_{v_2}$) in composition factors of top(rad(A) $P^*_{v_1}$), where $(\ )^* = \mathrm{Hom}_A(\ , A)$. Then the *quiver* $\mathscr{Q}(A)$ of A means the set $(\Gamma, \mathbf{d})$, where $\mathbf{d}$ stands for the set of d_{v_i,v_j} for $v_i, v_j \in \Gamma$. We use the symbols

$$\underset{v_i}{\circ} \xrightarrow{(d_{v_i,v_j},\, d'_{v_j,v_i})} \underset{v_j}{\circ} \qquad \text{for } d_{v_i,v_j} \neq 0,$$

and

$$\underset{v_i}{\circ} \longrightarrow \underset{v_j}{\circ} \qquad \text{simply for } d_{v_i,v_j} = d'_{v_j,v_i} = 1.$$

That is, the quiver means the *oriented valued graph* here.

The following proposition shows fundamental relations between the quiver of a hereditary Artinian ring and almost split sequences in mod T.

PROPOSITION 4.5. *Let A be a hereditary Artinian ring with a* QF-*module Q, T an extension over A with kernel Q and* $(\Gamma, \mathbf{d})$ *the quiver of A. Assume that both A and T have almost split sequences. Then for any vertex* $v \in \Gamma$, $\omega_T^{-1}(P_{v'}) \simeq \omega_A^{-1}(P_{v'})$ for $d_{v,v'} \neq 0$, *and there is an almost split sequence* $\mathbb{E}_{n,v}$ *in* mod T *for any integer n such that*

$$0 \to \omega_T^n(P_v) \to \left(\bigoplus_{d_{v,v'} \neq 0} \omega_T^{n-1}(P_{v'})^{(\ell_{v'})} \right) \oplus \left(\bigoplus_{d_{v'',v} \neq 0} \omega_T^n(P_{v''})^{(\ell_{v''})} \right) \oplus P$$
$$\to \omega_T^{n-1}(P_v) \to 0$$

for some integers $\ell_{v'}, \ell_{v''}$ (>0), *where* P *is zero or indecomposable projective right* T*-module. In particular,*

(1) *if* $\operatorname{top}(P_v)$ *is noninjective in* $\operatorname{mod} A$, $\omega_T^{-1}(P_v) \simeq \omega_A^{-1}(P_v)$ *and* $\mathbb{E}_{0,v}$ *is*

$$0 \to P_v \to \left(\bigoplus_{d_{v,v'} \neq 0} \omega_A^{-1}(P_{v'})^{(\ell_{v'})} \right) \oplus \left(\bigoplus_{d_{v'',v} \neq 0} (P_{v''})^{(\ell_{v''})} \right) \to \omega_A^{-1}(P_v) \to 0$$

(2) *if* $\operatorname{top}(P_v)$ *is injective in* $\operatorname{mod} A$, $\mathbb{E}_{0,v}$ *is*

$$0 \to P_v \to \bigoplus_{d_{v,v'} \neq 0} \omega_A^{-1}(P_{v'})^{(\ell_{v'})} \oplus P_1 \to \omega_T^{-1}(P_v) \to 0,$$

and there is an almost split sequence in $\operatorname{mod} T$ *such that*

$$0 \to \operatorname{rad}(P_2) \to \bigoplus_{d_{v,v'} \neq 0} (P_{v'})^{(\ell'_{v'})} \oplus P_2 \to P_v \to 0,$$

where P_1 *is zero or indecomposable projective in* $\operatorname{mod} T$ *and* P_2 *is indecomposable projective in* $\operatorname{mod} T$ *such that* $P_2/P_2 Q \simeq P_v$.

Remark. If A is a hereditary Artin algebra, it will be seen from the proof and [4, Sect. 2] that

$$\ell_{v'} = \ell'_{v'} = d_{v,v'} \qquad \text{and} \qquad \ell_{v''} = d'_{v,v''}.$$

Proof. It should be noted that if $0 \to X \to Y \to Z \to 0$ is an almost split sequence in $\operatorname{mod} T$ such that Y has a projective summand P, then P is indecomposable by (3.2).

(1) *The case* $\operatorname{top}(P_v)$ *is noninjective in* $\operatorname{mod} A$: Let X be an indecomposable nonprojective A-module. Then, since P_v is noninjective in $\operatorname{mod} A$, by (3.1) there is an irreducible morphism $P_v \to X$ in $\operatorname{mod} A$ iff there is an irreducible morphism $\omega_A(X) \to P_v$ in $\operatorname{mod} A$ iff there exists a vertex $v' \in V$ such that $\omega_A(X) \simeq P_{v'}$ and $d_{v,v'} \neq 0$, or equivalently, $X \simeq \omega_A^{-1}(P_{v'})$. On the other hand, in case X is indecomposable projective in $\operatorname{mod} A$, again by (3.1) there is an irreducible morphism $P_v \to X$ in $\operatorname{mod} A$ iff there is a vertex $v'' \in \Gamma$ such that $X \simeq P_{v''}$ and $d_{v'',v} \neq 0$. Thus we have an almost split sequence $\mathbb{E}_{0,v}$ in $\operatorname{mod} A$ such that for some integers $\ell_{v'}, \ell_{v''} > 0$,

$$0 \to P_v \to \left(\bigoplus_{d_{v,v'} \neq 0} \omega_A^{-1}(P_{v'})^{(\ell_{v'})} \right) \oplus \left(\bigoplus_{d_{v'',v} \neq 0} (P_{v''})^{(\ell_{v''})} \right) \to \omega_A^{-1}(P_v) \to 0. \qquad \text{(i)}$$

By (4.2) we know that $\mathbb{E}_{0,v}$ is also almost split in $\operatorname{mod} T$, so that $\omega_A^{-1}(P_v) \simeq \omega_T^{-1}(P_v)$ in particular. Furthermore it follows from (4.2) that $\omega_A^{-1}(P_{v'}) \simeq \omega_T^{-1}(P_{v'})$, because $P_{v'}$ is not injective in $\operatorname{mod} A$.

(2) *The case* $\operatorname{top}(P_v)$ *is injective in* $\operatorname{mod} A$: By (3.1), for a nonprojective indecomposable T-module X, there is an irreducible morphism $P_v \to X$ in mod T iff there is an irreducible morphism $\omega_T(X) \to P_v$ in mod T. Since X is nonprojective in mod T, $\omega_T(X)$ is nonprojective in mod T. Hence, if there is an irreducible morphism $f\colon \omega_T(X) \to P_v$ in mod T, $\omega_T(X)\,Q = 0$ by (4.4) and f is irreducible in $\operatorname{mod} A$, so that $\omega_T(X) \simeq P_{v'}$ for some $v' \in V$ with $d_{v,v'} \neq 0$, or equivalently $X \simeq \omega_T^{-1}(P_{v'})$.

Conversely assume that $f\colon P_{v'} \to P_v$ is irreducible in $\operatorname{mod} A$ for $v' \in \Gamma$ with $d_{v,v'} \neq 0$. Then f is also irreducible in mod T by (4.2). Hence by (3.1) there is an irreducible morphism $P_v \to \omega_T^{-1}(P_{v'})$ in mod T. Therefore, by (3.1), $\omega_T^{-1}(P_{v'})$ appears in the middle term of the almost split sequence

$$0 \to P_v \to M \to \omega_T^{-1}(P_v) \to 0$$

in mod T. Thus we know that there is an almost split sequence $\mathbb{E}_{0,v}$ in mod T such that

$$0 \to P_v \to \left(\bigoplus_{d_{v,v'} \neq 0} \omega_T^{-1}(P_{v'})^{(\ell_{v'})} \right) \oplus P_1 \to \omega_T^{-1}(P_v) \to 0 \qquad \text{(ii)}$$

for some integer $\ell_{v'} > 0$, where P_1 is zero or indecomposable projective in mod T. Moreover, by the same reason as in (1) it holds that $\omega_T^{-1}(P_{v'}) \simeq \omega_A^{-1}(P_{v'})$ for $d_{v,v'} \neq 0$.

Next we consider the second sequence. Let P_2 be a projective cover of P_v in mod T. Then $P_v \simeq P_2/P_2Q$ by (2.3). Since $\operatorname{top}(P_2) \simeq \operatorname{top}(P_v)$ and $\operatorname{top}(P_v)$ is injective in $\operatorname{mod} A$, it follows from (4.4) that $P_2Q = \operatorname{soc}(P_2)$ and hence $P_v \simeq P_2/\operatorname{soc}(P_2)$. Hence there is an almost split sequence in mod T such that

$$0 \to \operatorname{rad}(P_2) \to P_2 \oplus (\operatorname{rad}(P_2)/\operatorname{soc}(P_2)) \to P_v \to 0.$$

Further, since no indecomposable direct summand X of $\operatorname{rad}(P_2)/\operatorname{soc}(P_2)$ is projective in mod T by (3.2) but there is an irreducible morphism $X \to P_v$, by (4.4) it holds that $XQ = 0$ and $X \simeq P_{v'}$ for some $v' \in \Gamma$ with $d_{v,v'} \neq 0$ in view of (3.1). Conversely we easily see that for any $v' \in \Gamma$ with $d_{v,v'} \neq 0$, $P_{v'}$ appears in $\operatorname{rad}(P_2)/\operatorname{soc}(P_2)$ as a direct summand by (3.1) and (4.2). Thus we have the almost split sequence in mod T

$$0 \to \operatorname{rad}(P_2) \to \left(\bigoplus_{d_{v,v'} \neq 0} (P_{v'})^{(\ell'_{v'})} \right) \oplus P_2 \to P_v \to 0.$$

The rest immediately follows from (i) and (ii), in view of (3.1) and (3.2).

To show that a given set of isomorphism classes of indecomposable modules is an ω-basis, it is convenient to define "a distance ∂ from sources" for every vertex. Let A be a hereditary Artinian ring and $\mathscr{Q}(A) = (\Gamma, \mathbf{d})$ the quiver of A. We assume that $\mathscr{Q}(A)$ does not contain any quiver with cyclic orientation. Let $\{v^{(i)} \mid v^{(i)} \in \Gamma, 1 \leqslant i \leqslant s\}$ be the set of sources in $\mathscr{Q}(A)$ and, for $v \in \Gamma$, $\Gamma_i(v)$ be the set of vertices such that $v' \in \Gamma_i(v)$ if and only if there are arrows with $v - \cdots - v' - \cdots - v^{(i)}$, where $-$ denotes $\to$ or $\leftarrow$. Now we define a mapping

$$\partial_v^i \colon \Gamma_i(v) \times \Gamma_i(v) \to \mathbb{Z}, \qquad 1 \leqslant i \leqslant s,\ v \in \Gamma,$$

as follows; for $v_1, v_2 \in \Gamma_i(v)$

(1) *The case* $d_{v_1, v_2} \neq 0$:

$$\partial_v^i(v_1, v_2) = 1 \qquad \text{iff there are arrows } v^{(i)} - \cdots - v_1 \to v_2 - \cdots - v,$$

$$\partial_v^i(v_1, v_2) = -1 \qquad \text{iff there are arrows } v^{(i)} - \cdots - v_1 \leftarrow v_2 - \cdots - v.$$

(2) *The case* $d_{v_1, v_2} = 0$: we set $\partial_v^i(v_1, v_2) = 0$.

Let for $v \in \Gamma$

$$\partial^i(v) = \sum_{v_1, v_2 \in \Gamma_i(v)} \partial_v^i(v_1, v_2)$$

and

$$\partial(v) = \mathrm{Max}\{\partial^i(v) \mid 1 \leqslant i \leqslant s\}.$$

Then ∂ defines a mapping

$$\partial \colon \Gamma \to \mathbb{N},$$

where $\mathbb{N}$ denotes the set of nonnegative integers. For, $\partial^i(v^{(i)}) = 0$ for any $1 \leqslant i \leqslant s$, and for v a nonsource there is i such that there exist arrows with $v^{(i)} \to \cdots \to v$. Hence $\partial^i(v) > 0$ by definition, and hence we have that $\partial(v) > 0$ for all nonsources v in Γ.

LEMMA 4.6. *For any vertices $v_1, v_2 \in \Gamma$ such that $d_{v_1, v_2} \neq 0$, it holds that $\partial(v_2) - \partial(v_1) = 1$.*

Proof. Since $d_{v_1, v_2} \neq 0$, there is an arrow $v_1 \to v_2$. For the arrow there are two possible cases such that for a source $v^{(i)}$

$$v^{(i)} - \cdots - v_1 \to v_2 \qquad \text{or} \qquad v^{(i)} - \cdots - v_2 \leftarrow v_1.$$

However, in any case, it is easily seen from the definition that $\partial^i(v_2) - 1 = \partial^i(v_1)$. Hence it holds that $\partial^i(v_2) - \partial^i(v_1) = 1$ for any source $v^{(i)}$ such that $v^{(i)} - \cdots - v_j$ for $j = 1, 2$. Thus we have that $\partial(v_2) - \partial(v_1) = 1$.

For any integer $n \geqslant 0$ and vertex $v \in \Gamma$, we set

$$\begin{aligned} n_v &= \tfrac{1}{2}\partial(v) + n && \text{if} \quad \partial(v) \text{ is even,} \\ &= \tfrac{1}{2}(\partial(v) - 1) + n && \text{if} \quad \partial(v) \text{ is odd.} \end{aligned}$$

Then we have

LEMMA 4.7. *Assume that both A and T have almost split sequences. Let X and Y be nonprojective indecomposable T-modules and assume that there is an irreducible morphism $Y \to X$. Then for any integer $n \geqslant 0$,*

(I) *The following statements are equivalent.*

(1) *$X \simeq \omega_T^{-n_v}(P_v)$ for some $v \in \Gamma$ such that $\partial(v)$ is even.*

(2) *$Y \simeq \omega_T^{-n_{v'}}(P_{v'})$ for some $v' \in \Gamma$ such that $\partial(v')$ is odd.*

(II) *Let*

$$\bar{\mathbf{E}}_0 = \{[\omega_T^{-n_v}(P_v)] \mid v \in \Gamma, \partial(v) \textit{ is even}\}$$

and

$$\bar{\mathbf{E}}_1 = \{[\omega_T^{-n_v}(P_v)] \mid v \in \Gamma, \partial(v) \textit{ is odd}\}.$$

Then $(\bar{\mathbf{E}}_1, \bar{\mathbf{E}}_0)$ is reflexive.

Proof. Assertion (II) is obvious from (I).

(I) $(1) \Rightarrow (2)$. By (3.1) there is an almost split sequence in mod T

$$0 \to \omega_T^{-n_v+1}(P_v) \to Y \oplus Y' \to \omega_T^{-n_v}(P_v) \to 0,$$

where Y' is a T-module. It follows from (4.5) that either

$$\text{(i)} \qquad Y \simeq \omega_T^{-n_v}(P_{v'}) \qquad \text{for } d_{v,v'} \neq 0,$$

or

$$\text{(ii)} \qquad Y \simeq \omega_T^{-n_v+1}(P_{v''}) \qquad \text{for } d_{v'',v} \neq 0.$$

First we assume case (i). Then, by (4.6), $\partial(v') - \partial(v) = 1$ and so $\partial(v')$ is odd. Hence we have

$$n_{v'} = \tfrac{1}{2}(\partial(v') - 1) + n = \tfrac{1}{2}\partial(v) + n = n_v.$$

As a consequence, it holds that $Y \simeq \omega_T^{-n_{v'}}(P_{v'})$ for $\partial(v')$ odd.

Next we assume case (ii). By the same way as in the proof of (i), we then know that $\partial(v'')$ is odd and $n_{v''} = n_v - 1$. Hence it holds that $Y \simeq \omega_R^{-n_{v''}}(P_{v''})$ for $\partial(v'')$ odd.

$(2) \Rightarrow (1)$. By (3.1) there is an almost split sequence in mod T such that

$$0 \to \omega_T^{-n_{v'}}(P_{v'}) \to X \oplus X' \to \omega_T^{-n_{v'}-1}(P_{v'}) \to 0,$$

where X' is some T-module. Using this fact the implication $(2) \Rightarrow (1)$ is proved by the same argument as in the $(1) \Rightarrow (2)$, and we omit the rest.

Now we can prove the main theorem on a construction of indecomposable T-modules (cf. (5.3)).

THEOREM 4.8. *Let A be a hereditary Artinian ring with a QF-module Q and T an extension over A with kernel Q. Assume that A is of finite representation type. Then the set of nonisomorphic indecomposable projectie A-modules and nonisomorphic indecomposable projective T-modules is an ω_T-basis. Similarly the set of nonisomorphic indecomposable injective A-modules and nonisomorphic indecomposable injective T-modules is also an ω_T-basis.*

Proof. We my assume, without loss of generality, that A is an indecomposable ring. Since A is hereditary, T is also of finite representation type by (2.12). Hence, in particular, T has almost split sequences. If $\mathbf{L}(A) = 1$, A is semi-simple Artinian, so that T is clearly serial quasi-Frobenius ring with $\mathbf{L}(T) = 2$. Therefore nonprojective indecomposable T-modules are nothing but simple A-modules. Hence the assertions for this case are trivial. Now assume that $\mathbf{L}(A) \geqslant 2$. Then $\mathbf{L}(T) \geqslant 3$ clearly. Hence we can apply (3.4) for the ring T.

Let $\mathscr{Q}(A) = (\Gamma, \mathbf{d})$ be the quiver of A. Then $\mathrm{top}(P_v)$ denotes an arbitrary simple A-module for $v \in \Gamma$. On the other hand, from the proof given in [27, Theorem 1] we know that for any $v \in \Gamma$, there is a sequence of arrows in mod A such that

$$P_v \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \to X_{m(v)-1} \xrightarrow{f_{m(v)}} \mathrm{top}(P_v), \tag{1}$$

where each f_i denotes an irreducible morphism and X_i an indecomposable A-module. Let n be an arbitrary integer greater than $m(v)$ for every $v \in \Gamma$, and with the notations given in (4.7) before we put

$$\bar{\mathbf{E}}_0 = \{[\omega_T^{-n_v}(P_v)] \mid \partial(v) \text{ is even}\},$$

$$\bar{\mathbf{E}}_i = \bigcup \{\bar{\mathbf{E}}_i(X) \mid [X] \in \bar{\mathbf{E}}_0\}, \qquad \text{for } i \in \mathbb{Z}, \tag{2}$$

where each $\bar{\mathbf{E}}_i$ is taken in mod T. Then, it is easily seen from (4.7) that

$$\bar{\mathbf{E}}_1 = \{[\omega_T^{-n_v}(P_v)] \mid \partial(v) \text{ is odd}\} \tag{3}$$

and the ordered pair $(\bar{\mathbf{E}}_1, \bar{\mathbf{E}}_0)$ is reflexive. Now let

$$\bar{\mathbf{E}} = \bigcup_{i \in \mathbb{Z}} \bar{\mathbf{E}}_i.$$

Then for every vertex v, clearly $[P_v] \in \bar{\mathbf{E}}_k$ for $k = n_v$ or $n_v + 1$. It follows from this and (1) above that for these k

$$[\text{top}(P_v)] \in \bar{\mathbf{E}}_{k-m(v)}.$$

This shows that $\bar{\mathbf{E}}$ contains all isomorphism classes of simple T-modules. Therefore we know that

$$\bigcup_{i \geqslant 0} \{\mathbf{E}_i(S) \mid S \text{ is simple in mod } T\} \subset \bar{\mathbf{E}} \cup \mathbf{P}, \tag{4}$$

where $\mathbf{P}$ denotes the set of isomorphism classes of indecomposable projective T-modules. (See (3.4) before for the notations.) On the other hand, since T is of finite representation type, the left hand side in (4) consists of all isomorphism classes of indecomposable T-modules by [27]. Hence we have that

$$\bar{\mathbf{E}} \cup \mathbf{P} = \bigcup_{i \geqslant 0} \{\mathbf{E}_i(S) \mid S \text{ is simple}\},$$

and it is the set of all isomorphism classes of indecomposable T-modules. As a consequence, since $\bar{\mathbf{E}}_0 \cup \bar{\mathbf{E}}_1$ is an ω-generating set for $\bar{\mathbf{E}}$ by (3.4), $\bar{\mathbf{E}}_0 \cup \bar{\mathbf{E}}_1 \cup \mathbf{P}$ is an ω-basis. This shows that the set of isomorphism classes of indecomposable projective A-modules and of indecomposable projective T-modules is an ω-basis, in view of the above (2) and (3).

The case of the indecomposable injective modules is also proved in this way.

Let A be a hereditary Artin algebra of finite representation type. Then it is well known that every indecomposable A-module M is isomorphic to $(\text{Tr}_A \text{D})^m (P)$ and to $(\text{D}\,\text{Tr}_A)^n (E)$ for some nonnegative integers m and n, where P and E are indecomposable projective and indecomposable injective in mod A, respectively, Tr_A a transpose and D the usual duality functor [7, 12]. However, taking account of the fact [2] that $\text{D}\,\text{Tr}_A(X) \simeq \omega_A(X)$ and $\text{Tr}_A\, D(Y) \simeq \omega_A^{-1}(Y)$ for every indecomposable nonprojective module X and for every indecomposable noninjective module Y, we can now obtain this result as an easy consequence of (4.8):

COROLLARY 4.9. *Let A be a hereditary Artinian ring of finite representation type and with a* QF-*module. Then, for every indecomposable A-module M, there is an indecomposable projective A-module P and an indecomposable injective A-module E such that*

$$M \simeq \omega_A^{-m}(P) \qquad \textit{and} \qquad M \simeq \omega_A^{n}(E)$$

for some nonnegative integers m and n.

Proof. Let Q be a QF-module and T an extension over A with kernel Q. For an indecomposable A-module M, by (4.8) there is an indecomposable projective A-module P such that $M \simeq \omega_T^{-m}(P)$ for some $m \geqslant 0$. We will prove by induction on m that $M \simeq \omega_A^{-m'}(P)$ for some $m' \geqslant 0$.

Now, for an indecomposable A-module isomorphic to $\omega_T^{-0}(P) = P$, there is nothing to prove. For a given $m \geqslant 0$ assume that every indecomposable A-module which is isomorphic to $\omega_T^{-k}(P)$ $(0 \leqslant k \leqslant m)$ is isomorphic to $\omega_A^{-k'}(P)$ for some $k' \geqslant 0$. Let M be an indecomposable A-module which is isomorphic to $\omega_T^{-(m+1)}(P)$ but not isomorphic to $\omega_T^{-n}(P)$ for $0 \leqslant n \leqslant m$. Then clearly M is not projective in $\operatorname{mod} A$. Hence there exists an $\omega_A(M)$ in $\operatorname{mod} A$, so that $\omega_A(M) \simeq \omega_T(M)$ by (4.2). Since $M \simeq \omega_T^{-(m+1)}(P)$, it then follows that

$$\omega_A(M) \simeq \omega_T(\omega_T^{-(m+1)}(P)) \simeq \omega_T^{-m}(P).$$

This also shows that $\omega_T^{-m}(P)$ is an A-module. Therefore $\omega_T^{-m}(P) \simeq \omega_A^{-m'}(P)$ for some $m' \geqslant 0$ by induction hypothesis. Thus we have that

$$M \simeq \omega_A^{-1}(\omega_T^{-m}(P)) \simeq \omega_A^{-(m'+1)}(P),$$

which is a desired result.

The case of injective modules is proved in a similar way.

5. AUSLANDER–REITEN QUIVERS

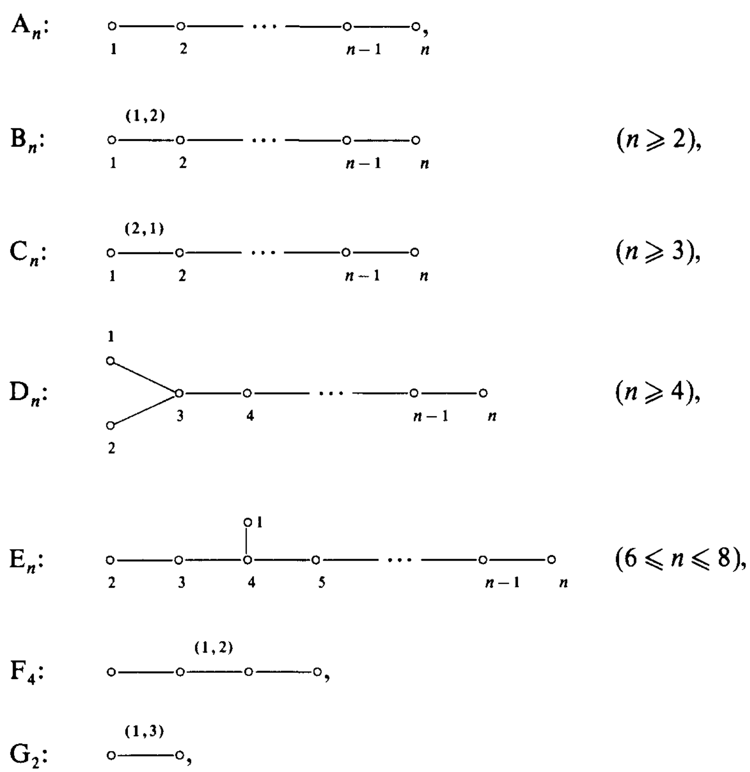
Let A be a hereditary Artin algebra with a QF-module Q and T an extension over A with kernel Q. In this final section we are concerned with the number of indecomposable direct summands of the middle term of an almost split sequence in mod T and the Auslander–Reiten quiver of T.

Following Refs. [6] and [22], we use the following notations. Let $0 \to Z \to Y \to X \to 0$ be an almost split sequence over an Artinian ring R and

$$Y = \bigoplus_{1 \leqslant i \leqslant \alpha(X)} Y_i,$$

where each Y_i is an indecomposable module. We denote by $\alpha(R)$ the maximal number among the $\alpha(X)$, where X ranges over the indecomposable nonprojective R-modules.

In Ref. [12] it is shown that the quiver of a hereditary Artin algebra A of finite representation type is a disjoint union of the following Dynkin diagrams.



where ○——○ means ○⟶○ or ○⟵○.

In the proof of (5.1) below, we determine for any Artin algebra A associated with any of these Dynkin diagrams, an almost split sequence in mod T whose middle term is a sum of $\alpha(T)$ indecomposable summands.

PROPOSITION 5.1. *Let A be a hereditary Artin algebra with a* QF-*module Q and T an extension over A with kernel Q. Then it holds that*

(1) $\alpha(A) \leqslant \alpha(T) \leqslant \alpha(A) + 1$.

(2) *If A is indecomposable and the quiver of A is one of the Dynkin diagrams, then we have the following table.*

the quivers of A	$\alpha(A)$	$\alpha(T)$
A_1	/	1
A_2	1	2
A_n $(n > 2)$	2	2 or 3
B_2	2	3
B_n $(n > 2)$	3	3 or 4
C_n $(n > 2)$	2	3
D_n $(n \geqslant 4)$	3	3 or 4
E_6, E_7, E_8	3	3 or 4
F_4	3	3 or 4
G_2	3	4

Proof. (1) The inequality $\alpha(A) \leqslant \alpha(T)$ is clear by (4.2). To show the other, we first consider the case that there is an injective and projective right A-module. In this case, it is clear that A is serial, because A is hereditary by assumption. Let P be an indecomposable projective right T-module, and let $p: P' \to \operatorname{soc}(P/PQ)$ be a projective cover in $\operatorname{mod} T$. Since $\operatorname{soc}(P/PQ)$ is projective in $\operatorname{mod} A$, it then follows that $P'Q = P' \operatorname{rad}(T)$. Let $f: P' \to P$ be the canonical morphism determined by the projectivity of P'. Then $f(P'Q)$ is injective in $\operatorname{mod} A$ and $f(P'Q) \subset PQ$, so that $f(P'Q) = PQ$. Hence $f(P') \operatorname{rad}(T) = PQ$. Thus we know that P is serial, i.e., T is serial, in view of the fact that P/PQ and PQ is serial. Hence $\alpha(A) \leqslant 2$ and $\alpha(T) \leqslant 2$ by [3, 4.12].

Next we consider the case that any projective A-module is not injective. Let $\mathbb{E}: 0 \to Z \to Y \to X \to 0$ be an almost split sequence in mod T. Then there are three posibilities: (a) $XQ = 0$ and X is nonprojective in $\operatorname{mod} A$, (b) $XQ = 0$ and X is projective in $\operatorname{mod} A$, and (c) $XQ \neq 0$. In case (a), $\mathbb{E}$ is an almost split sequence in $\operatorname{mod} A$ by (4.2). In case (b), $\Omega_T(X)\, Q = 0$, and $\Omega_T(X)$ is injective and hence not projective in $\operatorname{mod} A$ by (2.4). Hence $\Omega_T(\mathbb{E})$ is an almost split sequence in $\operatorname{mod} A$ by [4, 5.1] and (4.2), where $\Omega_T(\mathbb{E})$ denotes the short exact sequence in the top of the following canonical diagram ($\Omega_T^{-1}(\mathbb{E})$ is also similarly defined):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega_T(\mathbb{E}): 0 & \longrightarrow & \Omega_T(Z) & \longrightarrow & \operatorname{Ker} p & \longrightarrow & \Omega_T(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_0 \oplus P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\
 & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p_0}\downarrow & & \\
\mathbb{E}: 0 & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}
$$

where $p_0: P_0 \to X$ and $p_1: P_1 \to Z$ are projective covers in mod T and p is the canonical morphism defined by p_i. In case (c), $\Omega_T^{-1}(X)\, Q = 0$ and $\Omega_T^{-1}(X)$ is nonprojective in mod A by (2.9) and (2.4). Hence $\Omega_T^{-1}(\mathbb{E})$ is also an almost split sequence in mod A. Thus we know that the middle term of $\mathbb{E}$ in case (a), $\Omega_T(\mathbb{E})$ in (b), and $\Omega_T^{-1}(\mathbb{E})$ in (c) have $\alpha(A)$ summands at most. Hence the result follows from [4, 5.1] and (3.2).

(2) We will examine $\alpha(A)$ and $\alpha(T)$ for each quiver of A. Let $(\Gamma, \mathbf{d})$ be the quiver of A. Since every almost split sequence $\mathbb{E}$ in mod A is still almost split in mod T by (4.2), $\mathbb{E}$ is of the form

$$0 \to \omega_T^n(P_v) \to \left(\bigoplus_{d_{v,v'}\neq 0} \omega_T^{n-1}(P_{v'})^{(\ell_{v'})}\right) \oplus \left(\bigoplus_{d_{v'',v}\neq 0} \omega_T^n(P_{v''})^{(\ell_{v''})}\right)$$
$$\to \omega_T^{n-1}(P_v) \to 0,$$

where $\ell_{v'} = d_{v,v'}$ and $\ell_{v''} = d'_{v,v''}$ by (4.8), (4.5) and its Remark. Hence, for each quiver of A, $\alpha(A)$ will be easily obtained, i.e.,

$$\alpha(A) = \text{Max}\left\{\sum_{v'} d_{v,v'} + \sum_{v''} d_{v,v''} \mid v \in \Gamma\right\}.$$

On the other hand, it also follows from (4.2) that $\alpha(T) > \alpha(A)$, i.e., $\alpha(T) = \alpha(A) + 1$ by (1) if and only if there is an almost split sequence in mod T whose middle term contains an indecomposable projective T-module and has at least $\alpha(A) + 1$ direct summands. Thus, in order to know $\alpha(T)$, we have only to check almost split sequences in mod T with projective summands in the middle terms. For $v_i \in \Gamma$, P_i denotes a projective cover of P_{v_i} in mod T. For B_n, C_n, F_4, and G_2 it suffices to consider the orientations with

$$\circ \xrightarrow{(a,b)} \circ \qquad \text{for } a > b$$

because of the self-duality of A. We make use of (4.5).

(i) A_1: $\mathscr{Q}(A) = \overset{v_1}{\circ}$. $\qquad \alpha(T) = 1.$

Since indecomposable A-modules are P_{v_1} only, no almost split sequence exists in mod A, and since T is serial quasi-Frobenius, the almost split sequence in mod T is only

$$0 \to P_{v_1} \to T \to P_{v_1} \to 0.$$

(ii) A_2: $\mathscr{Q}(A) = \overset{v_1}{\circ} \longrightarrow \overset{v_2}{\circ}$. $\qquad \alpha(A) = 1, \qquad \alpha(T) = 2.$

Since $\mathrm{top}(P_1) \simeq \mathrm{top}(P_{v_1})$ and they are injective in mod A, by (4.5) there is an almost split sequence

$$0 \to \mathrm{rad}(P_1) \to P_1 \oplus P_{v_2} \to P_{v_1} \to 0.$$

Hence $\alpha(T) > 1$ and so $\alpha(T) = 2$.

(iii) (a) A_n $(n \geqslant 3)$:

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \longrightarrow \overset{v_2}{\circ} \longrightarrow \overset{v_3}{\circ} \longrightarrow \cdots \longrightarrow \overset{v_n}{\circ}, \qquad \alpha(A) = 2, \quad \alpha(T) = 2.$$

In this case, A is serial. Hence $\alpha(T) = 2$ (see the first part of the proof of (1)).

(b) A_n (n $\geqslant 3$):

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \text{———} \cdots \text{———} \overset{v_{k-1}}{\circ} \longleftarrow \overset{v_k}{\circ} \longrightarrow \overset{v_{k+1}}{\circ} \text{———} \cdots \text{———} \overset{v_n}{\circ},$$

$$\alpha(A) = 2, \qquad \alpha(T) = 3.$$

Since $\mathrm{top}(P_k) \simeq \mathrm{top}(P_{v_k})$ and these are injective in mod A, by (4.5) there is an almost split sequence in mod T

$$0 \to \mathrm{rad}(P_k) \to P_k \oplus P_{v_{k-1}} \oplus P_{v_{k+1}} \to P_{v_k} \to 0.$$

Hence $\alpha(T) \geqslant 3 > \alpha(A)$ (=2) and so $\alpha(T) = 3$.

(c) $\mathrm{A}_n (n \geqslant 3)$:

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \text{———} \cdots \text{———} \overset{v_{k-1}}{\circ} \longrightarrow \overset{v_k}{\circ} \longleftarrow \overset{v_{k+1}}{\circ} \text{———} \cdots \text{———} \overset{v_n}{\circ},$$

$$\alpha(A) = 2, \qquad \alpha(T) = 3.$$

This is a dual statement of (iii(b)).

(iv) B_2:

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \xrightarrow{(2,1)} \overset{v_2}{\circ}, \qquad \alpha(A) = 2, \quad \alpha(T) = 3.$$

Since $\mathrm{top}(P_2) \simeq \mathrm{top}(P_{v_2})$ and they are injective in mod A, by (4.5) there is an almost split sequence in mod T

$$0 \to \mathrm{rad}(P_2) \to P_2 \oplus P_{v_1} \oplus P_{v_1} \to P_{v_2} \to 0.$$

Hence $\alpha(T) \geqslant 3 > \alpha(A)$ (=2). So $\alpha(T) = 3$.

(v) (a) B_n $(n \geqslant 3)$:

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \text{———} \cdots \text{———} \overset{v_{n-2}}{\circ} \longleftarrow \overset{v_{n-1}}{\circ} \xrightarrow{(2,1)} \overset{v_n}{\circ}, \qquad \alpha(A) = 3, \quad \alpha(T) = 4.$$

The $\operatorname{top}(P_{n-1})$ is injective in $\operatorname{mod} A$, and hence there is an almost split sequence

$$0 \to \operatorname{rad}(P_{n-1}) \to P_{n-1} \oplus P_{v_{n-2}} \oplus P_{v_n} \oplus P_{v_n} \to P_{v_{n-1}} \to 0.$$

Hence we know that $\alpha(T) = 4$.

(b) B_n $(n \geqslant 3)$:

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \text{———} \cdots \text{———} \overset{v_{n-2}}{\circ} \longrightarrow \overset{v_{n-1}}{\circ} \xrightarrow{(2,1)} \overset{v_n}{\circ}, \qquad \alpha(A) = 3, \quad \alpha(T) = 3.$$

Every indecomposable injective A-module has the top with length 2 at most. Hence, in case $\operatorname{rad}(P_i) = P_i Q$,

$$|\operatorname{top}((\operatorname{rad}(P_i)/\operatorname{soc}(P_i))| = |\operatorname{top}(P_i Q/\operatorname{soc}(P_i Q))| \leqslant 2.$$

On the other hand, the radical of every indecomposable projective A-module has the top with length 2 at most. Hence, in case $\operatorname{rad}(P_i) \supsetneqq P_i Q$,

$$\begin{aligned} |\operatorname{top}(\operatorname{rad}(P_i)/\operatorname{soc}(P_i))| &= |\operatorname{top}(\operatorname{rad}(P_i/P_i Q))| \\ &= |\operatorname{top}(P_{v_i})| \leqslant 2. \end{aligned}$$

Thus, in any case, $\operatorname{rad}(P_i)/\operatorname{soc}(P_i)$ has at most two indecomposable direct summands. Hence $\alpha(T) \leqslant 3$ in view of (3.2), so that $\alpha(T) = \alpha(A)$.

(vi) C_n $(n \geqslant 3)$:

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \xrightarrow{(2,1)} \overset{v_2}{\circ} \text{———} \overset{v_3}{\circ} \text{———} \cdots \text{———} \overset{v_n}{\circ}, \qquad \alpha(A) = 2, \quad \alpha(T) = 3.$$

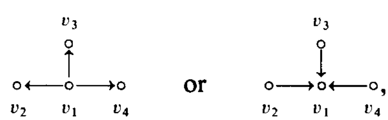
The $\operatorname{top}(P_1)$ is injective in $\operatorname{mod} A$ and so we have an almost split sequence by (4.5)

$$0 \to \operatorname{rad}(P_1) \to P_1 \oplus P_{v_2} \oplus P_{v_1} \to P_{v_1} \to 0.$$

Hence $\alpha(T) \geqslant 3 > \alpha(A)$ (=2) and so $\alpha(T) = 3$.

(vii) D_n $(n \geqslant 4)$, E_6, E_7, E_8:

(a) The case $\mathscr{Q}(A)$ contains



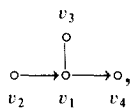
$\alpha(A) = 3$, $\alpha(T) = 4$.

Assume that $\mathscr{Q}(A)$ contains the first quiver. Then $\operatorname{top}(P_1)$ is injective in $\operatorname{mod} A$ and we have an almost split sequence

$$0 \to \operatorname{rad}(P_1) \to P_1 \oplus P_{v_2} \oplus P_{v_3} \oplus P_{v_4} \to P_{v_1} \to 0.$$

Hence $\alpha(T) \geqslant 4 > \alpha(A)$ $(=3)$, so that $\alpha(T) = 4$. The other case can be obtained by the dual argument.

(b) The case $\mathscr{Q}(A)$ contains



$\alpha(A) = 3$, $\alpha(T) = 3$.

This is proved by the same argument as for (v(b)).

(viii) (a) F_4:

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \longrightarrow \overset{v_2}{\circ} \xrightarrow{\;(2,1)\;} \overset{v_3}{\circ} \text{———} \overset{v_4}{\circ}, \qquad \alpha(A) = 3, \quad \alpha(T) = 3.$$

This is also proved by the same argument as for (v(b)).

(b) F_4:

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \longleftarrow \overset{v_2}{\circ} \xrightarrow{\;(2,1)\;} \overset{v_3}{\circ} \text{———} \overset{v_4}{\circ}, \qquad \alpha(A) = 3, \quad \alpha(T) = 3.$$

The $\operatorname{top}(P_2)$ is injective in $\operatorname{mod} A$ and hence we have an almost split sequence

$$0 \to \operatorname{rad}(P_2) \to P_2 \oplus P_{v_1} \oplus P_{v_3} \oplus P_{v_3} \to P_{v_2} \to 0.$$

Hence $\alpha(T) \geqslant 4 > \alpha(A)$ $(=3)$, and so $\alpha(T) = 4$.

(ix) G_2:

$$\mathscr{Q}(A) = \overset{v_1}{\circ} \xrightarrow{\;(3,1)\;} \overset{v_2}{\circ}, \qquad \alpha(A) = 3, \quad \alpha(T) = 4.$$

The $\operatorname{top}(P_1)$ is injective in $\operatorname{mod} A$ and hence we have an almost split sequence

$$0 \to \operatorname{rad}(P_1) \to P_1 \oplus P_{v_2} \oplus P_{v_2} \oplus P_{v_2} \to P_{v_1} \to 0.$$

Hence $\alpha(T) \geqslant 4 > \alpha(A)$ $(=3)$. So $\alpha(T) = 4$.

Remark 5.2. In Ref. [6] it has been proved that $\alpha(A) \leqslant 3$ for a hereditary algebra A of finite representation type. It is also given there an example such that A is a quasi-Frobenius algebra with Loewy length 3 and of finite representation type such that $A/\mathrm{rad}(A)^2$ is hereditary and that it satisfies that $\alpha(A)=4$. But this example is incorrect. In fact, there does not exist a quasi-Frobenius ring A with Loewy length 3 such that $A/\mathrm{rad}((A)^2$ is hereditary. To show this, assume that A is an indecomposable quasi-Frobenius ring with $\mathbf{L}(A)=3$, and suppose that $A/\mathrm{rad}(A)^2$ is hereditary. Let P be an indecomposable projective A-module with $\mathbf{L}(P)=3$. Then, since $P/\mathrm{soc}(P)$ is a projective module over $A/\mathrm{rad}(A)^2$, $\mathrm{rad}(P)/\mathrm{soc}(P)$ is projective in $\mathrm{mod}\, A/\mathrm{rad}(A)^2$. Let S be a simple summand of $\mathrm{rad}(P)/\mathrm{soc}(P)$ and P' a projective cover of S in $\mathrm{mod}\, A$. Since $P'/\mathrm{soc}(P')$ is a module over $A/\mathrm{rad}(A)^2$, $P'/\mathrm{soc}(P')$ is simple. For, S is a factor of $P'/\mathrm{soc}(P')$ and is projective in $\mathrm{mod}\, A/\mathrm{rad}(A)^2$. Hence the canonical morphism $P'/\mathrm{soc}(P')\to S$ splits and so it must be an isomorphism, which implies that the composition length of P' is 2, because $\mathrm{soc}(P')$ is simple. This means that $\mathbf{L}(P')=2$, but this contradicts (3.3).

Next we observe a construction of the Auslander–Reiten quiver of T. Let A be a hereditary Artin algebra, Q a QF-module, and T an extension over A with kernel Q. We conclude this paper by noting how to construct the subquiver of the Auslander–Reiten quiver of T which contains all indecomposable projective A-modules or all indecomposable injective A-modules. Here we recall a definition of the Auslander–Reiten quiver. Let R be an Artinian ring with almost split sequences. A quiver is said to be the *Auslander–Reiten quiver* Γ_R *of* R (or the *quiver for* $\mathrm{mod}\, R$) if it satisfies the following condition: the set of vertices are in a one-to-one correspondence with the set of isomorphism classes of indecomposable R-modules, and there is an arrow from a vertex i to another vertex j iff there is an irreducible morphism from M_i to M_j, where M_k is an indecomposable R-module corresponding to the vertex k. For two vertices i, j and the corresponding modules M_i, M_j, let $M_i\to M$ (resp. $N\to M_j$) be a minimal right (resp. minimal left) almost split morphism (cf. [3]). We denote by d_{ij} the multiplicity of M_j in direct summands of M, and d'_{ji} the multiplicity of M_i in direct summands of N. Then we use the symbol

$$\underset{i}{\circ}\xrightarrow{(d_{ij},d'_{ji})}\underset{j}{\circ}$$

if $d_{ij}\neq 0$ or equivalently $d'_{ji}\neq 0$. As usual, the symbol

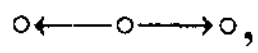
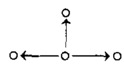
$$\underset{i}{\circ}\longrightarrow\underset{j}{\circ}$$

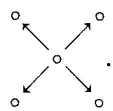
stands for

$$\underset{i}{\circ}\xrightarrow{(1,1)}\underset{j}{\circ}.$$

Now then, a construction of the Auslander–Reiten quiver of T is given by the following method: we first draw the Auslander–Reiten quiver $\mathbf{\Gamma}_A$ of A (for example, this is always possible for a hereditary algebra of finite representation type [12]). $\mathbf{\Gamma}_A$ is a subquiver of the quiver $\mathbf{\Gamma}_T$ for mod T by (4.2). Next double $\mathbf{\Gamma}_A$ by applying Ω_T, so we have a subquiver $\Omega_T(\mathbf{\Gamma}_A)$ by [4, 5.1]. Then fit the two together, and locate the T-projectives using (3.2). It follows from (2.9) that the quiver obtained by this method is the Auslander–Reiten quiver of T.

As examples, we will observe hereditary algebras A over a field K with the following quivers:

(1) A_3: , (2) D_4: ,

(3) $\tilde{D}_4$:  .

For a T-module M, we denote its dimension type by $(\mathbf{dim}\, M/MQ)/\mathbf{dim}\, MQ$ for $MQ \neq 0$, and $\mathbf{dim}\, M$ for $MQ = 0$, where $\mathbf{dim}$ stands for the usual dimension type for A-modules (cf. [12]). Now, following the method mentioned above, for (1) and (2) we have the Auslander–Reiten quivers $\mathbf{\Gamma}_T$ which are given in Figs. 1 and 2, respectively. For (3), A is not of finite representation type and so the subquivers $\mathbf{\Gamma}_{1_A}$ containing A-projectives is disjoint from the subquiver $\mathbf{\Gamma}_{2_A}$ containing A-injectives. However, both $\Omega_T(\mathbf{\Gamma}_{2_A}) \cup \mathbf{\Gamma}_{1_A}$ and $\mathbf{\Gamma}_{2_A} \cup \Omega_T(\mathbf{\Gamma}_{1_A})$ are connected and, locating T-projectives we have the subquivers $\mathbf{\Gamma}_{1_T}$ and $\mathbf{\Gamma}_{2_T}$, respectively. Those quivers are given in Fig. 3.

Here it should be noted that the subquivers without T-projectives of $\mathbf{\Gamma}_T$ in (1) and (2), and $\mathbf{\Gamma}_{i_T}$ in (3) are of types similar to that of graphs (without orientations) of the given quivers $\mathscr{Q}(A)$. To say this more explicitly, we recall some definitions. Let R be an Artinian ring with almost split sequences. The *stable* Auslander–Reiten quiver $\bar{\mathbf{\Gamma}}_R$ of R [23] is the full subquiver of $\mathbf{\Gamma}_R$ with vertices corresponding to the stable indecomposable R-modules, that is, to the indecomposable modules on which ω_R^n is defined for $n \in \mathbb{Z}$. For a subquiver $\mathbf{\Gamma}_0$ of $\mathbf{\Gamma}_R$, let $\mathscr{S}$ be the set of connected subquivers S of $\mathbf{\Gamma}_0$ such that every subquiver

$$[M_1] \xrightarrow{(d_1, d_2)} [M_2] \xrightarrow{(d_3, d_4)} [M_3] \subset S$$

satisfies that $M_1 \not\simeq \omega_R(M_3)$. $\mathscr{S}$ is clearly a partially ordered set by inclusion. Then a maximal element in $\mathscr{S}$ is called a *section* of $\mathbf{\Gamma}_0$ (in the sense of Bautista). Now then we have the following result. It is essentially the same statement as in (4.8). In fact it will be shown by making use of (4.7) as in (4.8) and so we omit the proof.

PROPOSITION 5.3. *Let A be an indecomposable hereditary Artin algebra with a* QF-*module Q and T an extension over A with kernel Q. Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be the subquivers of Γ_A consisting of A-projectives and A-injectives, respectively, and let $\bar{\Gamma}_{1_T}$ and $\bar{\Gamma}_{2_T}$ be the connected subquivers of $\bar{\Gamma}_T$ which contain A-projectives and A-injectives, respectively. Then $\mathbf{Q}_i$ is a section of $\bar{\Gamma}_{i_T}$, and every section of $\bar{\Gamma}_{i_T}$ has the same graph as $\mathbf{Q}_i$ for $i = 1, 2$.*

Again, as for the above examples, $\bar{\Gamma}_T$ in (1) and (2) are of type A_3 and D_4 in Fig. 4, respectively, and $\bar{\Gamma}_i$ $(i = 1, 2)$ in (3) are of type $\tilde{D}_4$ in Fig. 5.

Remark. Let A be an indecomposable hereditary Artinian ring with a QF-module Q, and T an extension over A with kernel Q. If we write

$$\underset{i}{\circ}\longrightarrow\underset{j}{\circ}$$

simply instead of

$$\underset{i}{\circ}\xrightarrow{(d_{ij},d'_{ji})}\underset{j}{\circ},$$

then we can also know the types of the subquivers, of the stable Auslander–Reiten quiver of T, which contain the A-projectives or A-injectives. For example, according to the diagrams A_n, B_n $(n \geqslant 2)$, C_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$, E_n $(n = 6, 7, 8)$, F_4, and G_2, associated with A, the stable quivers for mod T are of type A_n, A_n $(n \geqslant 2)$, A_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$, E_n $(n = 6, 7, 8)$, A_4, and A_2, respectively.

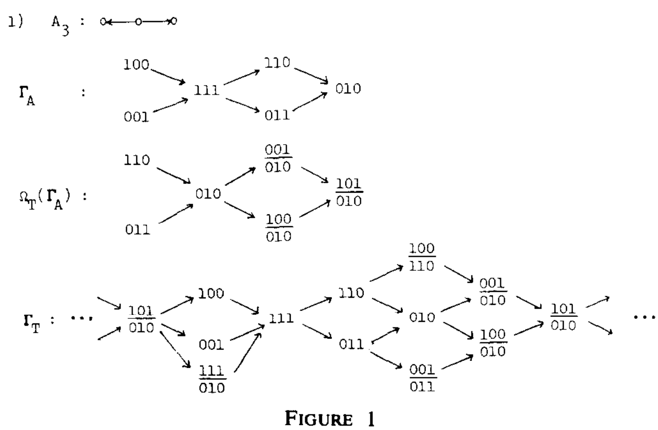


FIGURE 1

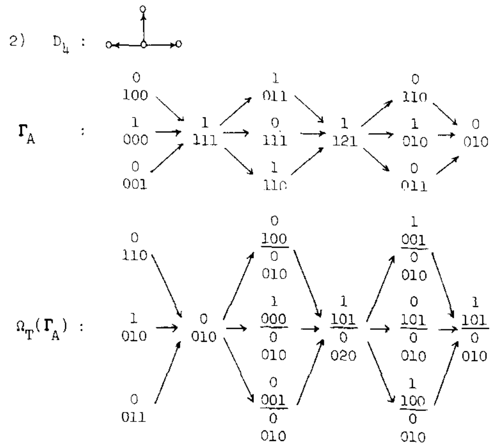


FIGURE 2(i)

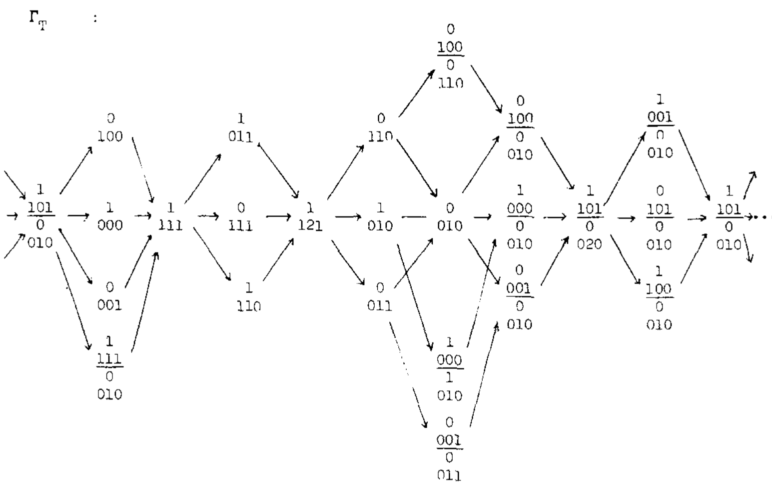
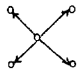
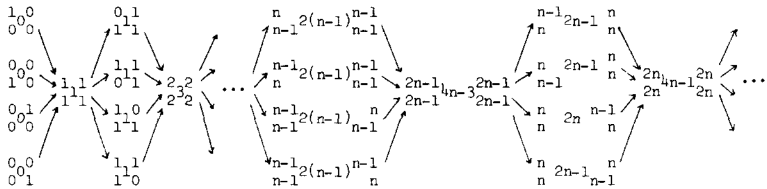


FIGURE 2(ii)

3) $\tilde{D}_4$:



Γ_{1_A} $(n \geq 1)$:



Γ_{2_A} $(n \geq 1)$:

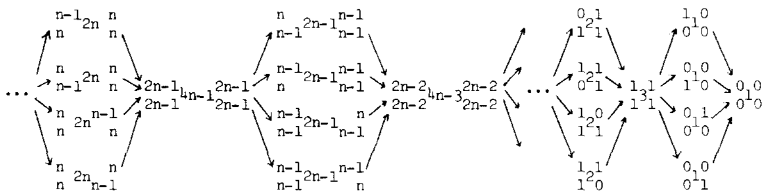
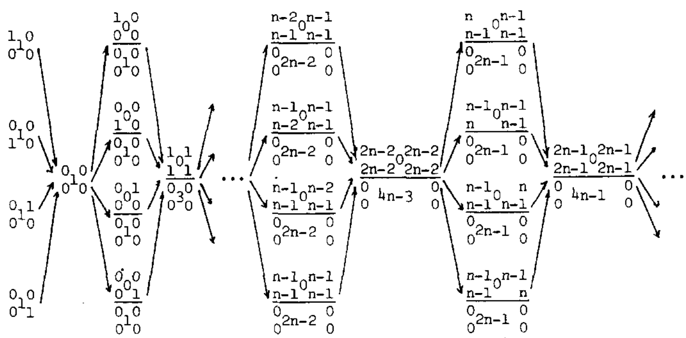


FIGURE 3(i)

$\Omega_T(\Gamma_{1_A})$ $(n \geq 2)$:



$\Omega_T(\Gamma_{2_A})$ $(n \geq 2)$:

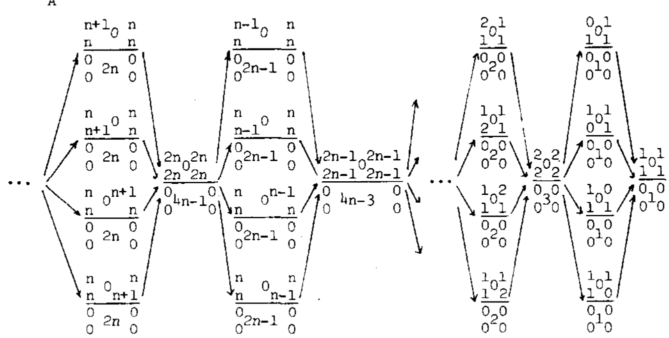
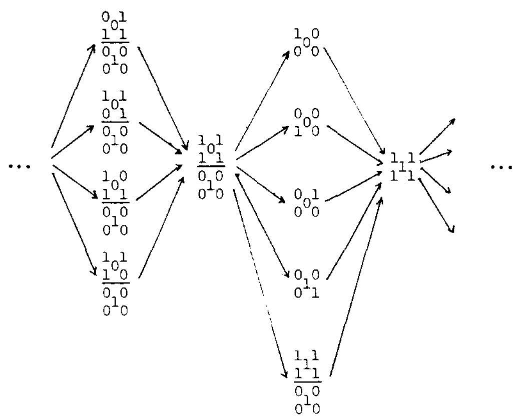


FIGURE 3(ii)

Γ_{1_T}



Γ_{2_T}

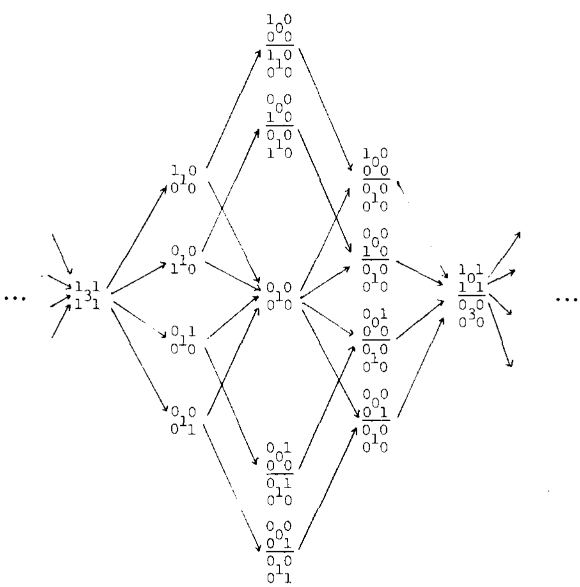


FIGURE 3(iii)

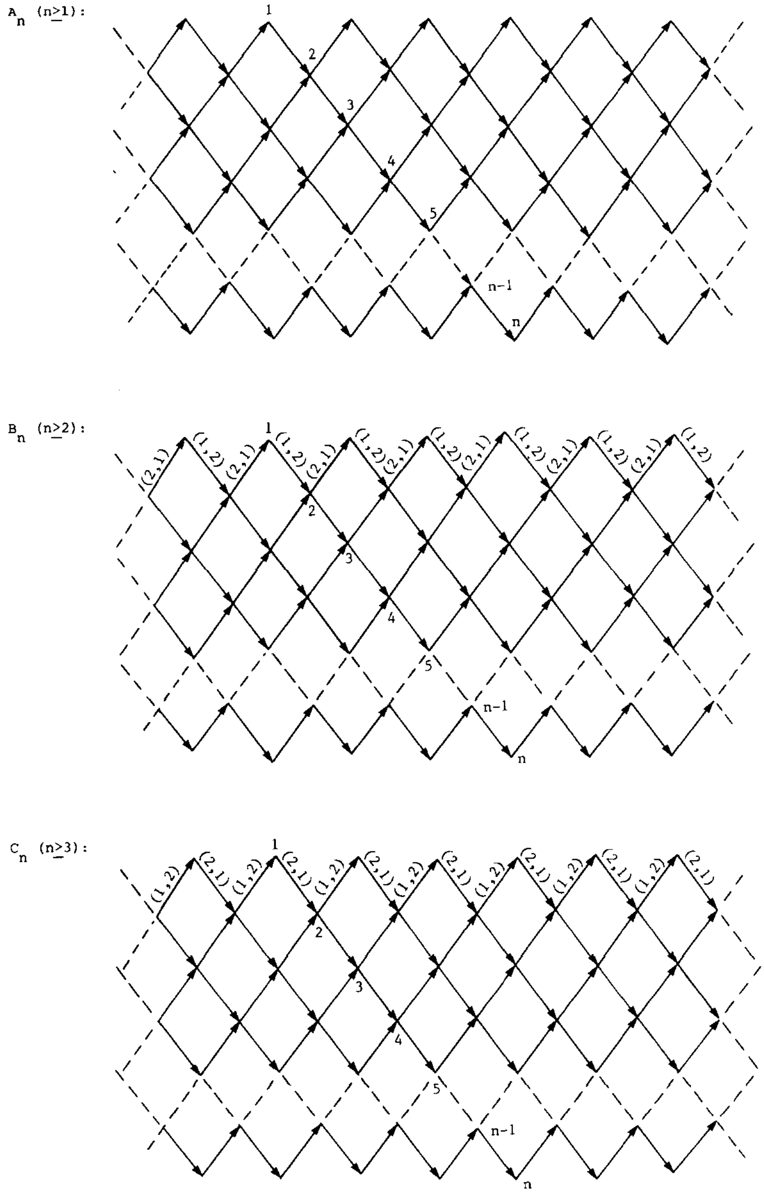


FIG. 4. Stable Auslander–Reiten quivers of extensions over heriditary Artin algebras of finite representation type.

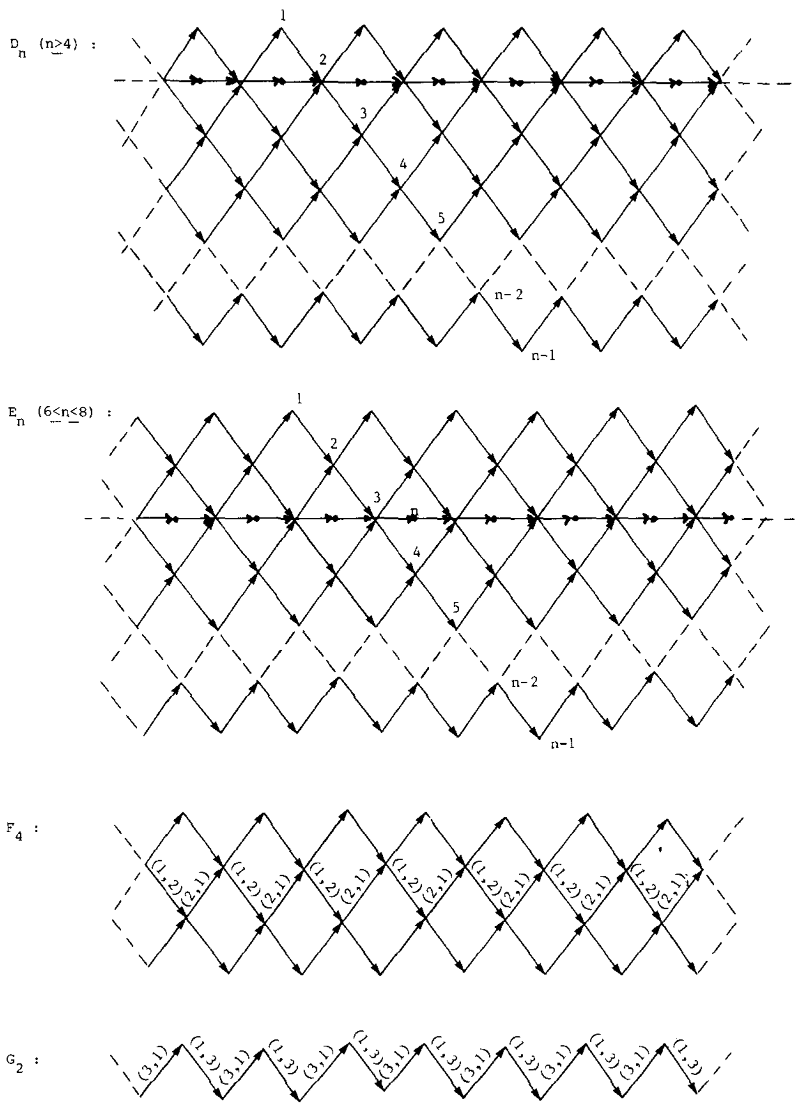


FIG. 4—*Continued.*

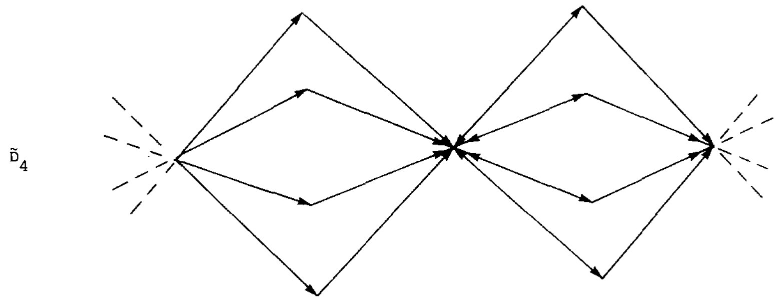


FIGURE 5

REFERENCES

1. M. AUSLANDER, On the dimension of modules and algebras, III, *Nagoya Math. J.* **9** (1955), 67–77.
2. M. AUSLANDER AND I. REITEN, Representation theory of Artin algebras, III, *Comm. Algebra* **3** (1975), 269–310.
3. M. AUSLANDER AND I. REITEN, Representation theory of Artin algebras, IV, *Comm. Algebra* **5** (1977), 443–518.
4. M. AUSLANDER AND I. REITEN, Representation theory of Artin algebras, V, *Comm. Algebra* **5** (1977), 519–554.
5. M. AUSLANDER AND I. REITEN, Representation theory of Artin algebras, VI, *Comm. Algebra* **6** (1978), 257–300.
6. M. AUSLANDER, R. BAUTISTA, M. I. PLATZECK, I. REITEN, AND S. O. SMALØ, Almost split sequences whose middle term has at most two indecomposable summands, *Canad. J. Math.* **31** (1979), 942–960.
7. M. AUSLANDER AND M. I. PLATZECK, "Representation Theory of Hereditary Artin Algebras" (R. Gordon, Ed.), Lecture Notes in Pure and Applied Math., Vol. 37, Dekker, New York, 1978.
8. G. AZUMAYA, A duality theory for injective modules, *Amer. J. Math.* **81** (1959), 249–278.
9. R. BAUTISTA, On algebras close to hereditary Artin algebras, preprint.
10. S. BRENNER AND M. C. R. BUTLER, The equivalence of certain functors occuring in the representation theory of Artin algebras and species, *J. London Math. Soc.* **14** (1976), 183–189.
11. H. CARTAN AND S. EILENBERG, "Homological Algebra," Princeton, N.J., Princeton Univ. Press, 1956.
12. V. DLAB AND C. M. RINGEL, "Representations of Graphs and Algebras," Memoirs of the A. M. S. No. 173, Amer. Math. Soc., Providence, R.I., July, 1976.
13. JU. A. DROZD, Coxeter transformations and representations of partially ordered sets, *Funct. Anal. Appl.* **8** (1974), 34–42.
14. S. EILENBERG, A. ROSENBERG, AND D. ZELINSKY, On the dimension of modules and algebras, VIII, *Nagoya Math. J.* **12** (1957), 71–93.
15. R. FOSSUM, P. GRIFFITH, AND I. REITEN, "Trivial Extensions of Abelian Categories," Lecture Notes in Math. No. 456, Springer-Verlag, Berlin/New York, 1975.
16. K. R. FULLER, Biserial rings, Proc. of Ring theory Conf. Waterloo, July, 1978.
17. M. IKEDA, On absolutely segregated algebras, *Nagoya Math. J.* **6** (1953), 63–75.

18. K. MORITA, Duality for modules and its applications to the theory of rings with minimum condition, *Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A* **6** (1958), 83–142.
19. W. MÜLLER, "On Artin rings of Finite Representation Type," Lecture Notes in Math. No. 488, Springer-Verlag, Berlin/New York, 1974.
20. T. NAKAYAMA, On Frobenius algebras, II, *Ann. of Math.* **42** (1941), 1–21.
21. T. NAKAYAMA AND G. AZUMAYA, "Algebra II. Ring Theory," Iwanami-Shoten, Tokyo, Japan, 1956 (in Japanese).
22. I. REITEN, Almost split sequences, Proc. of Ring Theory Conf. Antwerp, August, 1978.
23. CHR. RIEDTMANN, Algebren, Darstellungsköcher, Überlärungen und Zurück, preprint.
24. C. M. RINGEL, Finite dimensional hereditary algebras of wild representation type, *Math. Z.* **161** (1978), 235–255.
25. H. TACHIKAWA, Trivial extensions of finite representation type, *in* "Proc. Symp. on Representations of groups and algebras, University of Tsukuba, Ibaraki, 1978," pp. 69–80 (in Japanese).
26. R. B. WARFIELD, JR., A Krull-Schumidt theorem for infinite sums of modules, *Proc. Amer. Math. Soc.* **22** (1969), 460–465.
27. K. YAMAGATA, On Artinian rings of finite representation type, *J. Algebra* **50** (1978), 276–283.